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REALIZABLE REPETITION PATTERNS IN CONSTRAINED TOTAL NEGATION

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Abstract. Iteration of Bankston's total negation procedure generates a sequence of invariants that becomes repetitive in one of just seven patterns. When the discussion is restricted to some fixed family (the *constraint*) of spaces, this simple behavior can be destroyed, but not if the constraint satisfies natural side conditions. This note exhaustively explores how the constrained and unconstrained repetition patterns, starting with any given invariant, are then related.

1. General Principles of Total Negation

Total negation is a procedure, first formulated by Paul Bankston in 1979 [1], for generating from any given topological property $\mathcal{P}$ another (denoted by anti($\mathcal{P}$)) that is, in a precisely defined sense, "the opposite property." Iteration of the procedure generates a succession of properties that becomes repetitive in one of only seven patterns. Recent investigations have shown that when the procedure is constrained to take place within some fixed family (the *constraint*) of spaces, this simple iterative behavior can in general be lost, but not if the constraint satisfies some simple and natural hypotheses. Here we focus on how the patterns of repetition, under constraint and in the unconstrained (*universal*) setting, are related to one another for an arbitrary initial invariant by exhaustively listing the patterns that might arise from a given starting pattern once

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the analysis becomes constrained. We provide the critical proofs and examples to support the conclusions, several of which have been announced in conference proceedings [6]. A previous publication [7] also deals with possible constrained iteration patterns without discussing their relationship to any original pattern, and the proofs from there are not duplicated below. For completeness, however, we begin with a fairly full outline of relevant previous results, but without proofs which may be found in the references cited, especially [1] and [3].

Definition 1.1. Given a property $\mathcal{P}$, the *spectrum* of $\mathcal{P}$ is the class $\mathrm{spec}(\mathcal{P})$ of non-zero cardinal numbers λ such that every topology on a set of cardinality λ is a $\mathcal{P}$ topology.

Definition 1.2. Let $\mathcal{P}$ be a topological invariant. The *total negation* of $\mathcal{P}$ is the class $\mathrm{anti}(\mathcal{P})$ of topological spaces X such that, whenever $Y \subseteq X$, then Y is $\mathcal{P}$ if and only if $|Y| \in \mathrm{spec}(\mathcal{P})$. Further, we define

$$\begin{aligned} &\mathrm{anti}^0(\mathcal{P}) = \mathcal{P}; \\ &\mathrm{anti}^{n+1}(\mathcal{P}) = \mathrm{anti}(\mathrm{anti}^n(\mathcal{P})) \quad \text{if } n \geq 0. \end{aligned}$$

Definition 1.3. We also define

- the class of *indecisive* cardinals $\mathrm{ind}(\mathcal{P})$ to comprise those cardinals $\lambda \neq 0$ for which there exists both a $\mathcal{P}$ space and a non-$\mathcal{P}$ space on λ-many points;
- the class of *prohibitive* cardinals $\mathrm{proh}(\mathcal{P})$ to comprise those cardinals $\lambda \neq 0$ such that no $\mathcal{P}$ space on λ-many points exists.

The class $\mathrm{ind}(\mathcal{P})$ provides an alternative and convenient way of describing total negation.

Lemma 1.4. *Let $\mathcal{P}$ be a topological invariant. The* non-anti($\mathcal{P}$) *spaces are precisely those which possess a $\mathcal{P}$ subspace Y such that $|Y| \in \mathrm{ind}(\mathcal{P})$.*

Examples of anti-properties can be found in, for instance, [4] and [8].

Definition 1.5. The sequence of invariants

$$\mathcal{P}, \mathrm{anti}(\mathcal{P}), \mathrm{anti}^2(\mathcal{P}), \mathrm{anti}^3(\mathcal{P}), \ \ldots$$

is called the *Bankston iteration sequence.*

The structure of the Bankston iteration sequence is, in many cases, very simple. See I. L. Reilly and M. K. Vamanamurthy [8], for example.

