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JULIA SETS AS GROMOV BOUNDARIES FOLLOWING V. NEKRASHEVYCH

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ABSTRACT. We discuss work of Volodymyr Nekrashevych and give a re-exposition of his theorem and proof that Julia sets of postcritically finite rational maps arise as Gromov boundaries.

1. INTRODUCTION

In 2002, a remarkable preprint [1] by Laurent Bartholdi, Rostislaw Grigorchuk, and Volodymyr Nekrashevych appeared on the ArXiv. There the central object of study is termed a *self-similar, contracting group action* of a (usually finitely generated) group G acting on the set of infinite words in the finite alphabet

$$X^* = \{1, \ldots, d\}^{\mathbb{N}}.$$

Their paper contains a fascinating collection of assertions connecting many aspects of what one might call *finitely presented dynamical systems*, i.e., systems which are essentially determined by a finite amount of combinatorial data, such as a finite presentation of a group, a finite state automaton, subshift of finite type, etc. Some topics of their discussion, in random order, include groups of intermediate growth, amenability, regular languages, C^* algebras, spectral theory, and Gromov hyperbolic spaces.

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One particular point caught my attention: the theorem of Nekrashevych [8] identifying the Julia set of a postcritically finite rational function as the Gromov boundary of a certain Gromov hyperbolic one-complex. When I was a graduate student, Curt McMullen posed to me the problem of "finding combinatorial models for rational maps." Nekrashevych has found a very natural solution to this problem.

From the point of view of an impatient reader interested in, say, holomorphic dynamics, the algebraic point of view taken in the exposition of [1], [8], and [10] may be an obstacle to appreciating these interesting connections. (§§1.1 and 1.3 of [1] are entitled "Burnside groups" and "virtual endomorphisms and L-presentations," respectively, while [8] is more of a book aiming at a very general theory. We suggest first consulting [9] which is more motivated by dynamics but does not contain a proof of the theorem mentioned above.)

The goal of this note is to prove the theorem of Nekrashevych mentioned above, using only as much of the general theory as absolutely necessary and focusing on applications to rational maps. The proof is self-contained and follows the outline given in [1], which is not quite as elegant as combining the arguments given in [9] and [8]. I hope that this note will be a gentle introduction to the flavor of some of Nekrashevych's work.

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2. Definitions and main results

Definition 2.1 (Thurston map)**.** A *Thurston map* is an orientation-preserving, continuous, branched covering map $F : S^2 \to S^2$ of degree $d \geq 2$ such that the *postcritical set*

$$P_F = \bigcup_{n>0} F^{\circ n}(\{\text{critical points}\})$$

is finite. Let

$$P_F^a = \{p \in P_F | \exists k \geq 0 \text{s.t.} F^{\circ k}(p) \ \text{ is a periodic critical point}\}$$

and let $P_F^r = P_F - P_F^a$. F is called *expanding* if it is C^1 and there exists a Riemannian metric $||\cdot||$ on $S^2 - P_F$ such that

(1) there exist constants $C > 0$ and $\rho > 1$ such that $\forall n$, $\forall x \in S^2 - F^{-n}(P_F)$, $\forall v \in T_x S^2$,
$$||DF_x^{\circ n}(v)|| \geq C\rho^n ||v||;$$
(2) any compact piecewise smooth curve in $S^2 - P_F^a$ has finite length;
(3) $S^2 - P_F^a$ equipped with the induced "length metric"
$$d_{||\cdot||}(x,y) = \inf\{l_{||\cdot||}(\gamma) \mid \gamma \text{ joins } x \text{ to } y\}$$
is complete.

If a basepoint b in $S^2 - P_F$ is chosen, condition (2) implies that points in P_F^r can be joined to b by paths of finite length, while condition (3) implies for points in P_F^a no paths to b of finite length exist.

Example. Let $F(z) = z^2 - 2$. Then $P_F = \{-2, 2, \infty\}$ and the orbifold $\mathcal{O}_F$ is the $(2,2,\infty)$-orbifold. $P_F^a = \{\infty\}$, and F is expanding with respect to the Euclidean (orbifold) metric.

Expanding maps have the following crucial property. Let $\gamma : [0,1] \to S^2 - P_F$ be a curve and $\widetilde{\gamma}$ any lift of γ under F^{-n}. Then
$$l_{||\cdot||}(\widetilde{\gamma}) \leq C^{-1}\rho^{-n} l_{||\cdot||}(\gamma).$$

Atushi Kameyama [7] has analyzed the topological dynamics of expanding Thurston maps and has shown the following. For expanding Thurston maps, cycles containing critical points are topological attractors. The *Julia set* J_F of an expanding Thurston map can be defined as the complement of the union of the basins of all such attractors; equivalently, for every $x \in S^2$ (with at most two possible exceptions), J_F is the set of limit points of $\cup_{n>0} F^{-n}(x)$.

Let F be a Thurston map. Let
$$V = \bigcup_{n\geq 0} V^n, \quad \text{where} \quad V^n = F^{-n}(b) \times \{n\}.$$
Then V is the collection of all iterated preimages of b under F; note that V^n has d^n elements. Given $v \in V^n$ we denote $|v| = n$.

