Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 243-265

APPROXIMATE SEQUENCES AND HAUSDORFF DIMENSION

TAKAHISA MIYATA AND TADASHI WATANABE

Abstract. In this paper, we introduce a new approach using normal sequences and approximate sequences to study Hausdorff dimension for compact metrizable spaces. Using this approach, for each $r > 0$, we construct a Cantor set X_r with Hausdorff dimension r in the cube $[0,1]^N$, where N is the least integer that is greater than or equal to $\frac{\log 3}{\log 2}(r+1)+1$.

1. Introduction

For each subset F of $\mathbb{R}^m$ and for each $s > 0$, the *s-dimensional Hausdorff measure* of F is defined as $\mathrm{H}^s(F) = \lim_{\delta\to 0} \mathrm{H}^s_\delta(F)$ where for each $\delta > 0$,

$$\mathrm{H}^s_\delta(F) = \inf \sum_{U\in\mathfrak{U}} |U|^s \tag{1.1}$$

where the infimum is taken over all countable (possibly finite) coverings $\mathfrak{U}$ of F by open balls U with radius at most δ. Here $|U|$ denotes the diameter of the set U. The *Hausdorff dimension* of F is defined as $\dim_{\mathrm{H}} F = \sup\{s : \mathrm{H}^s(F) = \infty\}$ $(= \inf\{s : \mathrm{H}^s(F) = 0\})$ [2]. The present paper concerns Hausdorff dimension for non-Euclidean spaces. More precisely, we develop a systematic approach using normal sequences and approximate sequences to study Hausdorff dimension for compact metrizable spaces.

2000 *Mathematics Subject Classification.* Primary: 54E35, 54E40; Secondary: 28A78, 28A80, 54F45, 54C56.

Key words and phrases. approximate sequence, Cantor set, Hausdorff dimension, normal sequence.

In the theory of inverse systems, in order to study properties of X, one expands the space X into an inverse system $\boldsymbol{X}$ whose limit is X and investigates $\boldsymbol{X}$. It is known that the notion of approximate sequence is a useful tool for studying the properties of spaces and maps. An approximate sequence is an approximate system with the index set being the set of natural numbers. An approximate system is a generalization of an inverse system, and it was first introduced by Sibe Mardešić and Leonard R. Rubin [4] to deal with more general spaces than compact metrizable spaces. (See also [9], [15] for more general versions and [4], [5], [6], [7], [14], [16] for its applications).

Hausdorff dimension is not topologically invariant since for any $r > 0$ there is a Cantor set with Hausdorff dimension r. However, under an appropriate setting, the notion of approximate sequence becomes a useful tool for studying Hausdorff dimension. In this paper we introduce a new way of using approximate sequences in the study of Hausdorff dimension.

There are two reasons for using approximate sequences. First of all, motivated by the construction of a metric by P. Alexandroff and P. Urysohn [1] (see also [13, Theorem 2-16]), for each compact metrizable space X and for each approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of X with an approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$, we can define a new metric $\mathrm{d}_{\boldsymbol{p}}$ that induces the original topology. Secondly, for any map $f : X \to Y$ between spaces (even compact metrizable spaces), if approximate sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ are chosen in advance so that their limits are the spaces X and Y, respectively, then there is a system of maps between $\boldsymbol{X}$ and $\boldsymbol{Y}$ whose limit is f [8],[15]. Those facts are quite useful. For example, Lipschitz maps between spaces with metrics induced by approximate resolutions are characterized by some properties on approximate sequences [10], and the notion of box-counting dimension of spaces with such metrics is also studied by approximate sequences [11].

A little more simplified setting can be given by normal sequences. Given a compact metrizable space X and a normal sequence $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ on X, there exist a trivial approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$, where $X_i = X$ for all i, and $p_{i,i+1} = \mathrm{id}_X$, and an approximate resolution $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X}$ of X, where $p_i = \mathrm{id}_X$. Thus, given a normal sequence $\mathbb{U}$ with some reasonable property on a compact metrizable space X, one obtains a metric $\mathrm{d}_{\mathbb{U}}$

on X. Considering $\mathbb{U}$ as a "ruler" for X, one can study geometric properties of the metric space $(X, \mathrm{d}_{\mathbb{U}})$.

This paper consists of three primary parts. In section 3 (the first primary part), given a σ-compact metrizable space X with a normal sequence $\mathbb{U}$, we define the Hausdorff measure $\mathrm{H}^s_{\mathbb{U}}(F)$ and the Hausdorff dimension $\dim^{\mathbb{U}}_{\mathrm{H}} F$ for any subsets F of X and investigate their properties. The Hausdorff dimension which is defined in this way coincides with the usual Hausdorff dimension for subsets of Euclidean spaces as a special case if some particular normal sequence is taken (Theorem 3.3). Fundamental properties, such as the subset and sum theorems, hold for our Hausdorff dimension (Theorem 3.5). For normal sequences $\mathbb{U}$ and $\mathbb{V}$ on metrizable spaces X and Y, respectively, one can speak of Lipschitz maps with respect to the metrics $\mathrm{d}_{\mathbb{U}}$ and $\mathrm{d}_{\mathbb{V}}$. Indeed, they were characterized by a property on the normal sequences $\mathbb{U}$ and $\mathbb{V}$ [10]. We show that our Hausdorff dimension is Lipschitz invariant (Corollary 3.8).

In section 4 (the second primary part), we establish a more general approximate sequence approach to Hausdorff dimension. More precisely, given a compact metrizable space X and an approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ with an approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$, we define the Hausdorff dimensions for the approximate resolution $\boldsymbol{p}$ and also for the approximate sequence $\boldsymbol{X}$. This approach gives a characterization of the Hausdorff dimension in the above situation in terms of approximate sequences (Theorem 4.8). More precisely, if $\boldsymbol{p} : X \to \boldsymbol{X}$ is some nice approximate resolution with an approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$, each compact subset F of X corresponds to an approximate sequence $\boldsymbol{F} = (F_i, \mathfrak{U}_i|F_i, p_{i,i+1}|F_{i+1})$, which is a subsystem of $\boldsymbol{X}$; i.e., the coordinate spaces F_i are subspaces of X_i and the open coverings and the bonding maps are the restrictions, so that the equalities $\dim_{\mathrm{H}}(\boldsymbol{p}|F) = \dim_{\mathrm{H}} \boldsymbol{F} = \dim^{\mathbb{U}}_{\mathrm{H}} F$ hold where $\mathbb{U}$ is the normal sequence $\{p_i^{-1}\mathfrak{U}_i : i \in \mathbb{N}\}$. It is also shown that such defined Hausdorff dimension for approximate resolutions is bounded above by the box-counting dimension in the sense of [11] (Theorem 4.4).

Finally, in section 5 (the third primary part), we give an example. It is well-known that if X is a compact metrizable space with covering dimension n, then X can be embedded in $[0,1]^{2n+1}$ [3, Theorem V 2]. Motivated by this result, we consider the following problem: For each $r > 0$, find the least integer N for which there exists a Cantor set with Hausdorff dimension r in the cube $[0,1]^N$. Using our approximate sequence approach, we show that $N \leq \left[\frac{\log 3}{\log 2}(r+1)+1\right]$ (Theorem 5.1). Here, for each $r > 0$, let $[r]$ denote the least integer that is greater than or equal to r.

2. Normal sequences, approximate sequences, and metrics

Throughout the paper, *all spaces are assumed to be metrizable, and map means continuous function.*

For any space X, let $\mathrm{Cov}(X)$ denote the family of all open coverings of X. For any $\mathfrak{U}, \mathfrak{V} \in \mathrm{Cov}(X)$, $\mathfrak{U}$ is a *refinement* of $\mathfrak{V}$, in notation, $\mathfrak{U} < \mathfrak{V}$, if for each $U \in \mathfrak{U}$ there is $V \in \mathfrak{V}$ such that $U \subseteq V$. For any subset A of X and $\mathfrak{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A, \mathfrak{U}) = \bigcup\{U \in \mathfrak{U} : U \cap A \neq \emptyset\}$ and $\mathfrak{U}|A = \{U \cap A : U \in \mathfrak{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x, \mathfrak{U})$ for $\mathrm{st}(\{x\}, \mathfrak{U})$. For each $\mathfrak{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathfrak{U} = \{\mathrm{st}(U, \mathfrak{U}) : U \in \mathfrak{U}\}$. Let $\mathrm{st}^1\mathfrak{U} = \mathrm{st}\,\mathfrak{U}$ and $\mathrm{st}^{n+1}\mathfrak{U} = \mathrm{st}(\mathrm{st}^n\mathfrak{U})$ for each $n \in \mathbb{N}$. For any metric space (X, d), $x \in X$, and $r > 0$, let $\mathrm{U_d}(x, r) = \{y \in X : \mathrm{d}(x, y) < r\}$, and for each subset A of X, let $|A|$ denote the diameter of A. For any $\mathfrak{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are $\mathfrak{U}$-*near*, denoted $(x, x') < \mathfrak{U}$, provided $x, x' \in U$ for some $U \in \mathfrak{U}$. For any $\mathfrak{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between spaces are $\mathfrak{V}$-*near*, denoted $(f, g) < \mathfrak{V}$, provided $(f(x), g(x)) < \mathfrak{V}$ for each $x \in X$. For each $\mathfrak{U} \in \mathrm{Cov}(X)$ and $\mathfrak{V} \in \mathrm{Cov}(Y)$, let $f(\mathfrak{U}) = \{f(U) : U \in \mathfrak{U}\}$ and $f^{-1}(\mathfrak{V}) = \{f^{-1}(V) : V \in \mathfrak{V}\}$. Let I denote the closed interval $[0,1]$, and let $\mathbb{N}$ denote the set of all positive integers.