J. Matier and T. B. M. McMaster [3] showed that the Bankston iteration sequence is never very complicated, as shown by the following results. We shall use the symbol $\mathcal{U}$ to represent the class of all topological spaces.

Lemma 1.6. *Let $\mathcal{P}$ be a topological invariant. Then*

$$\text{anti}(\mathcal{P}) = \mathcal{U} \Leftrightarrow \text{ind}(\mathcal{P}) = \emptyset.$$

We observe from this that, in particular, $\text{anti}(\mathcal{U}) = \mathcal{U}$. It is readily verified that no other property maps to itself under total negation.

Lemma 1.7. *Let $\mathcal{P}$ be a topological invariant. Then*

$$\text{anti}(\mathcal{P}) = \mathcal{P} \Leftrightarrow \mathcal{P} = \mathcal{U}.$$

When the total negation operation is repeated, it can be shown that the generated sequence will follow one of only seven possible patterns.

Theorem 1.8. *The pattern of repetition in the Bankston iteration sequence (beginning with an arbitrary topological invariant) is one of the following*

(1) $(\mathcal{U})'$	*(5)* $(\mathcal{P}, \mathcal{Q})'$
(2) $\mathcal{P}, (\mathcal{U})'$	*(6)* $\mathcal{P}, (\mathcal{Q}, \mathcal{R})'$
(3) $\mathcal{P}, \mathcal{Q}, (\mathcal{U})'$	*(7)* $\mathcal{P}, \mathcal{Q}, (\mathcal{R}, \mathcal{S})'$
(4) $\mathcal{P}, \mathcal{Q}, \mathcal{R}, (\mathcal{U})'$	

where we use $(*)'$ as a short form for an unending repetition of the sequence segment $*$ enclosed in parentheses, and we observe the convention that $\mathcal{P}$, $\mathcal{Q}$, etc., denote distinct, non-universal invariants. Furthermore, (4) and (7) cannot occur when $\mathcal{P}$ is hereditary.

2. Constrained Total Negation

We next consider the effect upon the process of total negation when it is carried out in a restricted family of spaces $\mathcal{C}$, called the *constraint,* instead of in the family $\mathcal{U}$ of all topological spaces.

Definition 2.1. If $\mathcal{P}$ is a class of topological spaces and $\mathcal{C}$ is a constraint, then we shall employ the following notation:

- $\mathcal{C}$-spec($\mathcal{P}$) is the class of cardinal numbers λ such that
  - $\lambda \notin$ proh($\mathcal{C}$); and
  - all $\mathcal{C}$ spaces on λ-many points are $\mathcal{P}$.
- $\mathcal{C}$-ind($\mathcal{P}$) is the class of cardinal numbers λ such that
  - $\lambda \notin$ proh($\mathcal{C}$); and
  - there exist topological spaces $X \in \mathcal{C} \cap \mathcal{P}$ and $Y \in \mathcal{C} \setminus \mathcal{P}$ with $|X| = |Y| = \lambda$.
- $\mathcal{C}$-proh($\mathcal{P}$) is the class of cardinal numbers λ such that
  - $\lambda \notin$ proh($\mathcal{C}$); and
  - no $\mathcal{C}$ spaces on λ-many points are $\mathcal{P}$.
- $\mathcal{C}$-anti($\mathcal{P}$) is the class of topological spaces X for which
  - X is a $\mathcal{C}$ space;
  - if Y is a $\mathcal{C}$ and $\mathcal{P}$ subspace of X, then $|Y| \in \mathcal{C}$-spec($\mathcal{P}$).

Clearly, when the constraint $\mathcal{C}$ is taken to be $\mathcal{U}$, these definitions all collapse to their "classical" counterparts.

It turns out to be possible to generalize many results, such as lemmas 1.4, 1.6, and 1.7.

