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Mail: Topology Proceedings
Department of Mathematics & Statistics
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ON SQUARES OF SPACES AND F_σ-SETS

ARNOLD W. MILLER

Abstract. We show that the continuum hypothesis implies there exists a Lindelöf space X such that X^2 is the union of two metrizable subspaces but X is not metrizable. This gives a consistent solution to a problem of Z. Balogh, G. Gruenhage, and V. Tkachuk ("Additivity of Metrizability and Related Problems.") The main lemma is that assuming the continuum hypothesis there exist disjoint sets of reals X and Y such that X is Borel concentrated on Y, i.e., for any Borel set B if $Y \subseteq B$ then $X \setminus B$ is countable, but $X^2 \setminus \Delta$ is relatively F_σ in $X^2 \cup Y^2$.

In [1], Zoltan Balogh, Gary Gruenhage, and Vladimir Tkachuk ask the following question:

> **Question 4.1.** Let X be a regular paracompact space X such that $X \times X$ is the union of two metrizable subspaces. Must X be metrizable? What if X is Lindelöf? [p. 102]

Theorem 1. *Assume the continuum hypothesis. Then there exists a nonmetrizable regular Lindelöf space X such that X^2 is the union of two metrizable subspaces.*

We first prove the following lemma.

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Lemma 2. *(CH) There are uncountable disjoint sets $X, Y \subseteq 2^\omega$ such that*

1. *X is Borel concentrated on Y, i.e., every Borel set in 2^ω containing Y contains all but countably many elements of X;*
2. *$Y^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$; and*
3. *$X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$.*

Here, $\Delta = \{(x, x) : x \in 2^\omega\}$.

Proof: We identify the Cantor space 2^ω with the power set $P(\omega)$ of ω. We use $[\omega]^\omega$ to stand for the infinite subsets of ω. Define, for $y \in [\omega]^\omega$,

$$[y]^{*\omega} = \{x \in [\omega]^\omega : x \subseteq^* y\},$$

where $\subseteq^*$ stands for inclusion mod finite. Let $\langle B_\alpha : \alpha < \omega_1 \rangle$ be all Borel subsets of $[\omega]^\omega$. We construct y_α for $\alpha < \omega_1$ so that

1. $\alpha < \beta$ implies $y_\beta \subseteq^* y_\alpha$ and $y_\beta \neq^* y_\alpha$ and
2. either $y_\alpha \notin B_\alpha$ or $[y_\alpha]^{*\omega} \subseteq B_\alpha$.

These conditions are easy to get. Given y_β for $\beta < \alpha$ and B_α let $y \in [\omega]^\omega$ be arbitrary with $y \subseteq^* y_\beta$ but $y_\beta \neq^* y$ for each $\beta < \alpha$. If $[y]^{*\omega}$ is not a subset of B_α, then simply take $y_\alpha \in [y]^{*\omega} \setminus B_\alpha$; otherwise, take $y_\alpha = y$.

Let

$$X = \{y_\alpha \setminus y_{\alpha+1} : \alpha < \omega_1\} \text{ and } Y = \{y_\alpha : \alpha < \omega_1\}$$

If B is any Borel set containing Y, then choose α so that $B = B_\alpha$. At stage α of the construction, it must have been that $[y_\alpha]^{*\omega} \subseteq B_\alpha$. But this means that $x_\beta \in B_\alpha$ for all $\beta \geq \alpha$. So X is Borel concentrated on Y.

If we define

$$F = \{(u, v) \in P(\omega) \times P(\omega) : (u \subseteq^* v \text{ or } v \subseteq^* u) \text{ and } u \neq v\},$$

then F is an F_σ set and

$$F \cap (X^2 \cup Y^2) = (Y^2 \setminus \Delta).$$

Also, if we define

$$H = \{(u, v) \in P(\omega) \times P(\omega) : u \cap v =^* \emptyset\},$$

then H is an F_σ set and

$$H \cap (X^2 \cup Y^2) = (X^2 \setminus \Delta).$$

This finishes the proof of the lemma. □

We continue with the proof of Theorem 1. Define the following Michael-line-like topology. Suppose that M is a topological space and $X \subseteq M$. Then $M(X)$ is the topological space on the same set but with the following topology. For $x \in X$ we make x an isolated point, i.e., add $\{x\}$ to the topology of $M(X)$. For any point $y \in M \setminus X$, neighborhoods in M form a neighborhood basis for y in $M(X)$. It is easy to see that $M(X)$ is regular for any regular space M and subset $X \subseteq M$.

The following is Exercise 5.5.2 from [2]:

Proposition 3. *Suppose M is a metric space and $X \subseteq M$. Then $M(X)$ is metrizable iff X is an F_σ set in M.*

Our example is $M(X)$ where X and Y are from Lemma 2 and $M = X \cup Y$ has its usual (separable metric) topology as a subspace of 2^ω. It follows from the Proposition 3 that $M(X)$ is not metrizable.

