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SMALL LOCALLY COMPACT LINEARLY LINDELÖF SPACES

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Abstract. There is a locally compact Hausdorff space of weight $\aleph_\omega$ which is linearly Lindelöf and not Lindelöf.

We shall prove:

Theorem 1. *There is a compact Hausdorff space X and a point p in X such that:*

(1) $\chi(p, X) = w(X) = \aleph_\omega$.
(2) *For all regular $\kappa > \omega$, no κ-sequence of points distinct from p converges to p.*

As usual, $\chi(p, X)$, the *character* of p in X, is the least size of a local base at p, and $w(X)$, the *weight* of X, is the least size of a base for X. This theorem with "$\beth_\omega$" replacing "$\aleph_\omega$" was proved in [11]. A. V. Arhangel'skii and R. Z. Buzyakova [1] point out that if X, p satisfy (2) of the theorem, then the space $X\backslash\{p\}$ is linearly Lindelöf and locally compact; if, in addition, $\chi(p, X) > \aleph_0$, then $X\backslash\{p\}$ is not Lindelöf. (2) requires $\mathrm{cf}(\chi(p, X)) = \omega$, because there must be a sequence of type $\mathrm{cf}(\chi(p, X))$ converging to p. Thus, in (1) of the theorem, $\aleph_\omega$ is the smallest possible uncountable value for $\chi(p, X)$ and $w(X)$.

As in [11], the X of the theorem will be constructed as an inverse limit, using the following terminology:

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Definition 2. An *inverse system* is a sequence $\langle X_n, \pi_n^{n+1} : n \in \omega \rangle$, where each X_n is a compact Hausdorff space, and each π_n^{n+1} is a continuous map from X_{n+1} onto X_n.

Such an inverse system yields a compact Hausdorff space, $X_\omega = \varprojlim_n X_n$, and maps $\pi_m^\omega : X_\omega \twoheadrightarrow X_m$ for $m < \omega$ and $\pi_m^n : X_n \twoheadrightarrow X_m$ for $m \leq n < \omega$. Exactly as in [11], one easily proves:

Lemma 3. *Suppose that $\langle X_n, \pi_n^{n+1} : n \in \omega \rangle$ is an inverse system and $p \in X = X_\omega$, with the $p_n = \pi_n^\omega(p) \in X_n$ satisfying:*

(A) *Each p_n is a weak $P_{\aleph_n}$-point in X_n.*
(B) *Each $w(X_n) < \aleph_\omega$.*
(C) *Each $(\pi_0^n)^{-1}\{p_0\}$ is nowhere dense in X_n.*

Then X, p satisfy Theorem 1.

As usual, $y \in Y$ is a *weak* P_κ-point iff y is not in the closure of any subset of $Y \backslash \{y\}$ of size less than κ, and y is a P_κ-point iff the intersection of fewer than κ neighborhoods of y is always a neighborhood of y. These properties are trivial for $\kappa = \aleph_0$. The terms "P-point" and "weak P-point" denote "$P_{\aleph_1}$-point" and "weak $P_{\aleph_1}$-point," respectively.

Every P_κ-point is a weak P_κ-point, but as pointed out in [11], one cannot have each p_n being a $P_{\aleph_n}$-point, as that would contradict (C). In the construction we describe, it will be natural to make every p_n fail to be a P-point in X_n.