Since $F(P_F) \subset P_F$, the covering spaces $F^{\circ n} : S^2 - F^{-n}(P_F) \to S^2 - P_F$ are unramified. By path-lifting of loops, there is a natural homomorphism of groups

$$\Phi : \pi_1(S^2 - P_F, b) \to \mathrm{Sym}(V)$$

where $\mathrm{Sym}(V)$ is the group of bijections of X^* to itself.

Definition 2.2 (Iterated monodromy group)**.** The quotient $\pi_1(S^2 - P_F, b)/\ker(\Phi)$ is called the *iterated monodromy group* of F, denoted $\mathrm{IMG}(F)$.

Definition 2.3 (Defining data)**.** Let F be a Thurston map. A *defining data* is a triple (b, S, L) where

- $b \in S^2 - P_F$ is a basepoint; write $F^{-1}(b) = \{x_1, \ldots, x_d\}$;
- S is a finite generating set for $\mathrm{IMG}(F)$ such that $S = S^{-1} = \{s^{-1} : s \in S\}$;
- $L = \{l_i = [\lambda_i]\}_{i=1}^{d}$, where $\lambda_i : [0, 1] \to S^2 - P_F$ is a continuous path from b to x_i in $S^2 - P_F$, and $[\cdot]$ denotes the homotopy class of this path in $S^2 - P_F$ relative to its endpoints.

By path-lifting, each $s \in S$ determines a *bijection* $v \mapsto v^s$ of V to itself which preserves $|v|$, and each $l \in L$ determines an *injection* $v \mapsto l(v)$ from V to itself[1] which increases $|v|$ by 1.

Definition 2.4 (self-similarity complex)**.** Let F be a Thurston map. The *self-similarity complex* $\Sigma(F; b, S, L)$ associated with the data (b, S, L) is the infinite abstract 1-complex whose vertices are V and whose edges are oriented, labeled, and of the following two types:

- *horizontal edges labeled* s joining v to v^s, $s \in S$;
- *vertical edges labeled* l joining v to $l(v)$, $l \in L$.

Σ is endowed with the length metric such that each edge is isometric to the Euclidean unit interval. $\Sigma(F; b, S, L)$ is equipped with a preferred basepoint $(b, 0)$.

The goal of this note is to prove

Theorem 2.5 (Julia sets as Gromov boundaries)**.** *Let F be an expanding Thurston map, (b, S, L) a set of defining data, and Σ the associated self-similarity complex. Then*

[1] The monodromy action of loops is naturally a right action. However, it will be convenient to write the action of elements of L as a left action.

(1) Σ *is a complete, proper, geodesic metric space which is an infinite, uniform valence one-complex. Its quasi-isometry type is independent of the choice of defining data. Moreover, for all* $N \geq 1$, *the complexes for* F *and* $F^{\circ N}$ *are quasi-isometric.*
(2) Σ *is Gromov hyperbolic.*
(3) F *induces a cellular, degree* $\deg(F)$ *unramified covering map* $F_\Sigma : \Sigma - \{$*level zero edges*$\} \to \Sigma$.
(4) F_Σ *extends to a continuous map* $\partial F_\Sigma : \partial\Sigma \to \partial\Sigma$.
(5) *The defining data determines a homeomorphism* $h : \partial\Sigma \to J_F$ *conjugating* ∂F_Σ *to* F.

As a consequence of Theorem 2.5, the large body of techniques and results from the theory of Gromov hyperbolic spaces may now be applied to the study of complex dynamics. For example, in [4], the existence of a canonical *conformal metric gauge* on $\partial_\infty\Sigma$ (in the sense of [5]) is exploited to give new characterizations of rational maps among expanding Thurston maps without periodic critical points. We remark that Theorem 2.5 is true in vastly greater generality, and we refer the reader to Nekrashevych's work for details.

3. Quasi-isometric geometry and augmented trees

§3.1 is devoted to a rapid survey of concepts and basic facts from the theories of large-scale geometry of metric spaces and hyperbolic metric spaces; it can be safely skipped by readers familiar with these topics. §3.2 applies these concepts to the analysis of certain infinite one-complexes, *augmented trees*, which arise naturally in dynamics.

3.1. Quasi-isometric geometry

For details, see [2]. Given a fixed metric space X and $x, y \in X$, the distance from x to y is denoted $|x - y|$.

Definition 3.1. A metric space $(X, |\cdot|)$ is *proper* if its closed balls are compact. A *geodesic segment* is a map $\gamma : [0, a] \to X$ such that for all $0 \leq t \leq a$, $|\gamma(0) - \gamma(t)| = t$. X is *geodesic* if each pair of points $x, y \in \Sigma$ is joined by a geodesic segment $[x, y]$ of length $|x - y|$.

Geodesic segments need *not* be unique. In a geodesic metric space X, it makes sense to talk about triangles $\Delta xyz = [x, y] \cup$

$[y,z] \cup [z,x]$ where $[x,y],[y,z],[z,x]$ are given geodesic segments, called *sides*.