2.1. Metrics induced by normal spaces

A family $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ of open coverings on a space X is said to be a *normal sequence* on X provided $\mathrm{st}\,\mathfrak{U}_{i+1} < \mathfrak{U}_i$ for each i. Let $\Sigma\mathbb{U}$ denote the normal sequence $\{\mathfrak{V}_i : \mathfrak{V}_i = \mathfrak{U}_{i+1}, i \in \mathbb{N}\}$ and $\mathrm{st}\,\mathbb{U}$ the normal sequence $\{\mathrm{st}\,\mathfrak{U}_i : i \in \mathbb{N}\}$. Also, let $\Sigma^1\mathbb{U} = \mathbb{U}$ and $\Sigma^{n+1}\mathbb{U} = \Sigma(\Sigma^n\mathbb{U})$ for $n \in \mathbb{N}$. For any normal sequences $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ and $\mathbb{V} = \{\mathfrak{V}_i : i \in \mathbb{N}\}$, we write $\mathbb{U} < \mathbb{V}$ provided $\mathfrak{U}_i < \mathfrak{V}_i$ for each i. For each map $f : X \to Y$ and for each normal sequence $\mathbb{V} = \{\mathfrak{V}_i : i \in \mathbb{N}\}$ on Y, let $f^{-1}(\mathbb{V})$ denote the normal sequence $\{f^{-1}(\mathfrak{V}_i) : i \in \mathbb{N}\}$ on X. For any subspace A of X and normal sequence $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ on X, let $\mathbb{U}|A$ denote the normal sequence $\{\mathfrak{U}_i|A : i \in \mathbb{N}\}$ on A.

Let $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ be any normal sequence on a space X with the following property:

(B) For each $x \in X$, $\{\mathrm{st}(x, \mathfrak{U}_i) : i \in \mathbb{N}\}$ is a base at x.

Following the approach by Alexandroff and Urysohn [1], we define the metric $\mathrm{d}_{\mathbb{U}}$ on X as follows:

$$\mathrm{d}_{\mathbb{U}}(x, x') = \inf\{\mathrm{D}_{\mathbb{U}}(x, x_1) + \mathrm{D}_{\mathbb{U}}(x_1, x_2) + \cdots + \mathrm{D}_{\mathbb{U}}(x_n, x')\}$$

where the infimum is taken over all points $x_1, x_2, ..., x_n$ in X, and

$$\mathrm{D}_{\mathbb{U}}(y, z) = \begin{cases} 9 & \text{if } (y, z) \not< \mathfrak{U}_1; \\ \dfrac{1}{3^{i-2}} & \text{if } (y, z) < \mathfrak{U}_i \text{ but } (y, z) \not< \mathfrak{U}_{i+1}\ ; \\ 0 & \text{if } (y, z) < \mathfrak{U}_i \text{ for all } i \in \mathbb{N}\ . \end{cases}$$

Then the metric $\mathrm{d}_{\mathbb{U}}$ has the property

$$\mathrm{st}(x, \mathfrak{U}_{i+3}) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x, \frac{1}{3^i}) \subseteq \mathrm{st}(x, \mathfrak{U}_i) \text{ for each } x \in X \text{ and } i.$$

In particular, if $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ is the normal sequence on a metric space (X, d) such that $\mathfrak{U}_i = \{\mathrm{U}_{\mathrm{d}}(x, \frac{1}{3^i}) : x \in X\}$, then the metric $\mathrm{d}_{\mathbb{U}}$ induces a uniformity which is equivalent to that induced by the metric d. Moreover, it is proven in [10, Proposition 3.7] that if X is a convex subset of a normed linear space, and if X is equipped with the metric d which is induced by the norm, then $\mathrm{d}_{\mathbb{U}}$ is isometric up to a constant multiple of the metric d. Here, we say that two metrics d_1 and d_2 on a set X are isometric up to a constant multiple if there is a constant $C > 0$ such that $\mathrm{d}_1(x, x') = C\,\mathrm{d}_2(x, x')$ for $x, x' \in X$.

Example 2.1. Let $\mathbb{U} = \{\mathfrak{U}_i\}$ be the normal sequence on $\mathbb{R}^2$ with $\mathfrak{U}_i = \{\mathrm{U}_{\mathrm{d}}(x, \frac{1}{3^i}) : x \in \mathbb{R}^2\}$, where d is the usual metric on $\mathbb{R}^2$. Then the metric $\mathrm{d}_{\mathbb{U}|\mathrm{I}}$ is isometric up to a constant multiple of the usual metric on I. Let $X = \{(x, 0) :\ 0 \le x \le 1\} \cup \{(x, 1) :\ 0 \le x \le 1\} \cup \{(0, y) :\ 0 \le y \le 1\}$, and consider the points $\mathrm{O} = (0, 0), \mathrm{P} = (1, 0), \mathrm{Q} = (1, 1)$. Then $\mathrm{d}(\mathrm{O}, \mathrm{P}) = 1 = \mathrm{d}(\mathrm{P}, \mathrm{Q})$ in the usual metric d, but $\mathrm{d}_{\mathbb{U}|X}(\mathrm{O}, \mathrm{P}) = 3\,\mathrm{d}_{\mathbb{U}|X}(\mathrm{P}, \mathrm{Q})$.

Throughout the rest of the paper, *every normal sequence is assumed to have property* (B).

Proposition 2.2. Let $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ and $\mathbb{V} = \{\mathfrak{V}_i : i \in \mathbb{N}\}$ be normal sequences on a space X, and let $x, x' \in X$. Then the following properties hold:

(1) if $\mathbb{U} < \mathbb{V}$, $\mathrm{d}_{\mathbb{U}}(x, x') \geq \mathrm{d}_{\mathbb{V}}(x, x')$;
(2) if $\mathrm{d}_{\mathbb{U}}(x, x') \leq 1$, $\mathrm{d}_{\Sigma\mathbb{U}}(x, x') = 3\,\mathrm{d}_{\mathbb{U}}(x, x')$;
(3) if $\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x, x') \leq 1$, $\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x, x') \leq \mathrm{d}_{\mathbb{U}}(x, x') \leq 3\,\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x, x')$.

Proof: (1) immediately follows from the definition of $\mathrm{d}_{\mathbb{U}}$. To show (2), let $\varepsilon > 0$ be sufficiently small. Then there exist points $x_1, x_2, ..., x_n \in X$ so that

$$\mathrm{D}_{\mathbb{U}}(x, x_1) + \mathrm{D}_{\mathbb{U}}(x_1, x_2) + \cdots + \mathrm{D}_{\mathbb{U}}(x_n, x') < \mathrm{d}_{\mathbb{U}}(x, x') + \varepsilon/3.$$

Since $\mathrm{d}_{\mathbb{U}}(x, x') \leq 1$, $\mathrm{D}_{\mathbb{U}}(x_i, x_{i+1}) \leq 1 + \varepsilon/3$ for $i = 0, 1, ..., n$ where $x_0 = x$ and $x_{n+1} = x'$. So, $\mathrm{D}_{\mathbb{U}}(x_i, x_{i+1}) = \frac{1}{3^{k_i-2}}$ for some $k_i \geq 2$, and $\mathrm{D}_{\Sigma\mathbb{U}}(x_i, x_{i+1}) = \frac{1}{3^{(k_i-1)-2}} = 3\,\mathrm{D}_{\mathbb{U}}(x_i, x_{i+1})$. Thus,

$$\begin{aligned} \mathrm{d}_{\Sigma\mathbb{U}}(x, x') &\leq \mathrm{D}_{\Sigma\mathbb{U}}(x, x_1) + \mathrm{D}_{\Sigma\mathbb{U}}(x_1, x_2) + \cdots + \mathrm{D}_{\Sigma\mathbb{U}}(x_n, x') \\ &< 3\,\mathrm{d}_{\mathbb{U}}(x, x') + \varepsilon, \end{aligned}$$

and hence, $\mathrm{d}_{\Sigma\mathbb{U}}(x, x') \leq 3\,\mathrm{d}_{\mathbb{U}}(x, x')$. For the other inequality, let $\varepsilon > 0$ be sufficiently small again. Then there exist points $x_1, x_2, ..., x_n \in X$ so that

$$\mathrm{D}_{\Sigma\mathbb{U}}(x, x_1) + \mathrm{D}_{\Sigma\mathbb{U}}(x_1, x_2) + \cdots + \mathrm{D}_{\Sigma\mathbb{U}}(x_n, x') < \mathrm{d}_{\Sigma\mathbb{U}}(x, x') + \varepsilon.$$

But $\mathrm{d}_{\Sigma\mathbb{U}}(x, x') \leq 3\,\mathrm{d}_{\mathbb{U}}(x, x') \leq 3$, so $\mathrm{D}_{\Sigma\mathbb{U}}(x_i, x_{i+1}) = \frac{1}{3^{k_i-2}}$ for some $k_i \geq 1$, and $\mathrm{D}_{\mathbb{U}}(x_i, x_{i+1}) = \frac{1}{3^{(k_i+1)-2}} = \frac{1}{3}\mathrm{D}_{\Sigma\mathbb{U}}(x_i, x_{i+1})$. Thus,

$$\begin{aligned} 3\,\mathrm{d}_{\mathbb{U}}(x, x') &\leq 3\,\mathrm{D}_{\mathbb{U}}(x, x_1) + 3\,\mathrm{D}_{\mathbb{U}}(x_1, x_2) + \cdots + 3\,\mathrm{D}_{\mathbb{U}}(x_n, x') \\ &< \mathrm{d}_{\Sigma\mathbb{U}}(x, x') + \varepsilon, \end{aligned}$$

and hence, $3\,\mathrm{d}_{\mathbb{U}}(x, x') \leq \mathrm{d}_{\Sigma\mathbb{U}}(x, x')$. This completes the proof of (2).

(3) follows from (1), (2), and the fact that $\Sigma\,\mathrm{st}\,\mathbb{U} < \mathbb{U} < \mathrm{st}\,\mathbb{U}$. □

2.2. Approximate sequences and resolutions

An inverse sequence $(X_i, p_{i,i+1})$ consists of spaces X_i, called *coordinate spaces*, and maps $p_{i,i+1} : X_{i+1} \to X_i$, $i \in \mathbb{N}$. We write p_{ij} for the composite $p_{i,i+1}p_{i+1,i+2} \cdots p_{j-1,j}$ if $i < j$, and let $p_{ii} = 1_{X_i}$, and call the maps p_{ij} *bonding maps*. An *approximate inverse sequence* (*approximate sequence*, in short) $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ consists of an inverse sequence $(X_i, p_{i,i+1})$ and $\mathfrak{U}_i \in \operatorname{Cov}(X_i)$, $i \in \mathbb{N}$, and must satisfy the following condition:

(AI) For each $i \in \mathbb{N}$ and $\mathfrak{U} \in \operatorname{Cov}(X_i)$, there exists $i' > i$ such that $\mathfrak{U}_{i''} < p_{ii''}^{-1}\mathfrak{U}$ for $i'' > i'$.