CLAIM 1. $M(X)$ is a Lindelöf space.

Take any open cover $\mathcal{U}$ of $M(X)$. Open sets in $M(X)$ have the form $U \cup Z$ where U is open in M and $Z \subseteq X$ is arbitrary. Then since Y has its standard topology, countably many elements of $\mathcal{U}$ will cover Y, say

$$\{(U_n \cup X_n : n < \omega\} \subseteq \mathcal{U},$$

where each U_n is open in M and $X_n \subseteq X$. But since X is Borel concentrated on Y we have that $X \setminus \cup\{U_n : n < \omega\}$ is countable, so we need only add countably many more elements of $\mathcal{U}$ to cover all of $M(X)$.

CLAIM 2. $M(X)^2$ is the union of two metrizable subspaces.

Let

$M_1 = (X^2 \setminus \Delta) \cup Y^2$ and

$M_2 = (X \times Y) \cup (Y \times X) \cup (X^2 \cap \Delta)$.

Note that M_1 is $N(X^2 \setminus \Delta)$ where $N = (X^2 \setminus \Delta) \cup Y^2$ in its separable metric topology as a subspace of $2^\omega \times 2^\omega$. By Lemma 2 we have that $X^2 \setminus \Delta$ is relatively F_σ in N, and so by Proposition 3, M_1 is metrizable.

To see that M_2 is metrizable use the Bing Metrization Theorem:

> A topological space is metrizable iff it is regular and has a σ-discrete basis.

A family $\mathcal{B}$ of subsets of X is discrete iff every point of X has a neighborhood meeting at most one element of $\mathcal{B}$. σ-discrete means the countable union of discrete families.

Note that for each $x \in X$ the sets $\{x\} \times Y$ and $Y \times \{x\}$ are open in M_2. Let $\mathcal{B}$ be a countable open basis for Y. Then

$$\mathcal{C} = \{U \times \{x\},\ \ \{x\} \times U,\ \ \{(x,x)\}\ :\ \ x \in X, U \in \mathcal{B}\}$$

is an open basis for M_2. It is σ-discrete. The family $\{\{(x,x)\} : x \in X\}$ is discrete in M_2 since $X^2 \cap \Delta$ is closed in M_2. And for each fixed $U \in \mathcal{B}$ the family $\{U \times \{x\}\ :\ x \in X\}$ is discrete in M_2. (For $(x,x) \in X$ use the neighborhood $\{x\} \times \{x\}$. For (y,x) with $y \in Y$ and $x \in X$ use the neighborhood $Y \times \{x\}$ and for (x,y) use the neighborhood $\{x\} \times Y$.) Similarly, for each $U \in \mathcal{B}$ the family $\{\{x\} \times U\ \ :\ \ x \in X\}$ is discrete in M_2. Since $\mathcal{B}$ is countable, M_2 has a σ-discrete basis and is therefore metrizable.

This proves Theorem 1. $\square$

The next theorem is an easy generalization of Theorem 1 using the tower cardinal $\mathfrak{t}$ which is defined as follows. $\mathfrak{t}$ is the minimum cardinality of a set $T \subseteq [\omega]^\omega$ which is linearly ordered by $\subseteq^*$ but there does not exist $z \in [\omega]^\omega$ with $z \subseteq^* y$ for every $y \in T$. Martin's axiom implies that $\mathfrak{t} = \mathfrak{c}$.

Theorem 4. *Suppose $\mathfrak{t} = \mathfrak{c}$. Then there exists a nonmetrizable regular paracompact space X such that X^2 is the union of two metrizable subspaces.*

Proof: The main lemma changes to:

Lemma 5. *($\mathfrak{t} = \mathfrak{c}$) There are disjoint sets $X, Y \subseteq 2^\omega$ of cardinality $\mathfrak{c}$ such that*

(1) *X is Borel $\mathfrak{c}$-concentrated on Y, i.e., for every Borel set B in 2^ω, if $Y \subseteq B$ then $|X \setminus B| < \mathfrak{c}$;*
(2) *$Y^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$; and*
(3) *$X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$.*

The proof is similar. The space $M = X \cup Y$ is the same. Since X is not relatively Borel in M, we have by Proposition 3 that $M(X)$

is not metrizable. But $M(X)$ is regular and paracompact for any $X \subseteq M$ and metric M; see example 5.1.22 in [2]. $\square$

Remark 6. The Michael line is the topological space $M(X)$ where M is the unit interval, $[0,1]$, and X the irrationals in $[0,1]$. Michael Granado, in an unpublished work, has shown that the square of the Michael line is not the union of two metrizable subspaces.

Question 7. Do there exist (in ZFC) disjoint sets of reals X and Y such that X is not F_σ in $X \cup Y$ but $X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$?

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University of Wisconsin-Madison; Department of Mathematics; Van Vleck Hall; 480 Lincoln Drive; Madison, Wisconsin 53706-1388

E-mail address: miller@math.wisc.edu