We shall build the X_n and p_n inductively using the following:

Lemma 4. *Assume that $y \in F \subseteq Y$, where Y is compact Hausdorff, $w(Y) \leq \aleph_n$, and $\mathrm{int}(F) = \emptyset$. Then there is a compact Hausdorff space X, a point $x \in X$, and a continuous $g : X \to Y$ such that:*

(1) *$g(X) = Y$ and $g(x) = y$.*
(2) *$g^{-1}(F)$ is nowhere dense in X.*
(3) *$w(X) = \aleph_n$.*
(4) *In X, x is a weak $P_{\aleph_n}$-point and not a P-point.*

Proof of Theorem 1: Inductively build an inverse system as in Lemma 3, with each $w(X_n) = \aleph_n$. X_0 can be the Cantor set. When $n > 0$ and we are given X_{n-1}, p_{n-1}, we apply Lemma 4 with $F = (\pi_0^{n-1})^{-1}\{p_0\}$. □

Of course, we still need to prove Lemma 4. We remark that we do not assume that F is closed, although that was true in our proof of Theorem 1. Even if F is dense in Y in Lemma 4, we still get (2)— that is, $\mathrm{int}(\mathrm{cl}(g^{-1}(F))) = \emptyset$.

In Lemma 4, n can be 0, although this case is not used in the proof of Theorem 1. For this case, the "weak $P_{\aleph_0}$-point" is trivial, and the lemma is easily proved by an Aleksandrov duplicate construction. A more convoluted proof is: Let $D \subseteq Y \backslash F$ be dense in Y and countable. Let g map ω onto D and extend g to a map $\beta g : \beta\omega \twoheadrightarrow Y$. Choosing x to be any point in $(\beta g)^{-1}(\{y\})$ yields (1), (2), and (4), but $\beta\omega$ has weight $2^{\aleph_0}$. Now, we can take a countable elementary submodel of the whole construction to get an X of weight $\aleph_0$. Our proof for a general n will follow this pattern.

As usual, $\beta\kappa$ denotes the Čech compactification of a discrete κ, and $\kappa^* = \beta\kappa\backslash\kappa$. Equivalently, $\beta\kappa$ is the space of ultrafilters on κ, and κ^* is the space of nonprincipal ultrafilters. If $g : \kappa \to Y$, where Y is compact Hausdorff, then βg denotes the unique extension of g to a continuous map from $\beta\kappa$ to Y. Our weak P_κ-point in Lemma 4 will be a *good* ultrafilter in the sense of H. Jerome Keisler [9]:

Definition 5. An ultrafilter x on κ is *good* iff for all $H : [\kappa]^{<\omega} \to x$, there is a $K : \kappa \to x$ such that $K(\alpha_1) \cap \cdots \cap K(\alpha_n) \subseteq H(s)$ for each $s = \{\alpha_1, \ldots, \alpha_n\} \in [\kappa]^{<\omega}$.

The following is well-known:

Lemma 6. *Let κ be any infinite cardinal.*

(1) *There are ultrafilters x on κ which are both good and countably incomplete.*
(2) *Any x as in (1) is a weak P_κ point and not a P-point in $\beta\kappa$.*

In (2), x is not a P-point by countable incompleteness, and proofs that it is a weak P_κ point can be found in [2], [3], and [5]. For (1), see [4, Theorem 6.1.4]; also, [2] and [3] construct good ultrafilters with various additional properties.

We first point out (Lemma 9) that taking x to be a good ultrafilter on ω_n will give us (1), (2), and (4) of Lemma 4. Unfortunately, $w(\beta\omega_n) = 2^{\aleph_n}$, so we shall take an elementary submodel to bring the weight down. Omitting the elementary submodel, our argument is as in [11], which obtained the X of Theorem 1 with $w(X) = \beth_\omega$, rather than $\aleph_\omega$. A related use of elementary submodels to reduce the weight occurs in [7].

Before we consider the weight problem, we explain how to map the good ultrafilter onto the given point y. This part of the argument works for any regular ultrafilter.

Definition 7. An ultrafilter x on κ is *regular* iff there are $E_\alpha \in x$ for $\alpha < \kappa$ such that $\{\alpha : \xi \in E_\alpha\}$ is finite for all $\xi < \kappa$.