An *approximate map* $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X}$ of a compact space X into an approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ consists of maps $p_i : X \to X_i$ for $i \in \mathbb{N}$, called *projection maps*, such that $p_i = p_{ij}p_j$ for $i < j$, and it is an *approximate resolution* of X if it satisfies the following two conditions:

(R1) For each ANR P, $\mathfrak{V} \in \operatorname{Cov}(P)$ and map $f : X \to P$, there exist $i \in \mathbb{N}$ and a map $g : X_i \to P$ such that $(gp_i, f) < \mathfrak{V}$, and

(R2) for each ANR P and $\mathfrak{V} \in \operatorname{Cov}(P)$, there exists $\mathfrak{V}' \in \operatorname{Cov}(P)$ such that whenever $i \in \mathbb{N}$ and $g, g' : X_i \to P$ are maps with $(gp_i, g'p_i) < \mathfrak{V}'$, then $(gp_{ii'}, g'p_{ii'}) < \mathfrak{V}$ for some $i' > i$.

The following is a useful characterization.

Theorem 2.3 ([9, Theorem 2.8])**.** *An approximate map* $\boldsymbol{p} = (p_i) :$ $X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ *is an approximate resolution of* X *if and only if it satisfies the following two conditions:*

(B1) *For each* $\mathfrak{U} \in \operatorname{Cov}(X)$*, there exists* $i_0 \in \mathbb{N}$ *such that* $p_i^{-1}\mathfrak{U}_i <$ $\mathfrak{U}$ *for* $i > i_0$*, and*

(B2) *for each* $i \in \mathbb{N}$ *and* $\mathfrak{U} \in \operatorname{Cov}(X_i)$*, there exists* $i_0 > i$ *such that* $p_{ii'}(X_{i'}) \subseteq \operatorname{st}(p_i(X), \mathfrak{U})$ *for* $i' > i_0$*.*

An approximate resolution $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X}$ is said to be *normal* if the family $\mathbb{U} = \{p_i^{-1}\mathfrak{U}_i : i \in \mathbb{N}\}$ is a normal sequence. Note that property (B1) implies that the normal sequence $\mathbb{U}$ has property (B). Hence, each normal approximate resolution $\boldsymbol{p}$ induces a metric $\mathrm{d}_{\mathbb{U}}$, which will be denoted by $\mathrm{d}_{\boldsymbol{p}}$.

Theorem 2.4 ([9])**.** *Every compact space* X *admits a normal approximate resolution* $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ *such that all coordinate spaces* X_i *are finite polyhedra.*

Proof: By [15, Theorem 3.15], there is an approximate resolution $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ of X such that all coordinate spaces X_i are finite polyhedra. Let $\mathfrak{U} \in \operatorname{Cov}(X)$ be such that $\operatorname{st}\mathfrak{U} < p_1^{-1}\mathfrak{U}_1$. Then by (B1) there is $i_1 > 1$ so that $p_{i_1}^{-1}\mathfrak{U}_{i_1} < \mathfrak{U}$, and so $\operatorname{st} p_{i_1}^{-1}\mathfrak{U}_{i_1} < p_1^{-1}\mathfrak{U}_1$. By the same argument, we can find a subsequence i_k so that $\boldsymbol{p}' = (p_{i_k}) : X \to \boldsymbol{X}' = (X_{i_k}, \mathfrak{U}_{i_k}, p_{i_k,i_{k+1}})$ is a normal approximate resolution of X. □

Throughout the paper, *every normal approximate resolution is assumed to have the property of Theorem 2.4.*

Remark 2.5. The notion of approximate system was first introduced by Mardešić and Rubin [4] in a more general setting. Instead of requiring commutativity $p_{ij}p_{jk} = p_{ik}$ for $i < j < k$, it requires only approximate commutativity, i.e., $p_{ij}p_{jk}$ and p_{ik} are different but controlled by some number. The most general treatment of inverse systems, which is for studying arbitrary topological spaces and maps, is found in [9]. However, since our primary concern is compact metrizable spaces, it suffices to use simpler definitions than the original ones. Our notion of approximate resolution is also a special case of the corresponding notion in [15].

3. Hausdorff dimension

Let X be a space with a normal sequence $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$. In what follows, by an open covering of a subset F of X, we mean a covering of F by sets open in X. We assume that the normal sequence $\mathbb{U}$ satisfies the following condition:

$$\mathfrak{U}_i \cap \mathfrak{U}_j = \emptyset \text{ for } i \neq j. \tag{3.1}$$

For each subset F of X and for each $i \in \mathbb{N}$, let $\mathrm{Cov}_{\mathbb{U},i}(F)$ denote the set of all open coverings $\{U_k : k \in \Lambda\}$ of F indexed by a finite or countably infinite set Λ such that each $k \in \Lambda$ admits $m \geq i$ with $U_k \in \mathfrak{U}_m$. If $\{U_k : k \in \Lambda\} \in \mathrm{Cov}_{\mathbb{U},i}(F)$, (3.1) guarantees that for each $k \in \Lambda$ such m is uniquely determined, and this m is denoted by $\sigma(U_k; \mathbb{U}, i)$. For each $s > 0$, we then define

$$\mathrm{H}^s_{\mathbb{U},i}(F) = \inf\left\{\sum_{k\in\Lambda}\left(\frac{1}{3^{\sigma(U_k;\mathbb{U},i)}}\right)^s : \{U_k : k \in \Lambda\} \in \mathrm{Cov}_{\mathbb{U},i}(F)\right\},$$

and

$$\mathrm{H}^s_{\mathbb{U}}(F) = \lim_{i\to\infty} \mathrm{H}^s_{\mathbb{U},i}(F).$$

Then $\mathrm{H}^t_{\mathbb{U},i}(F) \leq \left(\frac{1}{3^i}\right)^{t-s}\mathrm{H}^s_{\mathbb{U},i}(F)$ for $s < t$ and for all i, and hence, if $\mathrm{H}^s_{\mathbb{U}}(F) < \infty$, then $\mathrm{H}^t_{\mathbb{U}}(F) = 0$ for $t > s$. Thus, there exists a unique $s_0 \in [0,\infty]$ so that $\mathrm{H}^s_{\mathbb{U}}(F) = \infty$ for $s < s_0$ and $\mathrm{H}^s_{\mathbb{U}}(F) = 0$ for $s > s_0$. We call this value (possibly ∞) the *Hausdorff dimension of F with respect to* $\mathbb{U}$ and denote it by $\dim^{\mathbb{U}}_{\mathrm{H}} F$.

Remark 3.1. For a space X to have a normal sequence $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ with property (3.1), X must be a space with no isolated point. Indeed, suppose that X has an isolated point x. Take any $\varepsilon > 0$ so that $B_\varepsilon(x) \cap X \setminus \{x\} = \emptyset$. For any normal sequence $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ on X, by property (B), there is $i_0 \in \mathbb{N}$ such that $\mathrm{st}(x, \mathfrak{U}_i) \subseteq B_\varepsilon(x)$ for $i \geq i_0$. Then $\{x\} \in \mathfrak{U}_i$ for $i \geq i_0$.

Theorem 3.2. *For each $s > 0$, there exists a metric outer measure $\mathrm{H}^s_{\mathbb{U}}$ on X with respect to the metric $\mathrm{d}_{\mathbb{U}}$.*

Proof: First, we show that $\mathrm{H}^s_{\mathbb{U},i}$ is an outer measure on X for each i. Clearly, $\mathrm{H}^s_{\mathbb{U},i}(\emptyset) = 0$ and $\mathrm{H}^s_{\mathbb{U},i}(A) \leq \mathrm{H}^s_{\mathbb{U},i}(B)$ for any subsets A, B of X with $A \subseteq B$. To show $\mathrm{H}^s_{\mathbb{U},i}(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mathrm{H}^s_{\mathbb{U},i}(A_j)$ for any subsets A_j of X, let $\varepsilon > 0$. For each $j \geq 1$, there exists $\{U_k : k \in \Lambda_j\} \in \mathrm{Cov}_{\mathbb{U},i}(A_j)$ such that $\sum_{k\in\Lambda_j} \left(\frac{1}{3^{\sigma(U_k;\mathbb{U},i)}}\right)^s < \mathrm{H}^s_{\mathbb{U},i}(A_j) + 2^{-j}\varepsilon$. So, $\sum_{j=1}^{\infty}\sum_{k\in\Lambda_j} \left(\frac{1}{3^{\sigma(U_k;\mathbb{U},i)}}\right)^s \leq \sum_{j=1}^{\infty} \mathrm{H}^s_{\mathbb{U},i}(A_j) + \varepsilon$. But $\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k\in\Lambda_j} U_k$, and hence, $\mathrm{H}^s_{\mathbb{U},i}(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mathrm{H}^s_{\mathbb{U},i}(A_j) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have the required inequality, showing that $\mathrm{H}^s_{\mathbb{U},i}$ is an outer measure on X. This immediately implies that $\mathrm{H}^s_{\mathbb{U}}$ is an outer measure on X. It remains to show $\mathrm{H}^s_{\mathbb{U}}(A) + \mathrm{H}^s_{\mathbb{U}}(B) = \mathrm{H}^s_{\mathbb{U}}(A \cup B)$ for any subsets A, B of X such that $\mathrm{d}_{\mathbb{U}}(A,B) = \inf\{\mathrm{d}_{\mathbb{U}}(x,x') : x \in A, x' \in B\} > 0$. Indeed, if $\mathrm{d}_{\mathbb{U}}(A,B) > 0$, then there is i_0 such that $\mathrm{st}(A,\mathfrak{U}_i) \cap \mathrm{st}(B,\mathfrak{U}_i) = \emptyset$ for $i > i_0$. For, if not, each i admits $x_i \in A$ and $x'_i \in B$ such that $\mathrm{d}_{\mathbb{U}}(x_i, x'_i) \leq \frac{2}{3^{i-2}}$, and hence, $\mathrm{d}_{\mathbb{U}}(A,B)$ would be 0. Then, for $i > i_0$, $\mathrm{H}^s_{\mathbb{U},i}(A) + \mathrm{H}^s_{\mathbb{U},i}(B) \leq \mathrm{H}^s_{\mathbb{U},i}(A \cup B)$. Letting $i \to \infty$, we have $\mathrm{H}^s_{\mathbb{U}}(A) + \mathrm{H}^s_{\mathbb{U}}(B) \leq \mathrm{H}^s_{\mathbb{U}}(A \cup B)$. Since the other inequality holds by the above argument, we have the equality. □

Hence, $\mathrm{H}^s_{\mathbb{U}}$ defines a measure on the Borel subsets of X, which we call the *s-dimensional Hausdorff measure with respect to* $\mathbb{U}$ (or *s-dimensional Hausdorff $\mathbb{U}$-measure*) on X.