Definition 3.2 (δ-hyperbolic)**.** Let X be a geodesic metric space and $\delta \geq 0$. A triangle is called δ*-thin* if each side is contained in a δ-neighborhood of the union of the two other sides:

$$[x,z] \subset N_\delta([x,y] \cup [y,z]).$$

X is called δ*-hyperbolic* if all triangles are δ-thin. X is called *hyperbolic* if it is δ-hyperbolic for some $\delta > 0$.

Example.

(1) A simplicial tree with edges isometric to unit intervals is 0-hyperbolic.
(2) Hyperbolic n-space is δ-hyperbolic with δ independent of n.
(3) Euclidean n-space is not hyperbolic when $n \geq 2$.
(4) Any compact space is δ-hyperbolic for δ equal to the diameter of the space.

The above notion of hyperbolicity is meant to capture large-scale, rather than small-scale, features.

Definition 3.3 (cobounded)**.** Let X be a metric space, $A \subset X$, and $k > 0$. A is k*-cobounded in* X if each $x \in X$ is at most distance k from some point in A.

Definition 3.4 (quasi-isometry)**.** Let X, Y be metric spaces and $f : X \to Y$ a function, not necessarily continuous. Let $\lambda \geq 1$ and $k \geq 0$. The map f is called a (λ, k)*-quasi-isometry* if $f(X)$ is k-cobounded in Y and if for all $x, y \in X$

$$\lambda^{-1}|x-y| - k \leq |f(x) - f(y)| \leq \lambda|x-y| + k.$$

Two metric spaces are called *quasi-isometric* if there is a (λ, k)-quasi-isometry between them.

Quasi-isometries behave like a bilipschitz map on points which are far enough apart.

Facts. (see [2])

(1) Quasi-isometric is an equivalence relation.
(2) Any metric space of finite diameter is quasi-isometric to a point.

(3) If A is k-cobounded in X and is equipped with the induced (not necessarily geodesic) metric, then A and X are $(1, k)$-quasi-isometric. Thus, $\mathbb{Z}$ is $(1, 1/2)$-quasi-isometric to $\mathbb{R}$.
(4) The property of being hyperbolic is a quasi-isometry invariant of a proper, geodesic metric space.

Definition 3.5 (Cayley graph)**.** Let G be a finitely generated group and S a finite generating set with $S = S^{-1}$. The *Cayley graph* $\Sigma(G, S)$ is the graph with vertex set G and edge set $\{(g_1, g_2)|g_1 g_2^{-1} \in S\}$. The *word metric* with respect to S is given by metrizing every edge to have length one. Thus, $||g - h||_S$ is the minimal length of a word in the generators S representing gh^{-1}.

While the Cayley graph depends on a choice of generators, its quasi-isometry type does not.

Example. Let G be the fundamental group of a compact, Riemannian manifold M. Then G is quasi-isometric to $\widetilde{M}$, the universal cover of M.

Definition 3.6 (Boundary at infinity)**.** Let X be a proper, geodesic, δ-hyperbolic metric space and $b \in X$ a basepoint. The *boundary at infinity* ∂X, as a set, is the set of infinite geodesic rays emanating from b, modulo: two rays are *equivalent* if there exists a constant $C > 0$ such that each is contained in a C-neighborhood of the other. In fact, two rays from b are equivalent iff the Hausdorff distance between them is $\leq 8\delta$ [2, Ch. 7, Corollary 3].

There are a couple of other useful characterizations which are important in practice; we refer the reader to [2].

The boundary at infinity inherits a natural topology such that ∂X is a compact, metrizable space. Basically, two points on the boundary are close if representing geodesic rays stay close for a long time.

Fact. Let X, Y be proper, geodesic, hyperbolic metric spaces.

(1) Given X, the boundary admits a canonical description as a set which is independent of basepoint, and the topology on ∂X is independent of basepoint.
(2) A quasi-isometry $f : X \to Y$ induces a homeomorphism $\partial f : \partial X \to \partial Y$.

3.2. Augmented trees

When F is expanding, we will show that it is always possible to choose S so that Σ has some additional structure. The following discussion, and the outline of the proof of the characterization theorem 3.11, are due to Vadim A. Kaimanovich [6].

Remark. Our definition of augmented rooted tree allows loops from a vertex to itself and multiple edges joining vertices and is thus slightly different from Kaimanovich's.

A *rooted tree* is a simplicial tree T with a basepoint o. Metrize T so each edge has length one. The set of vertices V is a disjoint union $V = \sqcup_n V^n$ where V^n is the set of vertices distance n from o. If $v \in V^n$ we write $|v| = n$. Given $v \in V$ there is a unique geodesic $[o, v]$ joining o to v. Thus, given $v \in V^n$ and $0 \leq k \leq n$, there is a unique vertex $v^{[-k]} \in [o, v]$ at distance k from v.

By "down" we mean toward the basepoint; "up" means away from the basepoint. So $v^{[-1]}$ is one unit "below" v.

Definition 3.7 (Augmented rooted tree)**.** An *augmented rooted tree* is a simplicial 1-complex $\mathcal{T}$ with uniformly bounded valence obtained by starting with a rooted tree (T, o) (whose edges are called *vertical*) and adding, or "augmenting," this edge set with a set of *horizontal* edges, subject to the following: if a horizontal edge joins u and v, then (i) $|u| = |v|$ and, (ii) the vertices $u^{[-1]}$ and $v^{[-1]}$ either coincide or are also joined by a horizontal edge. An augmented tree is metrized so that each edge has length one.