Lemma 2.2. *Let $\mathcal{C}$ and $\mathcal{P}$ be topological invariants. The* non $\mathcal{C}$-anti($\mathcal{P}$) *spaces are those which are not $\mathcal{C}$* or *which possess a $\mathcal{C}$ and $\mathcal{P}$ subspace Y such that $|Y| \in \mathcal{C}$-ind($\mathcal{P}$).*

Lemma 2.3. *Let $\mathcal{C}$ and $\mathcal{P}$ be topological invariants such that $\mathcal{C} \cap \mathcal{P} \neq \emptyset$. Then*

$$\mathcal{C}\text{-anti}(\mathcal{P}) = \mathcal{C} \Leftrightarrow \mathcal{C}\text{-ind}(\mathcal{P}) = \emptyset;$$
$$\mathcal{C}\text{-anti}(\mathcal{C}) = \mathcal{C};$$
$$\mathcal{C}\text{-anti}(\mathcal{P}) = \mathcal{C} \cap \mathcal{P} \Leftrightarrow \mathcal{C} \cap \mathcal{P} = \mathcal{C}.$$

We can produce an iteration theorem like Theorem 1.8 but, like many results in constrained total negation, it relies on the hypothesis that the constraint $\mathcal{C}$ be hereditary.

Definition 2.4. If $\mathcal{P}$ and $\mathcal{C}$ are topological invariants, we say that $\mathcal{P}$ is *$\mathcal{C}$-hereditary* if and only if whenever X is a $\mathcal{P}$ (and $\mathcal{C}$) space, then every $\mathcal{C}$ subspace of X is $\mathcal{P}$.

Theorem 2.5. *Let $\mathcal{C}$ be a hereditary topological invariant, and let $\mathcal{P}$ be a topological invariant. Then the constrained iteration pattern of $\mathcal{P}$ within $\mathcal{C}$ will be one of the following:*

(1) $(\mathcal{C})'$
(2) $\mathcal{C} \cap \mathcal{P}, (\mathcal{C})'$
(3) $\mathcal{C} \cap \mathcal{P}, \mathcal{Q}, (\mathcal{C})'$
(4) $\mathcal{C} \cap \mathcal{P}, \mathcal{Q}, \mathcal{R}, (\mathcal{C})'$
(5) $(\mathcal{C} \cap \mathcal{P}, \mathcal{Q})'$
(6) $\mathcal{C} \cap \mathcal{P}, (\mathcal{Q}, \mathcal{R})'$
(7) $\mathcal{C} \cap \mathcal{P}, \mathcal{Q}, (\mathcal{R}, \mathcal{S})'$

Furthermore, sequences (4) and (7) cannot occur when $\mathcal{P}$ is $\mathcal{C}$-hereditary. Note that the properties $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ are all subfamilies of the constraint family $\mathcal{C}$.

If we relax the hereditary constraint condition, we can produce within classical topology the sort of catastrophic example suggested by P. T. Matthews [5] in a more general category theory treatment; this example is discussed in detail in [6].

We emphasize that the iteration theorem is unaffected by the imposition of a hereditary constraint, which of course includes separation axioms, such as T_0, T_1, and T_2.

We now turn to examine the relationship between the "absolute" Bankston iteration sequence and its constrained counterpart for a specific property. Given that we know the iterative pattern in the universal context for a property, can we predict anything about the pattern of the constrained sequence?

Theorem 2.6. *Let $\mathcal{C}$ be a topological invariant. Then the universal topological invariant $\mathcal{U}$ has an iteration sequence within $\mathcal{C}$ of*

$$\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C}, \ \ldots.$$

Proof: Clearly, $\mathcal{C} \cap \mathcal{U} = \mathcal{C}$ and $\mathcal{C}$-anti$(\mathcal{C}) = \mathcal{C}$. □

Theorem 2.7. *Let $\mathcal{C}$ and $\mathcal{P}$ be topological invariants. Let $\mathcal{P}$ have an absolute iteration pattern*

$$\mathcal{P}, (\mathcal{U})' = \mathcal{P}, \mathcal{U}, \mathcal{U}, \mathcal{U}, \ \ldots.$$

Then within $\mathcal{C}, \mathcal{P}$ has an iteration pattern of either:

$$(\mathcal{C})' = \mathcal{C}, \mathcal{C}, \mathcal{C}, \ \ldots$$

or

$$\mathcal{C} \cap \mathcal{P}, (\mathcal{C})' = \mathcal{C} \cap \mathcal{P}, \mathcal{C}, \mathcal{C}, \ \ldots.$$