Theorem 3.3. *Let $\mathbb{B} = \{\mathfrak{B}_i : i \in \mathbb{N}\}$ be the normal sequence on $\mathbb{R}^n$ which consists of the open coverings $\mathfrak{B}_i$ by open balls with radius $\frac{1}{3^i}$. Then $\dim_{\mathrm{H}}^{\mathbb{B}} F = \dim_{\mathrm{H}} F$ holds for any subset F of $\mathbb{R}^n$.*

Proof: Fix $i \in \mathbb{N}$. For each $\{U_k : k \in \Lambda\} \in \mathrm{Cov}_{\mathbb{B},i}(F)$,

$$\mathrm{H}^s_{\frac{1}{3^i}}(F) \leq \sum_{k \in \Lambda} \left(\frac{2}{3^{\sigma(U_k;\mathbb{B},i)}} \right)^s,$$

and so,

$$\mathrm{H}^s_{\frac{1}{3^i}}(F) \leq 2^s \, \mathrm{H}^s_{\mathbb{B},i}(F). \tag{3.2}$$

Conversely, suppose that $\{U_{\mathrm{d}}(x_\lambda, \delta_\lambda) : \lambda \in \Lambda\}$ is an open covering of F by open balls with radius at most $\frac{1}{3^i}$, where the index set Λ is finite or countable. So, $\delta_\lambda \leq \frac{1}{3^i}$. If we choose $i_\lambda \in \mathbb{N}$ so that

$$\frac{1}{3^{i_\lambda+1}} < \delta_\lambda \leq \frac{1}{3^{i_\lambda}},$$

then $i_\lambda \geq i$. So, $\{U_{\mathrm{d}}(x_\lambda, \delta_\lambda) : \lambda \in \Lambda\} \in \mathrm{Cov}_{\mathbb{B},i}(F)$, and we have

$$\mathrm{H}^s_{\mathbb{B},i}(F) \leq \sum_{\lambda \in \Lambda} \left(\frac{1}{3^{i_\lambda}} \right)^s < 3^s \sum_{\lambda \in \Lambda} \delta_\lambda^s = \left(\frac{3}{2} \right)^s \sum_{\lambda \in \Lambda} |U_{\mathrm{d}}(x_\lambda, \delta_\lambda)|^s .$$

This implies

$$\mathrm{H}^s_{\mathbb{B},i}(F) \leq \left(\frac{3}{2} \right)^s \mathrm{H}^s_{\frac{1}{3^i}}(F). \tag{3.3}$$

(3.2) and (3.3) imply $\dim_{\mathrm{H}}^{\mathbb{B}} F = \dim_{\mathrm{H}} F$, as required. □

Theorem 3.4. *Let $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ and $\mathbb{V} = \{\mathfrak{V}_i : i \in \mathbb{N}\}$ be normal sequences on X with property (3.1), and let F be any subset of X. Then the following results hold:*

1. *If $\mathbb{V} < \mathbb{U}$, then $\mathrm{H}^s_{\mathbb{U}}(F) \leq \mathrm{H}^s_{\mathbb{V}}(F)$ and $\dim_{\mathrm{H}}^{\mathbb{U}} F \leq \dim_{\mathrm{H}}^{\mathbb{V}} F$.*
2. *$\mathrm{H}^s_{\Sigma\mathbb{U}}(F) = 3^s \, \mathrm{H}^s_{\mathbb{U}}(F)$ and $\dim_{\mathrm{H}}^{\Sigma\mathbb{U}} F = \dim_{\mathrm{H}}^{\mathbb{U}} F$.*
3. *$\mathrm{H}^s_{\mathrm{st}\,\mathbb{U}}(F) \leq \mathrm{H}^s_{\mathbb{U}}(F) \leq 3^s \, \mathrm{H}^s_{\mathrm{st}\,\mathbb{U}}(F)$ and $\dim_{\mathrm{H}}^{\mathrm{st}\,\mathbb{U}} F = \dim_{\mathrm{H}}^{\mathbb{U}} F$.*

Proof: (1) immediately follows from the definition. (2) follows from the fact that $\mathrm{H}^s_{\Sigma\mathbb{U},i}(F) = 3^s \, \mathrm{H}^s_{\mathbb{U},i+1}(F)$ for each $i \in \mathbb{N}$ and $s > 0$. (3) follows from (1) and (2) since $\Sigma\,\mathrm{st}\,\mathbb{U} < \mathbb{U} < \mathrm{st}\,\mathbb{U}$. □

Theorem 3.5. *Let $\mathbb{U}$ be a normal sequence on X with property (3.1). Then the following results hold:*

(1) *If* $F_1 \subseteq F_2 \subseteq X$*, then* $\mathrm{H}^s_{\mathbb{U}}(F_1) \leq \mathrm{H}^s_{\mathbb{U}}(F_2)$ *and* $\dim^{\mathbb{U}}_{\mathrm{H}} F_1 \leq \dim^{\mathbb{U}}_{\mathrm{H}} F_2$.
(2) $\dim^{\mathbb{U}}_{\mathrm{H}}(F_1 \cup F_2) = \max\{\dim^{\mathbb{U}}_{\mathrm{H}} F_1, \dim^{\mathbb{U}}_{\mathrm{H}} F_2\}$ *for any subsets* F_1, F_2 *of* X.

Proof: (1) is trivial. To see (2), it suffices to show the inequality "$\leq$." Suppose to the contrary that there is α such that

$$\max\{\dim^{\mathbb{U}}_{\mathrm{H}} F_1, \dim^{\mathbb{U}}_{\mathrm{H}} F_2\} < \alpha < \dim^{\mathbb{U}}_{\mathrm{H}}(F_1 \cup F_2).$$

Then $\mathrm{H}^\alpha_{\mathbb{U}}(F_1 \cup F_2) = \infty$ and $\mathrm{H}^\alpha_{\mathbb{U}}(F_1) = 0 = \mathrm{H}^\alpha_{\mathbb{U}}(F_2)$. But since $\mathrm{H}^\alpha_{\mathbb{U}}$ is an outer measure, $\mathrm{H}^\alpha_{\mathbb{U}}(F_1 \cup F_2) \leq \mathrm{H}^\alpha_{\mathbb{U}}(F_1) + \mathrm{H}^\alpha_{\mathbb{U}}(F_2)$, which leads to a contradiction. □

For any spaces X and Y with normal sequences $\mathbb{U} = \{U_i : i \in \mathbb{N}\}$ and $\mathbb{V} = \{V_i : i \in \mathbb{N}\}$, respectively, a map $f : X \to Y$ is called a $(\mathbb{U}, \mathbb{V})$-*Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \alpha\, \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X,$$

and it is a $(\mathbb{U}, \mathbb{V})$-*biLipschitz map* provided there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1\, \mathrm{d}_{\mathbb{U}}(x, x') \leq \mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \alpha_2\, \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

Lipschitz maps and biLipschitz maps are characterized in terms of normal sequences as follows:

Theorem 3.6 ([10, § 5, § 7] and [12, § 3])**.** *Let* $f : X \to Y$ *be a map between spaces* X *and* Y *with normal sequences* $\mathbb{U}$ *and* $\mathbb{V}$*, respectively. Consider the following conditions:*

$(\mathrm{L})_k$ $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^k\, \mathrm{d}_{\mathbb{U}}(x, x')$ *for* $x, x' \in X$*;*
$(\mathrm{L})^k$ $\mathrm{d}_{\mathbb{U}}(x, x') \leq 3^k\, \mathrm{d}_{\mathbb{V}}(f(x), f(x'))$ *for* $x, x' \in X$*;*
$(\mathrm{N})_{m,n}$ $\Sigma^m \mathbb{U} < f^{-1}(\Sigma^n \mathbb{V})$*;*
$(\mathrm{N})^{m,n}$ $f^{-1}(\Sigma^m \mathbb{V}) < \Sigma^n \mathbb{U}$*.*

Then for m, $n \geq 0$*, the following implications hold:*

(1) $(\mathrm{N})_{m,n} \Rightarrow (\mathrm{L})_{n-m}$; $(\mathrm{L})_m \Rightarrow (\mathrm{N})_{m+4,0}$; $(\mathrm{L})_{-m} \Rightarrow (\mathrm{N})_{4,m}$*, and*
(2) *if* f *is surjective, then*
$(\mathrm{L})^m \Rightarrow (\mathrm{N})^{m+4,0}$; $(\mathrm{L})^{-m} \Rightarrow (\mathrm{N})^{4,m}$; $(\mathrm{N})^{m,n} \Rightarrow (\mathrm{L})^{m-n}$.

We now show that the Hausdorff dimension in our sense is Lipschitz invariant.