Such an x is countably incomplete because $\bigcap_{n<\omega} E_n = \emptyset$. For the following, see [4, Exercise 6.1.3] or the proof of Lemma 2.1 in Keisler [10]:

Lemma 8. *If x is a countably incomplete good ultrafilter on κ, then x is regular.*

Lemma 9. *Let x be a regular ultrafilter on κ. Assume that $y \in F \subseteq Y$, where Y is compact Hausdorff, $w(Y) \leq \kappa$, and* $\mathrm{int}(F) = \emptyset$. *Then there is a map $g : \kappa \to Y$ such that:*

(A) *βg maps $\beta\kappa$ onto Y.*
(B) *$(\beta g)(x) = y$.*
(C) *$g(\xi) \notin F$ for all $\xi \in \kappa$.*
(D) *$g^{-1}(F)$ is nowhere dense in $\beta\kappa$.*

Proof: Of course, (D) follows from (C) because $g^{-1}(F) \subseteq \kappa^*$. Fix $A \subseteq \kappa$ with $A \notin x$ and $|A| = \kappa$. Let $\{E_\alpha : \alpha < \kappa\}$ be as in Definition 7, with each $E_\alpha \cap A = \emptyset$. Let $\{U_\alpha : \alpha < \kappa\}$ be an open base at y in Y. Let $D \subseteq Y \backslash F$ be dense in Y with $|D| \leq \kappa$. Choose $g : \kappa \to Y$ such that g maps A onto D (ensuring (A)) and each $g(\xi) \in \bigcap\{U_\alpha : \xi \in E_\alpha\} \setminus F$ (ensuring (B) and (C)). □

To apply the elementary submodel technique (as in [6]), we put the construction of Lemma 9 inside an $H(\theta)$, where θ is a suitably large regular cardinal. Let $M \prec H(\theta)$, with $\kappa \subset M$ and $|M| = \kappa$, such that M contains Y and its topology $\mathcal{T}$, along with F, g, x, y. Let $\mathcal{B} = \mathcal{P}(\kappa) \cap M$, let $\mathrm{st}(\mathcal{B})$ denote its Stone space, and let $\Gamma : \beta\kappa \twoheadrightarrow \mathrm{st}(\mathcal{B})$ be the natural map; so $\Gamma(x) = x \cap \mathcal{B} = x \cap M$. Since $\mathcal{T} \cap M$ is a base for Y (by $w(Y) \leq \kappa$), we have $\Gamma(z_1) = \Gamma(z_2) \to (\beta g)(z_1) = (\beta g)(z_2)$, so that βg yields a map $\widetilde{g} : \mathrm{st}(\mathcal{B}) \to Y$ with $\beta g = \widetilde{g} \circ \Gamma$. Note that $\mathcal{B}$ contains all finite subsets of κ, so that $\mathrm{st}(\mathcal{B})$ is some compactification of a discrete κ. It is easily seen that we still have (A)–(D), replacing βg by $\widetilde{g}$, $\beta\kappa$ by $\mathrm{st}(\mathcal{B})$, and x by

$\Gamma(x)$. Note that $\Gamma(x)$ must be countably incomplete by $M \prec H(\theta)$, so that $\Gamma(x)$ will not be a P-point in $\mathrm{st}(\mathcal{B})$. But to prove Lemma 4 (letting $\kappa = \aleph_n$), we also need $\Gamma(x)$ to be a weak P_κ-point in $\mathrm{st}(\mathcal{B})$. We may assume that $x \in \beta\kappa$ is good, so it is a weak P_κ-point there. But we need to show that in $\mathrm{st}(\mathcal{B})$, $\Gamma(x)$ is not a limit point of any set of size $\lambda < \kappa$. Our argument here needs to assume that M is λ-covering and that λ^+ is not a Jónsson cardinal. These two assumptions will cause no problems when $\lambda < \aleph_\omega$.

As usual, $M \prec H(\theta)$ is *λ-covering* iff for all $E \in [M]^\lambda$, there is an $F \in [M]^\lambda$ such that $E \subseteq F$ and $F \in M$. By taking a union of an elementary chain of type λ^+ (see [6, §3]), we see that there is an $M \prec H(\theta)$ with $|M| = \lambda^+$ such that M is λ-covering.