Proof: As anti$(\mathcal{P}) = \mathcal{U}$, ind$(\mathcal{P}) = \emptyset$. Therefore, $\mathcal{C}$-ind$(\mathcal{C} \cap \mathcal{P}) = \emptyset$. Hence, we deduce that $\mathcal{C}$-anti$^n(\mathcal{P}) = \mathcal{C}$ for all $n \geq 1$. Further, $\mathcal{C} \cap \mathcal{P}$ may also be $\mathcal{C}$ in the extreme case where $\mathcal{C} \subseteq \mathcal{P}$. □

So in these simple cases the Bankston iteration sequence produced within even an *arbitrary* constraint is, formally, a "truncation" of the absolute iteration sequence. It seems reasonable to ask if this will always be the case but, interestingly, even with "natural" constraints, we find that this is not invariably what occurs. However, under some additional hypotheses, we now prove a result that comes close to this expectation.

Theorem 2.8. *Let $\mathcal{C}$ be a hereditary topological invariant, and let $\mathcal{P}$ be a $\mathcal{C}$-hereditary topological invariant such that* $\mathcal{C}\text{-anti}(\mathcal{P}) = \mathcal{C} \cap \text{anti}(\mathcal{P})$. *Then, for each $n > 1$, either*

$$\mathcal{C}\text{-anti}^n(\mathcal{P}) = \mathcal{C}$$

or

$$\mathcal{C}\text{-anti}^n(\mathcal{P}) = \mathcal{C} \cap \text{anti}^n(\mathcal{P}).$$

Proof: Suppose that neither of these outcomes occurs for $n = 2$ and that we proceed towards a contradiction; then an appeal to recursion will deal with $n > 2$.

If neither of these outcomes occurs, then the following statements are true.

$$\mathcal{C} \supset \mathcal{C}\text{-anti}^2(\mathcal{C} \cap \mathcal{P}) \tag{1}$$

and

$$\mathcal{C}\text{-anti}^2(\mathcal{C} \cap \mathcal{P}) \supset \mathcal{C} \cap \text{anti}^2(\mathcal{P}) \tag{2}$$

where the set inclusions are proper.

Now by condition (2), we may choose $X \in \mathcal{C}$ such that $X \in \mathcal{C}\text{-anti}^2(\mathcal{C} \cap \mathcal{P})$ but $X \notin \text{anti}^2(\mathcal{P})$. Therefore, by Lemma 1.4, we may choose $Y, Z \subseteq X$ such that $|Z| = |Y| = \lambda$, say, and $Y \in \text{anti}(\mathcal{P})$, but $Z \notin \text{anti}(\mathcal{P})$. Note that Y is a $\mathcal{C}$ space.

Once again, by an application of Lemma 1.4, we may choose spaces $W, U \subseteq Z$ such that $|W| = |U| = \mu$, say, and that $W \in \mathcal{P}$, but $U \notin \mathcal{P}$.

Now $X \in \mathcal{C}\text{-anti}(\mathcal{C}\text{-anti}(\mathcal{C} \cap \mathcal{P}))$ and $Y \in \mathcal{C}\text{-anti}(\mathcal{C} \cap \mathcal{P})$; by Definition 1.2, we must conclude that $\lambda = |Y| \in \mathcal{C}\text{-spec}(\mathcal{C}\text{-anti}(\mathcal{C} \cap \mathcal{P}))$. Therefore, all $\mathcal{C}$ spaces on λ-many points are $\text{anti}(\mathcal{P})$.