The resulting metric space is complete, proper, and geodesic. We allow the possibility that a horizontal edge joins a vertex to itself, as well as multiple horizontal edges between two vertices. Note that *deleting loops and redundant edges yields a metric space which is* $(1,1)$*-quasi-isometric to the original one* and that *geodesic segments between distinct vertices never traverse loops from a vertex to itself.*

Let $\mathcal{T}$ be an augmented rooted tree. If $|u| = |v| = n$ we denote by $|u - v|_n$ the infimum of the length of an edge-path joining u to v through vertices at level n. Then $|u^{[-1]} - v^{[-1]}|_{n-1} \leq |u - v|_n$. This implies that geodesics have a special, "unimodal concave up" shape: if $[u_1, u_2, \ldots, u_d]$ is a geodesic segment, then there is some not necessarily unique i such that

$$|u_1| \geq |u_2| \geq \ldots \geq |u_i| \leq |u_{i+1}| \leq \ldots \leq |u_d|.$$

We call $|u_i|$ the *bottom level* of the geodesic. Also, geodesics, while not unique, can always be "modified" algorithmically so that they are in a certain normal form.

Definition 3.8 (Normal form)**.** An geodesic segment is said to be in *normal form* if it is the union of a vertical geodesic segment going down (toward o), a horizontal geodesic segment, and a vertical geodesic segment going up (away from o). Each piece is allowed to be empty.

Proposition 3.9. *Let $[u_0, u_1, \ldots, u_d]$ be a geodesic segment. Then there is an associated unique geodesic in normal form beginning at u_0 and ending at u_d. The associated normal form geodesic has the same number of horizontal edges as does the original geodesic.*

We emphasize that given a geodesic segment, the associated normal form geodesic segment is unique; given u_0, u_d there need not exist a unique normal form geodesic from u_0 to u_d.

Proof: If three consecutive vertices u_i, u_{i+1}, u_{i+2} satisfy $|u_i| = |u_{i+1}| = |u_{i+2}| + 1$ or $|u_i| + 1 = |u_{i+1}| = |u_{i+2}|$ then these three vertices form the corners of a square of size one with two opposite vertical and two opposite horizontal sides.

Here is the algorithm: if while traversing the given geodesic you go over one unit, then down one unit, replace this portion of the geodesic by the path which goes down one unit and then over one unit. If you go up one unit and then over one unit, replace this by going over one unit and then up one unit. Apply this replacement to the first vertex (i.e., the one with minimal index) at which a move is possible. Note that these moves preserve the number of horizontal edges and the bottom level. The algorithm terminates, since each move strictly decreases the *complexity*, defined as the sum, over all horizontal edges e on the geodesic, of the difference $|e| - N$ where N is the level of the bottom of the geodesic. $\square$

The following characterization of augmented rooted trees was announced by Kaimanovich [6].

Definition 3.10 (Geodesic square)**.** A *geodesic square of size k* in an augmented rooted tree $\mathcal{T}$ is an a quadruple of vertices u, v, u', v' such that

(1) $|u| = |v|$;

(2) $u' = u^{[-k]}, v' = v^{[-k]}$;
(3) $|u - v| = |u - u'| = |v - v'| = |u' - v'|$.

Said another way, a geodesic square is the image under an isometric embedding of a k-by-k piece of standard graph paper Q^k, where vertical edges are sent to vertical edges and horizontal edges to horizontal edges.

Theorem 3.11 (Characterization of hyperbolic augmented trees)**.** *An augmented rooted tree is hyperbolic if and only if there do not exist geodesic squares of arbitrarily large size.*

Proof: The necessity is clear: let x, y, z, w be the corners of a geodesic square of size K and suppose w is opposite y. Consider the triangle Δxyz whose boundary coincides with the boundary of the geodesic square. Then the distance of w to each of the two sides $[x, y]$ and $[y, z]$ is K.

The sufficiency is more involved. Suppose $\mathcal{T}$ is an augmented rooted tree such that each geodesic square has size $< K$. We first establish some lemmas.

Lemma 3.12. *The number of horizontal segments in any geodesic is* $\leq H = 2K^2 + K - 1$.

Proof: By modifying to normal form, which preserves the number of horizontal segments, it is enough to verify the claim for finite horizontal geodesic segments. Let $q > 0$ be an integer and let $[u_1, u_2, \ldots, u_{qK}]$ be a horizontal geodesic segment of length qK. Then $|u_1^{[-K]} - u_{qK}^{[-K]}| \leq q(K-1)$, since there do not exist geodesic squares of size K. Thus qK, the length of the original geodesic, must be less than $2K + q(K-1)$, the length of the curve which is obtained by going from u_1 down K to $u_1^{[-K]}$, then over to $u_{qK}^{[-K]}$, and back up K to u_{qK}. Hence, $q \leq 2K$ and the lemma follows. $\square$

Let $H = 2K^2 - K - 1$ be the constant given Lemma in 3.12.

Lemma 3.13. *The Hausdorff distance between a geodesic segment and its normal form is at most* H.