κ is called a *Jónsson cardinal* iff for all $\psi : [\kappa]^{<\omega} \to \kappa$, there is a $W \in [\kappa]^\kappa$ such that $\psi([W]^{<\omega})$ is a proper subset of κ.

By Jan Tryba [12] (or see [8]):

Lemma 10. *No successor to a regular cardinal is Jónsson.*

In particular, each $\aleph_n$ is not a Jónsson cardinal; this fact is much older and is easily proved by induction on n.

Lemma 11. *Let κ be infinite and $x \in \beta\kappa$ a good ultrafilter on κ. Fix an infinite $\lambda < \kappa$ and let $\theta > 2^\kappa$ be regular. Let $M \prec H(\theta)$, with $x, \kappa \in M$ and $\kappa \subset M$. Assume that M is λ-covering and λ^+ is not a Jónsson cardinal. Let $\mathcal{B} = \mathcal{P}(\kappa) \cap M$, and let $\Gamma : \beta\kappa \twoheadrightarrow \mathrm{st}(\mathcal{B})$ be the natural map. Then $\Gamma(x)$ is a weak P_{λ^+}-point of* $\mathrm{st}(\mathcal{B})$.

Proof: Fix $Z \subseteq \mathrm{st}(\mathcal{B}) \backslash \{\Gamma(x)\}$ with $|Z| \leq \lambda$. We shall show that $\Gamma(x)$ is not in the closure of Z. For each $z \in Z$, choose $F_z \in \Gamma(x) = x \cap \mathcal{B} = x \cap M$ such that $F_z \notin z$. Since M is λ-covering, we can get $\langle G_\xi : \xi < \lambda \rangle \in M$ such that each $G_\xi \in x$ and $\forall z \in Z$ $\exists \xi < \lambda \, [G_\xi = F_z]$. Since λ^+ is not Jónsson and $\lambda^+ \in M$, we can fix $\psi \in M$ such that $\psi : [\lambda^+]^{<\omega} \to \lambda$ and such that $\psi([W]^{<\omega}) = \lambda$ for all $W \in [\lambda^+]^{\lambda^+}$. Define $H(s) = G_{\psi(s)}$. Then $H \in M$ and $H : [\lambda^+]^{<\omega} \to \Gamma(x)$. Since x is good, we can find $\langle K_\alpha : \alpha < \lambda^+ \rangle \in M$ such that each K_α is in x (and hence in $\Gamma(x) = x \cap M$), and such that $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subseteq H(\{\alpha_1, \ldots, \alpha_n\})$ for each n and each $\alpha_1, \ldots, \alpha_n \in \lambda^+$.

Now (in V), we claim that $\exists \alpha < \lambda^+ \, \forall z \in Z \, [K_\alpha \notin z]$ (so that $\Gamma(x) \notin \mathrm{cl}(Z)$). If not, then we can fix $W \in [\lambda^+]^{\lambda^+}$ and $z \in Z$ such that $K_\alpha \in z$ for all $\alpha \in W$. Fix $\xi < \lambda$ such that $G_\xi = F_z$.

Since $\psi([W]^{<\omega}) = \lambda$, fix $s \in [W]^{<\omega}$ such that $\psi(s) = \xi$. Say $s = \{\alpha_1, \ldots, \alpha_n\}$. Then $G_\xi = G_{\psi(s)} = H(s) \supseteq K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \in z$, a contradiction, since $F_z \notin z$. □

Proof of Lemma 4: Use lemmas 11 and 9, with $\kappa = \lambda^+ = \aleph_n$. □

In view of Lemma 10, we can also prove Theorem 1, replacing $\aleph_\omega$ with any other singular cardinal of cofinality ω, since we can replace $\aleph_n$ in Lemma 4 by any successor to a regular cardinal.

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