Within $\mathcal{C}$ there are subspaces of Y of every cardinality less than λ, due to the hereditary nature of $\mathcal{C}$, and they are all $\text{anti}(\mathcal{P})$. The space Y inherits the property $\mathcal{C}\text{-anti}(\mathcal{C}\text{-anti}(\mathcal{C} \cap \mathcal{P}))$ which is the same as $\mathcal{C}$-anti$(\mathcal{C} \cap \text{anti}(\mathcal{P}))$ from X, so all these cardinals are $\mathcal{C}$-specific for anti$(\mathcal{P})$. So we strengthen our previous assertion to say that

(3) all $\mathcal{C}$ *spaces* on $\lambda -$ many points *or fewer* are anti$(\mathcal{P})$.

No $\mathcal{C}$ space on μ-many points can be $\mathcal{P}$, or it would be both $\mathcal{P}$ and anti$(\mathcal{P})$ forcing $\mu \in \text{spec}(\mathcal{P})$ which cannot be so because U is not $\mathcal{P}$, for example. If $\mathcal{C}$ has a $\mathcal{P}$ space on more than μ points, then by the $\mathcal{C}$-hereditary nature of $\mathcal{P}$ it would have, within $\mathcal{C}$, a subspace on μ points which would be $\mathcal{P}$ contradicting the start of this paragraph. Therefore, we conclude that

(4) all $\mathcal{C}$ spaces on $\mu -$ many points or more are not $\mathcal{P}$.

We know that $\mathcal{C}$-anti$(\mathcal{C} \cap \mathcal{P}) = \mathcal{C} \cap \text{anti}(\mathcal{P})$ has $\mathcal{C}$-indecisive cardinals. Let $\nu = \min \mathcal{C}\text{-ind}(\mathcal{C} \cap \text{anti}(\mathcal{P}))$. We may then choose $A, B \in \mathcal{C}$ such that $|A| = |B| = \nu$ and A is anti$(\mathcal{P})$ and B is not anti$(\mathcal{P})$.

So there exists a space $D \subseteq B$ such that D is $\mathcal{C}$ and $\mathcal{P}$, but $|D| \in \text{ind}(\mathcal{P})$. From (4), it follows that $|D| < \mu$ and therefore, it is clear that $|D| < \lambda$, yet this implies that D is anti$(\mathcal{P})$. Therefore by Definition 1.2, this forces $|D| \in \text{spec}(\mathcal{P})$, contradicting our choice of D and, ultimately, conditions (1) and (2). □

So under these conditions it appears at first sight that the constrained iteration sequence would just be the trace of the absolute one, or a truncation of it (going "universal" earlier). However, this conclusion is premature, for there could be a problem. Imagine for example that the absolute sequence runs

$$\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{Q}, \ \ldots$$

but that $\mathcal{P}$ and $\mathcal{R}$ had the same intersection with $\mathcal{C}$. Then, under constraint, we would move to

$$\mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q}, \mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q}, \ \ldots$$

which is a differently repeating kind of sequence. Fortunately, it transpires that such "pattern-shifting" can happen only in a few specific ways:

Theorem 2.9. *Let $\mathcal{C}$ be a hereditary topological invariant, and let $\mathcal{P}$ be a $\mathcal{C}$-hereditary topological invariant such that $\mathcal{C}$-*anti$(\mathcal{P}) = \mathcal{C} \cap \text{anti}(P)$. *Then the following table sets out, for each possible* *iteration pattern of the absolute Bankston iteration sequence, the list of possible iteration patterns within the constraint:*

$$(\mathcal{U})' \ldots\ldots\ldots \left\{ (\mathcal{C})' \right. \qquad \text{(i)}$$

$$\mathcal{P}, (\mathcal{U})' \ldots\ldots\ldots \left\{ \begin{array}{r} (\mathcal{C})' \qquad \text{(ii)} \\ \mathcal{C} \cap \mathcal{P}, (\mathcal{C})' \qquad \text{(iii)} \end{array} \right.$$

$$\mathcal{P}, \mathcal{Q}, (\mathcal{U})' \ldots\ldots\ldots \left\{ \begin{array}{r} (\mathcal{C})' \qquad \text{(iv)} \\ \mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q}, (\mathcal{C})' \qquad \text{(v)} \end{array} \right.$$