Proof: Suppose $[u_1, u_2, \ldots, u_d]$ is a geodesic. Then there is an index $1 \leq i \leq d$ such that $|u_1| \geq |u_2| \geq \ldots \geq |u_i| \leq |u_{i+1}| \leq |u_{i+2}| \ldots \leq |u_d|$. Let $j = |u_i|$ and set $u_1' = u_1^{[-(|u_1|-j)]}, u_d' =$

$u_d^{[-(|u_d|-j)]}$. The normal form of the associated geodesic is given by the geodesic running vertically from u_1 to u_1', horizontally to u_d', then vertically to u_d. Thus, $|u_i - u_1'| \leq H$ and $|u_i - u_d'| \leq H$. It follows that the original geodesic is contained in an H-neighborhood of the two vertical sides of the normal form geodesic, and that the horizontal segment of the normal form geodesic is within an H-neighborhood of u_i. □

Lemma 3.14 (Normal form geodesic triangles are $2H$-thin)**.** *Let Δ be a geodesic triangle whose sides are in normal form. Then Δ is $2H$-thin.*

Proof: Label the vertices x, y, z so that the following property holds: the levels of the horizontal segments of the sides opposite x, z, y are nonincreasing. More precisely, let

$$[z, y] = [z, z']_v \cup [z', y']_h \cup [y', y]_v,$$
$$[z, x] = [z, z'']_v \cup [z'', x'']_h \cup [x'', x]_v,$$
$$[x, y] = [x, x']_v \cup [x', y'']_h \cup [y'', y]_v$$

be the decomposition of each of the three sides into vertical, horizontal, then vertical segments. Then $|z'| = |y'| \geq |x'| = |y''| \geq |z''| = |x''|$.

It is enough to prove that the geodesic triangle with sides $A := [z', y'], B := [z', z''] \cup [z'', x''] \cup [x'', x'], C = [x', y''] \cup [y'', y']$ is $2H$-thin.

Since A has length at most H, $A \subset N_H(B \cup C)$.

Suppose $c \in C$ is on the vertical segment $[y', y'']$. Then $|c - z| \leq H$, where $z \in [z', z'']$ is on the same level as c, since $[z', y']$ is horizontal and has length $\leq H$. If $c \in C$ is on the horizontal segment $[x', y'']$, then it is within distance H of y'', which falls into the previous case. Thus, $C \subset N_{2H}(B)$.

Suppose $b \in B$ is on the vertical segment $[z', z''']$ where $|z'''| = |x'|$. Then $|b-y| \leq H$ where $y \in [y', y''] \subset C$ is on the same level as b; thus, for these b, we have $b \in N_H(C)$. In particular, $|z'''-y''| \leq H$ and hence, $|z''' - x'| \leq |z''' - y''| + |y'' - x'| \leq 2H$. Hence, the subsegment of B given by $[z''', x'] = [z''', z''] \cup [z'', x''] \cup [x'', x]$ has length $\leq 2H$, since the endpoints are at most $2H$ apart. Hence, the distance from any point on this segment to $x' \in C$ is at most $2H$. Thus, this subsegment is contained in $N_{2H}(C)$ and the proof is complete. □

Continuation of Proof of Theorem 3.11: Let Δ be any geodesic triangle. Then by Lemma 3.13, the sides of Δ lie within an H-neighborhood of a geodesic triangle Δ' with normal form sides. Δ' is $2H$-thin by Lemma 3.14. It follows that Δ is $2H+H+H=4H$-thin, and therefore that $\mathcal{T}$ is $4H=4(2K^2+K-1)$-hyperbolic. $\square$

4. Proofs

4.1. Properties of $\boldsymbol{\Sigma}$

Definition 4.1. The *level* of an edge is the minimum of the levels of its endpoints. The subcomplex consisting of vertical edges and their vertices is the *coding tree.* The correspondence

$$l_n l_{n-1} \cdots l_2 l_1 \leftrightarrow l_n \circ l_{n-1} \cdots l_2 \circ l_1 \leftrightarrow l_n l_{n-1} \cdots l_2 l_1(b) \tag{4.1}$$

defines a bijection between (i) L^n, the set of words of length n in the alphabet L, (ii) the set of compositions of n elements of maps drawn from L, and (iii) vertices V^n at level n in Σ. We will henceforth identify the set of vertices V^n with L^n and write $L^* = V$.

Remark.

(1) According to this definition, "loops" joining a vertex to itself are possible as horizontal edges, and there may be more than one horizontal edge joining two vertices.
(2) The subcomplex consisting of vertical edges is a rooted, infinite tree with uniform d-fold branching; in fact, it is the *coding tree* introduced by Feliks Przytycki [11].
(3) If $u, v \in L^*$ and are thought of as vertices of Σ, then uv lies above v in the coding tree. In particular, for all $l \in L$ and all $v \in L^*$, lv and v are joined by a vertical edge, i.e., $l(v) = lv$.
(4) If $u = l_n l_{n-1} \cdots l_2 l_1$ is a word in L^n, we denote by l_u the map $l_n \circ l_{n-1} \circ l_2 \circ l_1$. Thus,

$$l_u(v) = uv \quad \text{and} \quad l_{uv} = l_u \circ l_v. \tag{4.2}$$

(5)
$$\partial\Sigma = \{\vec{x} = (\ldots x_3 x_2 x_1) | x_n \in L\}/\sim$$

where $\vec{x} = (\dots x_3x_2x_1) \sim (\dots y_3y_2y_1) = \vec{y}$ if the Hausdorff distance of the corresponding rays from the basepoint (labeled by the empty word in L) is finite. Given $\vec{x}$ we denote by $\mathbf{x} = [\vec{x}] \in \partial\Sigma$ the point on the boundary which it represents.