Theorem 3.7. *Let $f : X \to Y$ be a map between spaces X and Y with normal sequences $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ and $\mathbb{V} = \{\mathfrak{V}_i : i \in \mathbb{N}\}$, respectively, both of which have property* (3.1)*, and let F be a subset of X. Consider the following conditions:*

(H)$_m$ $\mathrm{H}^s_{\mathbb{V}}(f(F)) \leq 3^{ms}\,\mathrm{H}^s_{\mathbb{U}}(F)$ *for $s > 0$, and*
(H)^m $\mathrm{H}^s_{\mathbb{U}}(F) \leq 3^{ms}\,\mathrm{H}^s_{\mathbb{V}}(f(F))$ *for $s > 0$.*

Then for $m \geq 0$, the following implications hold:

(1) (L)$_m$ $\Rightarrow$ (H)$_{m+4}$ $\Rightarrow \dim^{\mathbb{V}}_{\mathrm{H}} f(F) \leq \dim^{\mathbb{U}}_{\mathrm{H}} F$, *and*
(2) *if f is surjective, then*
(L)^m $\Rightarrow$ (H)$^{m+4}$ $\Rightarrow \dim^{\mathbb{V}}_{\mathrm{H}} f(F) \geq \dim^{\mathbb{U}}_{\mathrm{H}} F$.

Proof: To see the two implications in (1), it suffices to show (N)$_{m+4,0}$ $\Rightarrow$ (H)$_{m+4}$ by Theorem 3.6 (1). Let $i > m+4$. Suppose $\{U_k : k \in \Lambda\} \in \mathrm{Cov}_{\mathbb{U},i}(F)$. Fix $k \in \Lambda$. Then, $U_k \in \mathfrak{U}_{\sigma(U_k;\mathbb{U},i)}$. By (N)$_{m+4,0}$, there exists $V_k \in \mathfrak{V}_{\sigma(U_k;\mathbb{U},i)-m-4}$ such that $f(U_k) \subseteq V_k$. So,

$$\mathrm{H}^s_{\mathbb{V},i-m-4}(f(F)) \leq \sum_{k\in\Lambda}\left(\frac{1}{3^{\sigma(U_k;\mathbb{U},i)-m-4}}\right)^s = 3^{(m+4)s}\sum_{k\in\Lambda}\left(\frac{1}{3^{\sigma(U_k;\mathbb{U},i)}}\right)^s,$$

and hence, $\mathrm{H}^s_{\mathbb{V},i-m-4}(f(F)) \leq 3^{(m+4)s}\,\mathrm{H}^s_{\mathbb{U},i}(F)$. Taking limits as $i \to \infty$, we have (H)$_{m+4}$.

To see the two implications in (2), it suffices to show (N)$^{m+4,0}$ $\Rightarrow$ (H)$^{m+4}$ by Theorem 3.6 (2). This is proven similarly to the above. Let $i > m+4$. Suppose $\{V_k : k \in \Lambda\} \in \mathrm{Cov}_{\mathbb{V},i}(f(F))$. Fix $k \in \Lambda$. Then, $V_k \in \mathfrak{V}_{\sigma(V_k;\mathbb{V},i)}$. By (N)$^{m+4,0}$, there exists $U_k \in \mathfrak{U}_{\sigma(V_k;\mathbb{V},i)-m-4}$ such that $f^{-1}(V_k) \subseteq U_k$. So,

$$\mathrm{H}^s_{\mathbb{U},i-m-4}(F) \leq \sum_{k\in\Lambda}\left(\frac{1}{3^{\sigma(V_k;\mathbb{V},i)-m-4}}\right)^s = 3^{(m+4)s}\sum_{k\in\Lambda}\left(\frac{1}{3^{\sigma(V_k;\mathbb{V},i)}}\right)^s,$$

and hence, $\mathrm{H}^s_{\mathbb{U},i-m-4}(F) \leq 3^{(m+4)s}\,\mathrm{H}^s_{\mathbb{V},i}(f(F))$. Taking limits as $i \to \infty$, we have (H)$^{m+4}$. □

Corollary 3.8. *Let $f : X \to Y$ be a map between spaces X and Y with normal sequences $\mathbb{U} = \{\mathfrak{U}_i : i \in \mathbb{N}\}$ and $\mathbb{V} = \{\mathfrak{V}_i : i \in \mathbb{N}\}$, respectively, both of which have property* (3.1). *If f is a surjective $(\mathbb{U},\mathbb{V})$-biLipschitz map, then $\dim^{\mathbb{V}}_{\mathrm{H}} f(F) = \dim^{\mathbb{U}}_{\mathrm{H}} F$ for any subset F of X.*

Proof: Suppose that $f : X \to Y$ is a surjective $(\mathbb{U}, \mathbb{V})$-biLipschitz map. Then there exists a positive integer m for which both $(\mathrm{L})_m$ and $(\mathrm{L})^m$ hold. Thus, Theorem 3.7 implies the required equality. □

4. An approximate sequence approach

Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be an approximate resolution of a compact space X. We assume that the approximate sequence $\boldsymbol{X}$ has the following property:

(4.1) For each $j \in \mathbb{N}$, $p_{ij}^{-1}\mathfrak{U}_i \cap p_{i'j}^{-1}\mathfrak{U}_{i'} = \emptyset$ if $i, i' \leq j$ and $i \neq i'$.

For any $i, j \in \mathbb{N}$ with $i \leq j$ and for each compact subset F_j of X_j, let $\mathrm{Cov}_{\boldsymbol{X},i,j}(F_j)$ denote the set of all open coverings $\{V_k : k \in \Lambda\}$ of F_j indexed by a finite set Λ such that each k admits m with $i \leq m \leq j$ and $V_k \in p_{mj}^{-1}\mathfrak{U}_m$. By property (4.1), for each of these indexed coverings and $k \in \Lambda$, such m is uniquely determined, and it is denoted by $\tau(V_k; F_j, i, j)$. For each $s > 0$ and $i \in \mathbb{N}$, we define $\mathrm{H}_i^s(\boldsymbol{p})$ as

$$\inf \sum_{k \in \Lambda} \left(\frac{1}{3^{\tau(V_k; p_j(X), i, j)}} \right)^s,$$

where the infimum is over all finitely indexed $\{V_k : k \in \Lambda\}$ elements of $\mathrm{Cov}_{\boldsymbol{X},i,j}(p_j(X))$, $i \leq j$. For each $s > 0$, we then define the *s-dimensional Hausdorff measure* of $\boldsymbol{p}$ as $\mathrm{H}^s(\boldsymbol{p}) = \lim_{i \to \infty} \mathrm{H}_i^s(\boldsymbol{p})$. Similarly to $\dim_{\mathrm{H}}^{\mathbb{U}}$, there exists a unique $s_0 \in [0, \infty]$ such that $\mathrm{H}^s(\boldsymbol{p}) = \infty$ for $s < s_0$ and $\mathrm{H}^s(\boldsymbol{p}) = 0$ for $s > s_0$. We call this value the *Hausdorff dimension of* $\boldsymbol{p}$ and denote it by $\dim_{\mathrm{H}}(\boldsymbol{p})$.

Lemma 4.1. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution of a compact space X. If the approximate sequence $\boldsymbol{X}$ has property* (4.1)*, then the normal sequence $\mathbb{U} = \{p_i^{-1}\mathfrak{U}_i : i \in \mathbb{N}\}$ has property* (3.1)*.*

Proof: Let $i < i'$. Then property (4.1) for $\boldsymbol{X}$ implies that $p_i^{-1}\mathfrak{U}_i \cap p_{i'}^{-1}\mathfrak{U}_{i'} = p_{i'}^{-1}p_{ii'}^{-1}\mathfrak{U}_i \cap p_{i'}^{-1}\mathfrak{U}_{i'} = \emptyset$, which means that $\mathbb{U}$ satisfies (3.1). □

Lemma 4.2. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be an approximate resolution of X. Let Λ be a finite index set, and for each*

$k \in \Lambda$, let U_k be a subset of X_{i_k} for some $i_k \in \mathbb{N}$. If $X \subseteq \bigcup_{k\in\Lambda} p_{i_k}^{-1}(U_k)$ and if $j > i_k$ for $k \in \Lambda$, then $p_j(X) \subseteq \bigcup_{k\in\Lambda} p_{i_k j}^{-1}(U_k)$.

Proof: For each $j > i_k$,

$$p_{i_k j} p_j(X) \subseteq \bigcup_{k\in\Lambda} p_{i_k j} p_j(p_{i_k}^{-1}(U_k)) \subseteq \bigcup_{k\in\Lambda} U_k,$$

which implies

$$p_j(X) \subseteq \bigcup_{k\in\Lambda} p_{i_k j}^{-1}(U_k). \qquad \square$$

Lemma 4.3. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution of X such that the approximate sequence $\boldsymbol{X}$ has property (4.1), and let $\mathbb{U} = \{p_i^{-1}(\mathfrak{U}_i) : i \in \mathbb{N}\}$. Then $\mathrm{H}^s(\boldsymbol{p}) = \mathrm{H}^s_{\mathbb{U}}(X)$ for each $s > 0$, and $\dim_{\mathrm{H}}(\boldsymbol{p}) = \dim_{\mathrm{H}}^{\mathbb{U}} X$.*

Proof: It suffices to show $\mathrm{H}^s_i(\boldsymbol{p}) = \mathrm{H}^s_{\mathbb{U},i}(X)$ for each $i \in \mathbb{N}$. Suppose $\{V_k : k \in \Lambda\} \in \mathrm{Cov}_{\mathbb{U},i}(X)$, where Λ is a finite or countably infinite index set. Then $V_k = p_{i_k}^{-1}(U_k)$ for some $U_k \in \mathfrak{U}_{i_k}$, where $i_k = \sigma(V_k; \mathbb{U}, i)$. By Lemma 4.2, if $j > i_k$ for $k \in \Lambda$,

$$p_j(X) \subseteq \bigcup_{k\in\Lambda} p_{i_k j}^{-1}(U_k).$$

Since $p_j(X)$ is compact, there is a finite subcovering $\{p_{i_k j}^{-1}(U_k) : k \in \Lambda'\}$. Also $i_k = \tau(p_{i_k j}^{-1}(U_k); p_j(X), i, j)$ by property (4.1). So,

$$\mathrm{H}^s_i(\boldsymbol{p}) \le \sum_{k\in\Lambda'} \left(\frac{1}{3^{i_k}}\right)^s,$$

and hence, $H^s_i(\boldsymbol{p}) \le \mathrm{H}^s_{\mathbb{U},i}(X)$. It remains to show $H^s_i(\boldsymbol{p}) \ge \mathrm{H}^s_{\mathbb{U},i}(X)$. Let $j \ge i$, and suppose $\{W_k : k \in \Lambda\} \in \mathrm{Cov}_{\boldsymbol{X},i,j}(p_j(X))$, where Λ is a finite index set. For each $k \in \Lambda$, $W_k = p_{i_k j}^{-1}(U_k)$ for some $U_k \in \mathfrak{U}_{i_k}$, where $i_k = \tau(W_k; p_j(X), i, j)$, and $p_j(X) \subseteq \bigcup_{k\in\Lambda} W_k$. So,

$$X \subseteq \bigcup_{k\in\Lambda} p_j^{-1} p_{i_k j}^{-1}(U_k) = \bigcup_{k\in\Lambda} p_{i_k}^{-1}(U_k).$$

Also $i_k = \sigma(p_{i_k}^{-1}(U_k); \mathbb{U}, i)$ by Proposition 4.1. So,

$$\mathrm{H}^s_{\mathbb{U},i}(X) \le \sum_{k\in\Lambda} \left(\frac{1}{3^{i_k}}\right)^s,$$

and $\mathrm{H}^s_{\mathbb{U},i}(X) \le \mathrm{H}^s_i(\boldsymbol{p})$. $\square$

Next, we recall the definition of box-counting dimension for approximate resolutions [11], which will be needed in the next section.