$$(\mathcal{P}, \mathcal{Q})' \ldots\ldots\ldots \left\{ \begin{array}{r} (\mathcal{C})' \qquad \text{(vi)} \\ \mathcal{C} \cap \mathcal{P}, (\mathcal{C})' \qquad \text{(vii)} \\ (\mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q})' \qquad \text{(viii)} \end{array} \right.$$

$$\mathcal{P}, (\mathcal{Q}, \mathcal{R})' \ldots\ldots\ldots \left\{ \begin{array}{r} (\mathcal{C})' \qquad \text{(ix)} \\ \mathcal{C} \cap \mathcal{P}, (\mathcal{C})' \qquad \text{(x)} \\ \mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q}, (\mathcal{C})' \qquad \text{(xi)} \\ (\mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q})' \qquad \text{(xii)} \\ \mathcal{C} \cap \mathcal{P}, (\mathcal{C} \cap \mathcal{Q}, \mathcal{C} \cap \mathcal{R})' \qquad \text{(xiii)} \end{array} \right.$$

Proof: Since $\mathcal{P}$ is actually hereditary, its absolute iteration pattern, as detailed in Theorem 1.8, can be only 1, 2, 3, 5, or 6. For each of these in turn, Theorem 2.8 assures us that each term in the constrained iteration sequence is either $\mathcal{C}$ or the intersection with $\mathcal{C}$ of the corresponding absolute term. These considerations limit the variety of possible constrained iteration sequences as follows:

$$1: \ \ldots\ldots\ldots \left\{ (\mathcal{C})' \right.$$

$$2: \ \ldots\ldots\ldots \left\{ \begin{array}{r} (\mathcal{C})' \\ \mathcal{C} \cap \mathcal{P}, (\mathcal{C})' \end{array} \right.$$

$$3: \ \ldots\ldots\ldots \left\{ \begin{array}{r} (\mathcal{C})' \\ \mathcal{C} \cap \mathcal{P}, (\mathcal{C})' \\ \mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q}, (\mathcal{C})' \end{array} \right. \qquad [A]$$

$$5: \quad \ldots\ldots\ldots\ldots \quad \begin{cases} (\mathcal{C})' & \\ \mathcal{C}\cap\mathcal{P}, (\mathcal{C})' & \\ \mathcal{C}\cap\mathcal{P}, \mathcal{C}\cap\mathcal{Q}, (\mathcal{C})' & [\mathrm{B}] \\ \mathcal{C}\cap\mathcal{P}, \mathcal{C}\cap\mathcal{Q}, \mathcal{C}\cap\mathcal{P}, (\mathcal{C})' & [\mathrm{C}] \\ (\mathcal{C}\cap\mathcal{P}, \mathcal{C}\cap\mathcal{Q})' & \end{cases}$$

$$6: \quad \ldots\ldots\ldots\ldots \quad \begin{cases} (\mathcal{C})' & \\ \mathcal{C}\cap\mathcal{P}, (\mathcal{C})' & \\ \mathcal{C}\cap\mathcal{P}, \mathcal{C}\cap\mathcal{Q}, (\mathcal{C})' & \\ \mathcal{C}\cap\mathcal{P}, \mathcal{C}\cap\mathcal{Q}, \mathcal{C}\cap\mathcal{R}, (\mathcal{C})' & [\mathrm{D}] \\ \mathcal{C}\cap\mathcal{P}, (\mathcal{C}\cap\mathcal{Q}, \mathcal{C}\cap\mathcal{R})' & [\mathrm{E}] \end{cases}$$

For the full determination of the iteration *pattern*, as contrasted with the mere sequence of invariants, it is necessary to consider the possibility of additional coincidences: for example, in [E] we might have $\mathcal{C}\cap\mathcal{P} = \mathcal{C}\cap\mathcal{R}$, so that [E] generates (possibly) two patterns of iteration

$$\mathcal{C}\cap\mathcal{P}, (\mathcal{C}\cap\mathcal{Q}, \mathcal{C}\cap\mathcal{R})' \qquad [\mathrm{E1}],$$

$$(\mathcal{C}\cap\mathcal{P}, \mathcal{C}\cap\mathcal{Q})' \qquad [\mathrm{E2}].$$