4.2. Outline of proofs

Here is a brief outline of the proofs.

Definition of F_Σ. F defines a surjective cellular map $F_\Sigma : \Sigma - \{\text{level } 0 \text{ edges}\} \to \Sigma$ as follows. Given a vertex $u = (z, n+1)$, set $F_\Sigma((z, n+1)) = (F(z), n)$.

Proposition 4.2 (Properties of $\mathbf{F_\Sigma}$)**.**

(1) *For all $u, v \in L^*$*

$$(4.3)\ \forall s \in S \quad u^s = v \implies (F_\Sigma(u))^s = F_\Sigma(v)$$

$$(4.4)\ \forall l \in L \quad l(u) = v \implies l(F_\Sigma(u)) = F_\Sigma(v).$$

In particular, F_Σ is cellular, label-preserving, and orientation-preserving.

(2) *F_Σ acts as the right shift under our identification of vertices with words in L: $F_\Sigma(vl) = v$ for all $v \in L^*$ and all $l \in L$, and more generally,*

$$(4.5)\ \forall u, v \in L^*, \quad F_\Sigma^{\circ|v|}(uv) = u.$$

(3) *$F_\Sigma : \Sigma - \{$ level zero edges $\} \to \Sigma$ is a covering map.*

(4) *Let $B(v, r)$ denote the ball of radius r in Σ centered at v. If $r < |v|$ is an integer, then*

$$(4.6)\ F_\Sigma(B(v, r)) = B(F_\Sigma(v), r).$$

($F_\Sigma|B(v, r)$ need not be injective.)

Proof: (1) Suppose $u = (z, n+1), v = (w, n+1)$ and $v = u^s, s \in S$. Then by definition, there is a representative γ of s which lifts under F^{n+1} to a path $\tilde{\gamma}$ joining z to w. Then, $F(\tilde{\gamma})$ is a lift of γ which joins $F(z)$ to $F(w)$ and thus, $(F_\Sigma(u))^s = F_\Sigma(v)$.

The argument handling vertical edges is similar.

(2) Equation (4.4) of (1) above implies that for all $l \in L$, $F_\Sigma \circ l = l \circ F_\Sigma$ on the vertices at level ≥ 1. By induction on $|v|$, this implies $F_\Sigma \circ l_v = l_v \circ F_\Sigma$ on the set of vertices at level ≥ 1. Evaluating both sides of this equality at the vertex l gives $F_\Sigma(vl) = (F_\Sigma \circ l_v)(l) = (l_v \circ F_\Sigma)(l) = l_v(b) = v$.

The remaining assertions are straightforward consequences of the definitions. □

Definition of ∂F_Σ. Set

$$\partial F_\Sigma([(\ldots x_3 x_2 x_1)]) = [(\ldots x_3 x_2)].$$

Since F_Σ is cellular, it is 1-Lipschitz and hence, ∂F_Σ is well-defined.

Definition of h. For each $l \in L$, choose a representing arc $\lambda_l : [0,1] \to S^2 - P_F$. For each $s \in S$, choose a representing loop $\gamma_s : [0,1] \to S^2 - P_F$ for $s \in S$. Such a choice induces a map $p : \Sigma \to S^2$ defined as follows. Set $p((z,n)) = z \in S^2$. If $u, v \in L^n$ and $v = u^s$, then there is a unique lift $\tilde{\gamma}_s$ of γ_s under $F^{\circ n}$ based at $p(u)$ joining $p(u)$ to $p(v)$. A point in Σ on the edge joining u to v at distance $t \in [0,1]$ from u is sent to $\tilde{\gamma}_s(t)$. The image under p of a vertical edge joining u to $l(u), l \in L$ is defined similarly.

The proof of the following is straightforward.

Proposition 4.3. (1) *The functional equation $p \circ F_\Sigma^{\circ n} = F^{\circ n} \circ p$ holds on the subcomplex of Σ consisting of vertices and edges at level $\geq n$.*

(2) *Let e be any edge-path in Σ which is a loop based at b. Then the homotopy class of $p(e)$ in $S^2 - P_F$ relative to b, as a loop with basepoint b, is independent of the choice of representatives $\{\lambda_l\}_{l \in L}, \{\gamma_s\}_{s \in S}$.*

The expanding property implies the following, whose conclusion contains the definition of h.