If $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ is a normal approximate resolution of a compact space X, we define the *upper* and *lower box-counting dimensions* of $\boldsymbol{p}$ by

$$\overline{\dim}_{\mathrm{B}}(\boldsymbol{p}) = \overline{\lim_{i\to\infty}} \frac{\log_3 \beta_i(\boldsymbol{X})}{i} \text{ and } \underline{\dim}_{\mathrm{B}}(\boldsymbol{p}) = \underline{\lim_{i\to\infty}} \frac{\log_3 \beta_i(\boldsymbol{X})}{i},$$

where $\beta_i(\boldsymbol{X}) = \overline{\lim_{j\to\infty}} N_{p_{ij}^{-1}(\mathfrak{U}_i)}(X_j)$ for each $i \in \mathbb{N}$. Here, for any compact space Z and for any $\mathfrak{U} \in \mathrm{Cov}(Z)$, let $N_{\mathfrak{U}}(Z)$ be the minimum number of elements of $\mathfrak{U}$ that cover Z. If the two values coincide, the common value is denoted by $\dim_{\mathrm{B}}(\boldsymbol{p})$ and called the *box-counting dimension* of $\boldsymbol{p}$.

Theorem 4.4. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution of X. Then*

$$\dim_{\mathrm{H}}(\boldsymbol{p}) \le \underline{\dim}_{\mathrm{B}}(\boldsymbol{p}) \le \overline{\dim}_{\mathrm{B}}(\boldsymbol{p}).$$

Proof: It suffices to verify the first inequality. For each $i \in \mathbb{N}$ and $s > 0$,

$$\mathrm{H}_i^s(\boldsymbol{p}) \le \left(\frac{1}{3^i}\right)^s N_{p_{ij}^{-1}(\mathfrak{U}_i)}(p_j(X)) \le \left(\frac{1}{3^i}\right)^s N_{p_{ij}^{-1}(\mathfrak{U}_i)}(X_j) \text{ for } j \ge i.$$

So, $\mathrm{H}_i^s(\boldsymbol{p}) \le \left(\frac{1}{3^i}\right)^s \beta_i(\boldsymbol{X})$. If $\mathrm{H}^s(\boldsymbol{p}) = \infty$, then for a sufficiently large i, $1 < \mathrm{H}_i^s(\boldsymbol{p}) \le \left(\frac{1}{3^i}\right)^s \beta_i(\boldsymbol{X})$. This implies $s < \frac{\log_3 \beta_i(\boldsymbol{X})}{i}$, and so, $s \le \underline{\dim}_{\mathrm{B}}(\boldsymbol{p})$. This verifies the first inequality. □

For each approximate sequence $\boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ with property (4.1) and for each $s > 0$ and $i \in \mathbb{N}$, we define $\mathrm{H}_i^s(\boldsymbol{X})$ as

$$\inf \sum_{k\in\Lambda} \left(\frac{1}{3^{\tau(V_k; X_j, i, j)}}\right)^s,$$

where the infimum is over all finitely indexed $\{V_k : k \in \Lambda\}$ elements of $\mathrm{Cov}_{\boldsymbol{X},i,j}(X_j)$, $i \le j$. We then define the *s-dimensional Hausdorff measure of $\boldsymbol{X}$* as $\mathrm{H}^s(\boldsymbol{X}) = \lim_{i\to\infty} \mathrm{H}_i^s(\boldsymbol{X})$. Similarly to $\dim_{\mathrm{H}}^{\mathbb{U}}$, there exists a unique $s_0 \in [0, \infty]$ such that $\mathrm{H}^s(\boldsymbol{X}) = \infty$ for $s < s_0$ and $\mathrm{H}^s(\boldsymbol{X}) = 0$ for $s > s_0$. We call this value the *Hausdorff dimension of $\boldsymbol{X}$* and denote it by $\dim_{\mathrm{H}}(\boldsymbol{X})$. Note here that the definition of $\mathrm{H}_i^s(\boldsymbol{X})$ does not depend on the projection maps p_i.

Lemma 4.5. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution of X such that the approximate sequence $\boldsymbol{X}$ has the following property:*

$$\operatorname{st}\mathfrak{U}_j < p_{ij}^{-1}\mathfrak{U}_i \text{ for } i < j, \tag{4.2}$$

and let $F \subseteq X$ be compact. For each $i \in \mathbb{N}$, let F_i be a compact polyhedron such that $\overline{\operatorname{st}(p_i(F), \mathfrak{U}_i)} \subseteq F_i \subseteq \operatorname{st}(p_i(F), \operatorname{st}\mathfrak{U}_i)$. Then there is a well-defined approximate sequence $\boldsymbol{F} = (F_i, \mathfrak{U}_i|F_i, p_{i,i+1}|F_{i+1})$, and the restriction $\boldsymbol{p}|F = (p_i|F) : F \to \boldsymbol{F}$ is a normal approximate resolution of F. Moreover, if $\boldsymbol{X}$ has property (4.1), so does $\boldsymbol{F}$.

Proof: It suffices to show $p_{ij}(\operatorname{st}(p_j(F), \operatorname{st}\mathfrak{U}_j)) \subseteq \operatorname{st}(p_i(F), \mathfrak{U}_i)$ for $i < j$ since all the required properties (AI), (B1), and (B2) are deduced from the corresponding properties for $\boldsymbol{p} : X \to \boldsymbol{X}$. Let $x \in \operatorname{st}(p_j(F), \operatorname{st}\mathfrak{U}_j)$. Then $x \in U$ for some $U \in \operatorname{st}\mathfrak{U}_j$ such that $p_j(F) \cap U \neq \emptyset$. So there is $y \in F$ such that $p_j(y) \in U$. Then, both $p_{ij}(x)$ and $p_i(y) = p_{ij}p_j(y)$ belong to $p_{ij}(U)$, but by (4.2), $p_{ij}(U) \subseteq U'$ for some $U' \in \mathfrak{U}_i$. This shows that $p_{ij}(x) \in \operatorname{st}(p_i(F), \mathfrak{U}_i)$. □

Lemma 4.6. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution of X such that the approximate sequence $\boldsymbol{X}$ has property (4.1), and let $\mathbb{U} = \{p_i^{-1}(\mathfrak{U}_i) : i \in \mathbb{N}\}$. Suppose that F is a compact subset of X and that F_i, $i \in \mathbb{N}$, are compact polyhedra in X_i so that the restriction $\boldsymbol{p}|F = (p_i|F) : F \to \boldsymbol{F} = (F_i, \mathfrak{U}_i|F_i, p_{i,i+1}|F_{i+1})$ is an approximate resolution of F. Then $\mathrm{H}^s_{\mathbb{U}}(F) = \mathrm{H}^s(\boldsymbol{F})$ for each $s > 0$, and $\dim^{\mathbb{U}}_{\mathrm{H}} F = \dim_{\mathrm{H}} \boldsymbol{F}$.*

Proof: It suffices to show $\mathrm{H}^s_{\mathbb{U},i}(F) = \mathrm{H}^s_i(\boldsymbol{F})$ for each $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$, and suppose $\{V_k : k \in \Lambda\} \in \operatorname{Cov}_{\mathbb{U},i}(F)$, where Λ is a finite or a countably infinite index set. Then $V_k = p_{i_k}^{-1}(U_k)$ for some $U_k \in \mathfrak{U}_{i_k}$, where $i_k = \sigma(V_k; \mathbb{U}, i)$. By Lemma 4.2, we have

$$p_j(F) \subseteq \bigcup_{k\in\Lambda} p_{i_k j}^{-1}(U_k) \text{ for each } j > i_k.$$

This, together with property (B2) for $\boldsymbol{p}|F : F \to \boldsymbol{F}$ (Theorem 2.3), implies that there is $j' > j$ such that

$$p_{jj'}(F_{j'}) \subseteq \bigcup_{k\in\Lambda} p_{i_k j}^{-1}(U_k),$$

and hence,

$$F_{j'} \subseteq \bigcup_{k\in\Lambda} (p_{i_k j'}|F_{j'})^{-1}(U_k \cap F_{i_k}).$$