However, in almost all cases, such coincidences only create a pattern that duplicates one already in the list for that absolute pattern: recall from Lemma 2.3 that *adjacent* terms coincide only when they are context-universal. This analysis readily shows that, apart from the replacement of [E] by [E1] and [E2], the above list is an exhaustive account of conceivable constrained iteration patterns corresponding to the absolute patterns 1, 2, 3, 5, and 6. Yet we can immediately detect “spurious” possibilities here: [D] is prohibited by Theorem 2.5, while [C] is clearly impossible since it ascribes to $\mathcal{C}\cap\mathcal{P}$, two distinct constrained anti-properties. To complete the proof, we need to show that [A] and [B] are also unrealizable.

Suppose that [A] were realized by (hereditary) properties $\mathcal{P}$ and $\mathcal{C}$. Then $\mathcal{C}$-anti$(\mathcal{C}\cap\mathcal{P}) = \mathcal{C}\cap$ anti$(\mathcal{P}) = \mathcal{C}$, implying that $\mathcal{C} \subseteq$ anti$(\mathcal{P})$ and that $\mathcal{C}\cap\mathcal{P}$ has no indecisive cardinals in $\mathcal{C}$; yet ind$(\mathcal{P}) \neq \emptyset$. Put $\alpha =$ the least member of ind$(\mathcal{P})$ and observe that all spaces on fewer than α-many points are $\mathcal{P}$, while anti$(\mathcal{P})$ is precisely the family of spaces on fewer than α-many points. Yet this forces $\mathcal{C} \subseteq \mathcal{P}$

and yields the contradiction $\mathcal{C} \cap \mathcal{P} = \mathcal{C}$, and we conclude that pattern [A] cannot occur.

Lastly, suppose that (hereditary) invariants $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{C}$ generated the absolute pattern $\mathcal{P}, \mathcal{Q}, \mathcal{P}, \mathcal{Q}, \ldots$ and the constrained pattern

$$\mathcal{C} \cap \mathcal{P}, \mathcal{C} \cap \mathcal{Q}, \mathcal{C}, \mathcal{C}, \ldots.$$

Then $\mathcal{P}$ and $\mathcal{Q}$ will have the *same* family of specific cardinals, say $[1, \alpha)$, and α will be indecisive for both of them. Yet $\mathcal{C}$-anti$(\mathcal{C} \cap \mathcal{Q}) = \mathcal{C}$, so $\mathcal{C} \cap \mathcal{Q}$ is cardinally decisive within $\mathcal{C}$. Some spaces in $\mathcal{C}$ must have cardinalities $\geq \alpha$ (or else the constrained pattern would be $(\mathcal{C})'$, which it is not), so some $\mathcal{C}$-spaces will contain exactly α points. Either *all of these* or else *none of these* will be $\mathcal{Q}$. Recalling that $\mathcal{C} \cap \mathcal{Q} \neq \mathcal{C}$, we may find a $\mathcal{C}$-space X that is not $\mathcal{Q}$ and has at least α points. Now, X is not anti$(\mathcal{P})$, so one of its $\mathcal{P}$-subspaces has cardinality $\geq \alpha$, and one of *its* $\mathcal{P}$-subspaces Y has cardinality exactly α. This space Y belongs to $\mathcal{C}$ and cannot be $\mathcal{Q}$ (else the contradiction $\alpha \in \text{spec}(\mathcal{P})$ arises). Therefore, we conclude that $\mathcal{C}$ has *no* $\mathcal{Q}$-subspaces on α points.