Proposition 4.4. *Let M be the maximum $||\cdot||$-length of the λ_l's and γ_s's, $l \in L, s \in S$. Let $C > 0, \rho > 1$ denote the expansion constants as in the definition of expanding.*

Let $e_1 e_2 \ldots e_k \subset \Sigma$ be an edge-path in Σ ($k = \infty$ is permitted), and denote by n_i the level of e_i. Then

$$l_{||\cdot||}(p(e_1 e_2 \ldots e_k)) \leq C^{-1} M \textstyle\sum_{i=1}^{k} \rho^{-n_i}. \tag{4.7}$$

In particular, for any geodesic ray $\vec{x} = (\ldots x_3 x_2 x_1)$ regarded as an edge-path, the image $p(\vec{x}) \subset S^2$ has finite length; thus, $\lim_{n\to\infty} p(x_n)$ exists. Given $\mathbf{x} = [(\ldots x_3 x_2 x_1)] \in \partial\Sigma$, define

$$h(\mathbf{x}) = \lim_{n\to\infty} (p(x_n)) \in S^2.$$

Then h is well-defined on the Gromov boundary, continuous, surjective, commutes with the dynamics, and does not depend on the choice of representatives.

Proof: The bound (4.7) follows immediately from the expanding property of F. Fix $D \geq 0$. Suppose $\vec{x} = (\dots x_3x_2x_1)$ and $\vec{y} = (\dots y_3y_2y_1)$ are two geodesic rays, and that for some level n, $|x_n - y_n| \leq D$. The level of any edge on a geodesic segment e joining x_n and y_n is at least $n - D$. Then (4.4) applied to e implies that

$$d_{||\cdot||}(p(x_n), p(y_n)) \leq C^{-1}M\rho^D\rho^{-n}.$$

It follows that h is well-defined and continuous.

That h is surjective onto J_F follows immediately from the definition of the Julia set as the limit set of backward iterated preimages; see [7, Theorem 3.4]. It is obvious that h is a semiconjugacy from ∂F_Σ to F. If different representatives are used, the expanding property implies that the lengths of the traces of lifted homotopies will tend uniformly to zero as the level tends to infinity, and so h is independent of the choice of representatives. $\square$

To prove Theorem 2.5, then, the only remaining points to verify are

- *the injectivity of h*: This will follow from a finiteness result, Lemma 4.5, which asserts that there are at most finitely many elements of $\mathrm{IMG}(F)$ which are representable by loops whose length is at most a given constant.
- *the hyperbolicity of Σ*: Lemma 4.5 and an analysis of the "self-similar" nature of the action of $\mathrm{IMG}(F)$ developed in §4.5 will show that the action of $\mathrm{IMG}(F)$ has a crucial algebraic finiteness property ("contracting" in the sense of [1]). This will imply the existence of a finite set $S^{good} \supset S$ of generators of $\mathrm{IMG}(F)$ for which the associated self-similarity complex is an augmented tree without large geodesic squares.
- *quasi-isometric independence of Σ on the choices of S, L*: The argument is given in §4.6 and is quite similar to that used to prove the analogous statement for Cayley graphs.

4.3. A finiteness result

Lemma 4.5 (Finiteness lemma)**.** *For $B > 0$ let*

$$\Gamma(B) = \{\gamma | \gamma \subset S^2 - P_F \mid l_{||\cdot||}(\gamma) \leq B\}$$

where γ is a loop based at b, and let $G(B) \subset \mathrm{IMG}(F)$ denote the set of elements of $\mathrm{IMG}(F)$ *represented by elements of* $\Gamma(B)$*. Then $G(B)$ is finite.*

Proof: Since the length structure on $S^2 - P_F^a$ is complete, there is a neighborhood U of P_F^a such that for all $\gamma \in \Gamma(B)$, $\gamma \subset S^2 - U$.

Write $P_F^a = \{p_i\}$ and $P_F^r = \{q_j\}$. For each i, choose a counterclockwise oriented loop α_i in $S^2 - P_F$ which runs from b out to near the point p_i, around the point once via a loop in U, and back the way it came. For each j, choose a counterclockwise oriented loop β_j' which runs from b out to near the point, around the point, and back the way it came. Arrange so that these loops meet only at b.

The lemma will follow from the following:

Claim (1). Each β_j' represents an element of $\mathrm{IMG}(F)$ of finite order.

Claim (2). There exists a positive integer N depending on B such that any loop as in the statement of the lemma is homotopic to $\prod_{k=1}^{N} \delta_k \beta_{i_k}^{m_k}$, where δ_k is a product of at most $|P_F|$ $\alpha_i^{\pm 1}$'s and $\beta_j^{\pm 1}$s.

Claim (2) says that the only way to get lots of homotopy classes represented by curves of finite length in $S^2 - (U \cup P_F^r)$ is to wind around an element of P_F^r lots of times; claim (1) says that this arbitrarily long winding produces only finitely many distinct elements of $\mathrm{IMG}(F)$.

Claim (1) is obvious: iterates of F are uniformly ramified over points in P_F^r.

Claim (2) is pretty clear but here is a proof. Cut the sphere along the loops α_i and along arcs β_j which proceed via β_j'; then run into q_j to obtain a topological disk D with a cell structure on its boundary as shown in Figure 1.