Since $F_{j'}$ is compact, there is a finite subcovering $\{(p_{i_k j'}|F_{j'})^{-1}(U_k \cap F_{i_k}) : k \in \Lambda'\}$. Also, property (4.1) for $\boldsymbol{X}$ implies property (4.1) for $\boldsymbol{F}$, and so

$$i_k = \tau((p_{i_k j'}|F_{j'})^{-1}(U_k \cap F_{i_k}); F_{j'}, i, j').$$

So,

$$\mathrm{H}^s_i(\boldsymbol{F}) \leq \sum_{k \in \Lambda'} \left(\frac{1}{3^{i_k}}\right)^s,$$

and hence, $\mathrm{H}^s_i(\boldsymbol{F}) \leq \mathrm{H}^s_{\mathbb{U},i}(F)$. It remains to show $\mathrm{H}^s_i(\boldsymbol{F}) \geq \mathrm{H}^s_{\mathbb{U},i}(F)$. Let $j \geq i$, and suppose $\{W_k : k \in \Lambda\} \in \mathrm{Cov}_{\boldsymbol{F},i,j}(F_j)$, where Λ is a finite index set. Then $W_k = p^{-1}_{i_k j}(U_k \cap F_{i_k})$ for some $U_k \in \mathfrak{U}_{i_k}$, where $i_k = \tau(W_k; F_j, i, j)$, and $F_j \subseteq \bigcup_{k \in \Lambda} W_k$. So,

$$F \subseteq \bigcup_{k \in \Lambda} p_j^{-1} p^{-1}_{i_k j}(U_k) = \bigcup_{k \in \Lambda} p^{-1}_{i_k}(U_k).$$

Also $i_k = \sigma(p^{-1}_{i_k}(U_k); \mathbb{U}, i)$. So,

$$\mathrm{H}^s_{\mathbb{U},i}(F) \leq \sum_{k \in \Lambda} \left(\frac{1}{3^{i_k}}\right)^s,$$

and hence, $\mathrm{H}^s_{\mathbb{U},i}(F) \leq \mathrm{H}^s_i(\boldsymbol{F})$. □

Remark 4.7. Given any normal approximate resolution $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ of X, we can always find a normal approximate resolution $\boldsymbol{p}' = (p_{i_k}) : X \to \boldsymbol{X}' = (X_{i_k}, \mathfrak{U}_{i_k}, p_{i_k, i_{k+1}})$ of X so that $\boldsymbol{X}'$ has property (4.2).

By lemmas 4.5, 4.6, and 4.3, we have characterizations of $\dim^{\mathbb{U}}_{\mathrm{H}}$ in terms of an approximate sequence and in terms of an approximate resolution.

Theorem 4.8. *Let $\boldsymbol{p} = (p_i) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i, p_{i,i+1})$ be a normal approximate resolution of X such that the approximate sequence $\boldsymbol{X}$ has properties* (4.1) *and* (4.2)*, and let $\mathbb{U} = \{p_i^{-1}(\mathfrak{U}_i) : i \in \mathbb{N}\}$. For each compact subset F of X, there exists an approximate sequence $\boldsymbol{F} = (F_i, \mathfrak{U}_i|F_i, p_{i,i+1}|F_{i+1})$ with $F_i \subseteq X_i$, $i \in \mathbb{N}$, being compact polyhedra in X_i so that $\boldsymbol{p}|F = (p_i|F) : F \to \boldsymbol{F}$ forms a normal approximate resolution of F and $\dim_{\mathrm{H}} \boldsymbol{F} = \dim_{\mathrm{H}}(\boldsymbol{p}|F) = \dim^{\mathbb{U}}_{\mathrm{H}} F$.*

5. Cantor sets

For each $N \in \mathbb{N}$ and for each $i \in \mathbb{N}$, let $\mathrm{I}_i^N = \mathrm{I}^N$ with the usual metric d of the Euclidean space $\mathbb{R}^N$, let $\mathfrak{U}_i$ be the open covering by open $\frac{1}{3^{i+1}}$-balls, and let $q_{i,i+1} : \mathrm{I}_{i+1}^N \to \mathrm{I}_i^N$ be the identity map. Then it is easy to see that $\boldsymbol{I}^N = (\mathrm{I}_i^N, \mathfrak{U}_i, q_{i,i+1})$ is an approximate sequence with property (4.1). For each $i \geq 1$, let $q_i : \mathrm{I}^N \to \mathrm{I}_i^N$ be the identity map. Then the approximate map $\boldsymbol{q} = (q_i) : \mathrm{I}^N \to \boldsymbol{I}^N$ is a normal approximate resolution of I^N.

Theorem 5.1. *For each positive real number r, let*

$$N = \left[\frac{\log 3}{\log 2}(r+1) + 1\right].$$

Then there exist compact subsets X_i, $i \in \mathbb{N}$, of I_i^N such that $X = \bigcap_{i=1}^{\infty} X_i$ is a Cantor set, so that the restriction $\boldsymbol{p} = (q_i|X) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i|X_i, q_{i,i+1}|X_{i+1})$ is a normal approximate resolution of X, and $\dim_{\mathrm{H}}(\boldsymbol{p}) = \dim_{\mathrm{B}}(\boldsymbol{p}) = r$.

Proof: First, let us introduce some notation. Let $B_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and let $f_0, f_1 : \mathrm{I} \to \mathrm{I}$ be the maps defined by $f_0(x) = \frac{1}{3}x$ and $f_1(x) = \frac{1}{3}x + \frac{2}{3}$, respectively. Then we define B_n, $n \in \mathbb{N}$, inductively by $B_{n+1} = f_0(B_n) \cup f_1(B_n)$.

For each $n \in \mathbb{N}$, we let $B_n^N = \{(x_i) \in \mathrm{I}^N : x_i \in B_n, 1 \leq i \leq N\}$. Note that B_n^N consists of 2^{nN} components.

Now, let r be any positive real number. For each $n \in \mathbb{N}$, let a_n be the unique integer satisfying

$$a_n - 1 < nr \leq a_n. \tag{5.1}$$

Then $a_n \leq a_{n+1}$ and $\lim_{n\to\infty} \frac{a_n}{n} = r$. Define the sequence $\{b_n\}$ by $b_1 = a_1$ and $b_{n+1} = a_{n+1} - a_n$ (≥ 0). Then, by (5.1) for a_{n+1} and a_n,

$$b_n \leq r + 1 \text{ for } n \in \mathbb{N}. \tag{5.2}$$

Moreover, for each $n \in \mathbb{N}$, let c_n be the unique integer satisfying

$$c_n - 1 < \frac{\log 3}{\log 2} b_n \leq c_n. \tag{5.3}$$

Then we have

$$2^{c_n - 1} < 3^{b_n} \leq 2^{c_n} \text{ for } n \in \mathbb{N}. \tag{5.4}$$

By the first inequality of (5.3) and (5.2),

$$c_n < \frac{\log 3}{\log 2} b_n + 1 \le \frac{\log 3}{\log 2}(r+1) + 1 \text{ for } n \in \mathbb{N}. \tag{5.5}$$

By the second inequality of (5.4), (5.5), and the definition of N,

$$3^{b_n} \le 2^N \text{ for } n \in \mathbb{N}. \tag{5.6}$$

We wish to define compact subsets X_n of I_n^N $(n \in \mathbb{N})$ with the following properties:

(1) X_n is the disjoint sum of 3^{a_n} $(= 3^{b_n} \cdot \cdots \cdot 3^{b_1})$ components $C_{i_1 \cdots i_n}$ $(1 \le i_1 \le 3^{b_1}, \ldots, 1 \le i_n \le 3^{b_n})$ of B_n^N, and

(2) if C is any one of the $3^{a_{n-1}}$ components of X_{n-1}, then $C \cap X_n$ is the disjoint sum of 3^{b_n} components of X_n.

First, we take any 3^{b_1} components C_{i_1} $(1 \le i_1 \le 3^{b_1})$ of B_1^N, which is possible by (5.6), and let

$$X_1 = \bigcup \left\{ C_{i_1} : 1 \le i_1 \le 3^{b_1} \right\}.$$

Suppose that we have defined compact subsets X_l of I_l^N $(l \le n)$ with properties (1) and (2). Then

$$X_n = \bigcup \left\{ C_{i_1 \cdots i_n} : \ 1 \le i_k \le 3^{b_k}, \ k = 1, \ldots, n \right\},$$

where $C_{i_1 \cdots i_n}$ are components of B_n^N. For each $C_{i_1 \cdots i_n}$, $C_{i_1 \cdots i_n} \cap B_{n+1}^N$ consists of 2^N components of B_{n+1}^N. Take any $3^{b_{n+1}}$ components $C_{i_1 \cdots i_n i_{n+1}}$ $(1 \le i_{n+1} \le 3^{b_{n+1}})$ from those components. Define the compact subset X_{n+1} of I_{n+1}^N as

$$X_{n+1} = \bigcup \left\{ C_{i_1 \cdots i_n i_{n+1}} : \ 1 \le i_k \le 3^{b_k}, \ k = 1, \ldots, n, n+1 \right\}.$$

Then X_{n+1} clearly has the desired properties.

Note that $X_{n+1} \subseteq X_n$, and let $X = \bigcap_{n=1}^{\infty} X_n$. Thus, we have a well-defined normal approximate resolution $\boldsymbol{p} = (q_i | X) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i | X_i, q_{i,i+1} | X_{i+1})$.

Claim 1. There is $\alpha(N) \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ each component of X_n can be covered by $\alpha(N)$ open balls from $\mathfrak{U}_n$.

Fix $n \in \mathbb{N}$, and let D be a component of X_n. Then D is a cube with dimension N and each side having length $\frac{1}{3^n}$. Let $m = \left[3\sqrt{N} \right]$. Divide each of the N sides of D equally into m intervals, and obtain

m^N cubes $D_{i_1 \cdots i_N}$ $(1 \le i_1 \le m, \ldots, 1 \le i_N \le m)$ which subdivide D. Then each side of $D_{i_1 \cdots i_N}$ has length $\frac{1}{3^n m}$, and the diagonal of $D_{i_1 \cdots i_N}$ has length $\frac{\sqrt{N}}{3^n m}$. Since $m \ge 3\sqrt{N}$, we have $\frac{\sqrt{N}}{3^n m} \le \frac{1}{3^{n+1}}$. So, each $D_{i_1 \cdots i_N}$ is covered by a single open ball with radius $\frac{1}{3^{n+1}}$. Thus, if we let $\alpha(N) = m^N$, then D can be covered by $\alpha(N)$ open balls with radius $\frac{1}{3^{n+1}}$, which establishes the claim.