However, $\mathcal{C} \nsubseteq \mathcal{P}$, so there is a space Z in $\mathcal{C} \backslash \mathcal{P}$. Since Z is not anti$(\mathcal{Q})$, it has a $\mathcal{Q}$-subspace W with cardinality $\geq \alpha$. This, in turn, has a ($\mathcal{C}$- and $\mathcal{Q}$-) subspace V of cardinality α, yielding a final contradiction which prohibits outcome [B]. □

Example 2.10. *Finally, we demonstrate that all thirteen of the scenarios listed in Theorem 2.9 do actually occur for hereditary invariants $\mathcal{P}$ within hereditary constraints $\mathcal{C}$ and subject to the requirement that*

$$\mathcal{C}\text{-anti}(\mathcal{P}) = \mathcal{C} \cap \text{anti}(\mathcal{P}).$$

Proof: (a) By choosing $\mathcal{C} = \mathcal{U}$ and appealing to standard examples in the non-constrained theory, we access existing realizations of (i), (iii), (v), (viii), and (xiii).

(b) Take $\mathcal{C} = \mathcal{P} =$ any cardinally decisive, hereditary invariant (for example, the class of all finite spaces). It is trivial that $\mathcal{C}$-anti$(\mathcal{P}) = \mathcal{C} \cap \text{anti}(\mathcal{P}) = \mathcal{C}$ and that scenario (ii) is obtained.

(c) The invariant $\mathcal{P} =$ complete separability has the unconstrained pattern $\mathcal{P}, \mathcal{Q}, (\mathcal{U})'$. The constraint $\mathcal{C} =$ the class of finite spaces gives scenario (iv), just as in paragraph (b).

(d) The separation axioms T_1 and T_2 have unconstrained patterns $(\mathcal{P}, \mathcal{Q})'$ and $\mathcal{P}, (\mathcal{Q}, \mathcal{R})'$, respectively. Imposing the constraint $\mathcal{C} = \{\text{the singleton space}\}$ is consistent with Theorem 2.9's hypotheses and realizes (vi) and (ix) from them.

(e) Considering again $\mathcal{P} = T_1$ (and $\mathcal{Q}$ = nested for the unconstrained pattern $(\mathcal{P}, \mathcal{Q})'$), we now impose the constraint $\mathcal{C}$ comprising the singleton space and the Sierpiński two-point space. Then

$$\mathcal{C}\text{-anti}(\mathcal{C} \cap \mathcal{P}) = \mathcal{C}\text{-anti}(\text{singleton})$$

$$= \mathcal{C} = \mathcal{C} \cap \text{nested} = \mathcal{C} \cap \text{anti}(\mathcal{P})$$

as required for the theorem. Also $\mathcal{C} \cap \mathcal{P} \neq \mathcal{C}$, so the constrained pattern is $\mathcal{C} \cap \mathcal{P}, (\mathcal{C})'$, and we have (vii) this time. The same argument applied to T_2 in place of T_1 yields (x).

(f) Consider the invariant $\mathcal{P} = T_2$ within the constraint $\mathcal{C} = T_2 \cup \text{nested}$. Routine argument establishes that both $\mathcal{C} \cap \text{anti}(\mathcal{P})$ and $\mathcal{C}\text{-anti}(\mathcal{C} \cap \mathcal{P})$ simplify to nested and that $\mathcal{C}\text{-anti}(\text{nested}) = T_2$. Thus, scenario (xii) describes the pair of iteration patterns.

(g) Lastly, we take $\mathcal{P}$ to comprise the finite spaces and the discrete space on $\aleph_0$ elements. Then $\mathcal{Q} = \text{anti}(\mathcal{P})$ consists of all spaces that do not have an infinite discrete subspace (that is, those whose infinite subspaces all contain a copy of one of the other four Ginsburg and Sands spaces [2]), and $\mathcal{R} = \text{anti}^2(\mathcal{P})$ encompasses the spaces whose only Ginsburg and Sands subspace is the discrete one; the next iteration yields $\text{anti}(\mathcal{R}) = \mathcal{Q}$. Now consider the constraint $\mathcal{C} = \mathcal{R}$. Routine checks confirm that $\mathcal{C}\text{-anti}(\mathcal{P}) = \mathcal{C} \cap \text{anti}(\mathcal{P}) =$ finite and that $\mathcal{C}\text{-anti}^2(\mathcal{P}) = \mathcal{C}$. This realizes (xi), the final permissible scenario. □

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