Claim 2. Let $j \in \mathbb{N}$, and let $X_j = \bigcup_{k=1}^{n} q_{i_k j}^{-1}(U_k)$, where for each $k = 1, \ldots, n$, $U_k \in \mathfrak{U}_{i_k}$ for some $i_k \le j$. Then $\sum_{k=1}^{n} 3^{a_j - a_{i_k}} \ge 3^{a_j}$.

If C and C' are two different components of X_{i_k}, the distance between C and C' are $\ge \frac{1}{3^{i_k}}$ since they are also components of $B_{i_k}^N$. Since each $U_k \in \mathfrak{U}_{i_k}$ is an open ball with radius $\frac{1}{3^{i_k+1}}$, U_k intersects at most one component of X_{i_k}. Let Λ be the set of all k for which U_k intersects a component of X_{i_k}. For each $k \in \Lambda$, we call this component C_k. Then each component C of X_j is contained in some C_k. To see this, fix $x \in C$. Then $x \in U_k$ for some $k \in \Lambda$ since U_k, $k \in \Lambda$, cover X_j. C is contained in exactly one component C' of X_{i_k}, so $U_k \cap C' \ne \emptyset$. Since C_k is the only component of X_{i_k} that intersects U_k, then $C' = C_k$, so $C \subseteq C_k$. Thus, $\bigcup_{k \in \Lambda} C_k$ contains all the components of X_j. Each C_k consists of $3^{a_j - a_{i_k}}$ $(= 3^{b_j} \cdot \cdots \cdot 3^{b_{i_k+1}})$ components of X_j, and X_j consists of 3^{a_j} components of B_j^N. Thus, we have the required inequality.

Claim 3. $\dim_{\mathrm{H}}(\boldsymbol{p}) = \dim_{\mathrm{B}}(\boldsymbol{p}) = r$.

By Theorem 4.4, it suffices to show $\dim_{\mathrm{B}}(\boldsymbol{p}) = r$ and $r \le \dim_{\mathrm{H}}(\boldsymbol{p})$. For each $n \ge 1$, X_n consists of 3^{a_n} components of B_n^N, and each open ball of $\mathfrak{U}_n$ intersects at most one component of X_n. Hence, $3^{a_n} \le N_{q_{nj}^{-1}(\mathfrak{U}_n)}(X_j)$ for any $j \ge n$. On the other hand, by Claim 1, there is $\alpha(N) \in \mathbb{N}$ such that each component of X_n can be covered by $\alpha(N)$ open balls from $\mathfrak{U}_n$. So, $N_{q_{nj}^{-1}(\mathfrak{U}_n)}(X_j) \le \alpha(N) \cdot 3^{a_n}$ for any $j \ge n$. So, $3^{a_n} \le \beta_n(\boldsymbol{X}) \le \alpha(N) \cdot 3^{a_n}$, which implies

$$\frac{a_n}{n} \le \frac{\log_3 \beta_n(\boldsymbol{X})}{n} \le \frac{\log_3 \alpha(N)}{n} + \frac{a_n}{n}.$$

Thus, $\dim_{\mathrm{B}}(\boldsymbol{p}) = \lim_{n \to \infty} \frac{a_n}{n} = r$, as required.

It remains to verify $r \leq \dim_{\mathrm{H}}(\boldsymbol{p})$. Since $\dim_{\mathrm{H}}(\boldsymbol{p}) = \dim_{\mathrm{H}}(\boldsymbol{X})$ (Theorem 4.8), it suffices to show $r \leq \dim_{\mathrm{H}}(\boldsymbol{X})$. Let $i \in \mathbb{N}$, and suppose there exist i_k, j with $i \leq i_k \leq j$ and $U_k \in \mathfrak{U}_{i_k}$ for $k = 1, ..., n$ such that $X_j = \bigcup_{k=1}^{n} q_{i_k j}^{-1}(U_k)$. By the choices of a_{i_k} and a_j in (5.1), we have $i_k r \leq a_{i_k}$ and $a_j - 1 < jr$, which imply

$$-i_k r \geq -a_{i_k} \geq a_j - a_{i_k} - jr - 1.$$

So,

$$\sum_{k=1}^{n} \left(\frac{1}{3^{i_k}}\right)^r \geq 3^{-jr-1} \sum_{k=1}^{n} 3^{a_j - a_{i_k}}. \tag{5.7}$$

By Claim 2, we have

$$\sum_{k=1}^{n} 3^{a_j - a_{i_k}} \geq 3^{a_j}. \tag{5.8}$$

(5.7) and (5.8) together with the second inequality of (5.1) for a_j imply

$$\sum_{k=1}^{n} \left(\frac{1}{3^{i_k}}\right)^r \geq 3^{a_j - rj - 1} \geq \frac{1}{3}.$$

This means $\mathrm{H}_i^r(\boldsymbol{X}) \geq \frac{1}{3}$, and hence, $\dim_{\mathrm{H}}(\boldsymbol{X}) \geq r$, as required. This proves Claim 3, and hence completes the proof of the theorem. □

Corollary 5.2. *For each positive real number r, let*

$$N(r) = \begin{cases} [2r+1] & \text{if } r \geq \dfrac{\log 3}{2\log 2 - \log 3}, \\ [2r+2] & \text{otherwise.} \end{cases}$$

Then there exists a Cantor set X in $\mathrm{I}^{N(r)}$ such that $\dim_{\mathrm{H}} X = r$.

Proof: Let $M(r) = \left[\frac{\log 3}{\log 2}(r+1) + 1\right]$. Theorem 5.1 implies that there exist a Cantor set X in $\mathrm{I}^{M(r)}$ and a normal approximate resolution $\boldsymbol{p} = (q_i | X) : X \to \boldsymbol{X} = (X_i, \mathfrak{U}_i | X_i, q_{i,i+1} | X_{i+1})$ of X with $\dim_{\mathrm{H}}(\boldsymbol{p}) = r$ for some compact subsets X_i of $\mathrm{I}_i^{M(r)}$. For each $i \in \mathbb{N}$, $\mathfrak{U}_i | X = (q_i | X)^{-1}(\mathfrak{U}_i | X_i)$. If we let $\mathbb{U} = \{\mathfrak{U}_i | X : i \in \mathbb{N}\}$, by Lemma 4.3, $\dim_{\mathrm{H}}(\boldsymbol{p}) = \dim_{\mathrm{H}}^{\mathbb{U}} X$. If $\mathbb{B} = \{\mathfrak{B}_i : i \in \mathbb{N}\}$ is the normal sequence on $\mathbb{R}^{M(r)}$ which consists of the open coverings $\mathfrak{B}_i$ by open balls with radius $\frac{1}{3^i}$, then $\dim_{\mathrm{H}}^{\mathbb{U}} X = \dim_{\mathrm{H}}^{\Sigma\mathbb{B}} X$. Theorem 3.4(2) and Theorem 3.3 imply $\dim_{\mathrm{H}}^{\Sigma\mathbb{B}} X = \dim_{\mathrm{H}}^{\mathbb{B}} X = \dim_{\mathrm{H}} X$. Thus, $\dim_{\mathrm{H}} X = r$. Moreover, since $M(r) \leq N(r)$, X is a Cantor set in $\mathrm{I}^{N(r)}$. This proves the corollary. □

Acknowledgment. The authors would like to thank the referee for his/her very valuable comments on the paper.

References

[1] P. Alexandroff and P. Urysohn, "Une condition nécessaire et suffisante pour qu'une classe (L) soit une classe (B)," *C. R. Acad. Sci. Paris Sér. A-B* **177** (1923), 1274–1276.

[2] K. J. Falconer, *The Geometry of Fractal Sets.* Cambridge Tracts in Mathematics, 85. Cambridge, UK: Cambridge Univ. Press, 1985.

[3] Witold Hurewicz and Henry Wallman, *Dimension Theory.* Princeton Mathematical Series, V. 4. Princeton, NJ: Princeton University Press, 1941.

[4] Sibe Mardešić and Leonard R. Rubin, "Approximate inverse systems of compacta and covering dimension," *Pacific J. Math.* **138** (1989), no. 1, 129–144.

[5] ———, "Cell-like mappings and nonmetrizable compacta of finite cohomological dimension," *Trans. Amer. Math. Soc.* **313** (1989), no. 1, 53–79.

[6] Sibe Mardešić and Jack Segal, "*P*-like continua and approximate inverse limits," *Math. Japon.* **33** (1988), no 6, 895–908.

[7] ———, "Stability of almost commutative inverse systems of compacta," *Topology Appl.* **31** (1989), no. 3, 285–299.

[8] ———, "Mapping approximate inverse systems of compacta," *Fund. Math.* **134** (1990), no. 1, 73–91.

[9] S. Mardešić and T. Watanabe, "Approximate resolutions of spaces and maps," *Glas. Mat. Ser. III* **24(44)** (1989), no. 4, 587–637.

[10] Takahisa Miyata and Tadashi Watanabe, "Lipschitz functions and approximate resolutions," *Topology Appl.* **122** (2002), no. 1-2, 353–375.

[11] ———, "Approximate resolutions and box-counting dimension," *Topology Appl.* **132** (2003), no. 1, 49–69.

[12] ———, "Bi-Lipschitz maps and the category of approximate resolutions," *Glas. Mat. Ser. III* **38(58)** (2003), no. 1, 129–155.

[13] Keiô Nagami, *Dimension Theory.* Pure and Applied Mathematics, Vol. 37. New York-London: Academic Press, 1970.

[14] J. Segal and T. Watanabe, "Cosmic approximate limits and fixed points," *Trans. Amer. Math. Soc.* **333** (1992), no. 1, 1–61.

[15] Tadashi Watanabe, "Approximative shape I: Basic notions," *Tsukuba J. Math.* **11** (1987), no. 1, 17–59.

[16] ———, "Approximate resolutions and covering dimension," *Topology Appl.* **38** (1991), no. 1, 147–154.

Department of Mathematics and Informatics; Faculty of Human Development; Kobe University; 3-11 Tsurukabuto, Nada-Ku; Kobe, 657-8501 Japan
E-mail address: miyatt@u.washington.edu

Department of Mathematics and Information Sciences; Faculty of Education; Yamaguchi University; Yamaguchi-City, 753-8513 Japan
E-mail address: tadashi@yamaguchi-u.ac.jp