A loop in $S^2 - (U \cup P_F^r)$ based at b may be homotoped to a curve γ in $S^2 - (U \cup P_F^r)$ which meets the α_i's and β_j's transversely and minimally. (This is a routine PL topology argument.) The intersections of γ with D, called *pieces*, run from one side labeled

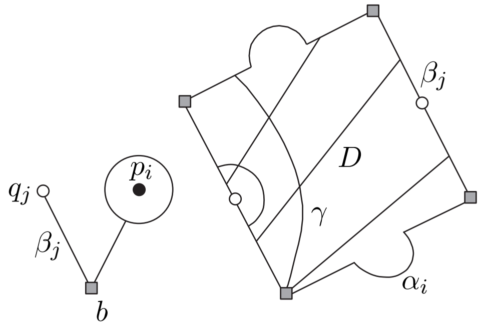

FIGURE 1. For a loop of bounded length to have long word length, it must wind around one of the q_j shown in white

$\alpha_i^{\pm 1}$ or $\beta_j^{\pm 1}$ to another such side but never from such a side to itself, by minimality. The $||\cdot||$-distance between any such pair of sides not of the form $(\beta_j^{\pm 1}, \beta_j^{\mp 1})$ is positive. Thus, a loop of length at most B has only finitely many pieces not connecting a pair of sides of the form $(\beta_j^{\pm 1}, \beta_j^{\mp 1})$. There is a homotopy of γ which slides each piece to an embedded arc connecting the vertices of ∂D corresponding to the basepoint b. The sequence of pairs of sides intersected by this loop determines the corresponding element of the free group $\pi_1(S^2 - P_F, b)$ and the claim follows. $\square$

4.4. INJECTIVITY OF h.

Suppose $\vec{x} = (\ldots x_3 x_2 x_1)$ and $\vec{y} = (\ldots y_3 y_2 y_1)$ are geodesic rays from b in Σ, and suppose $h(x) = h(y) = z \in J_F \subset S^2$. We will use Lemma 4.5 to conclude that $|x_n - y_n|$ is bounded independent of n and therefore that $\vec{x}$ and $\vec{y}$ represent the same point on $\partial\Sigma$.

First, assume P_F^r is empty. Choose representatives for L and S as in the definition of the projection map p; by Proposition 4.4, the map h is independent of such choices. Let C, ρ be as in the definition of expanding and M as in the statement of Proposition 4.4.

Given $n \geq 0$, let $[x_n, x_{n+1}, \ldots]$, $[y_n, y_{n+1}, \ldots]$ denote the geodesic subrays of $\vec{x}, \vec{y}$, respectively, which lie at and above level n. Then for each n, the set

$$p([x_n, x_{n+1}, \ldots]) \cup z \cup p([y_n, y_{n+1}, \ldots]) \tag{4.8}$$

is contained in $S^2 - F^{-n}(P_F)$, is connected, and joins $p(x_n)$ to $p(y_n)$. It is easy to see, by reparameterizing the edge-paths defining $[x_n, x_{n+1}, \ldots]$ and $[y_n, y_{n+1}, \ldots]$, that this set is also the image of a rectifiable arc $\widetilde{\gamma}_n : [0,1] \to S^2 - F^{-n}(P_F)$ joining $p(x_n)$ to $p(y_n)$. The arc $\widetilde{\gamma}_n$ runs from $p(x_n)$ along the rectifiable arc $p([x_n, x_{n+1}, \ldots])$ to z, then runs from z along the reverse of the rectifiable arc $p([y_n, y_{n+1}, \ldots])$ to $p(y_n)$. By construction, the arc

$$\gamma_n = F^{\circ n}(\widetilde{\gamma_n})$$

has the following properties:

(1) γ_n is a loop based at b in $S^2 - P_F$. Therefore, γ_n represents an element of $\pi_1(S^2 - P_F, b)$ and hence an element $g_n \in \mathrm{IMG}(F)$.
(2) $x_n^{g_n} = y_n$, since by construction there is a lift $\widetilde{\gamma}_n$ of γ_n based at $p(x_n)$ which joins $p(x_n)$ to $p(y_n)$.
(3) By the functional equation in Proposition 4.3, we have $F^{\circ n} \circ p([x_n, x_{n+1}, \ldots]) = p \circ F_\Sigma^{\circ n}([x_n, x_{n+1}, \ldots])$ (and similarly for the y's). Hence, γ_n is the closure of the image under p of two infinite rays from b in Σ. By Proposition 4.4, the length of γ_n is at most $B = 2CM(1 - 1/\rho)^{-1}$. Hence for all n, $\gamma_n \in G(B) \subset \mathrm{IMG}(F)$. By Lemma 4.5, $G(B)$ is finite and therefore, $r = \max_n\{||g_n||_{S^{good}}\} < \infty$.

For all n we have, by (2), that $x_n^{g_n} = y_n$ and, by (3), that $|x_n - y_n| \leq r$. Hence, the rays $\vec{x}$ and $\vec{y}$ represent the same point of $\partial\Sigma$.

If P_F^r is nonempty, it may happen that $z \in F^{-n}(P_F^r)$ for some n. We modify the definition of $\widetilde{\gamma_n}$ as follows. Choose a tiny disk neighborhood of P_F^r and lift this under F^{-n} to a neighborhood V of z. Replace $\widetilde{\gamma_n} \cap V$ with a curve which stays in V and avoids z. Then $\gamma_n = F^{\circ n}(\widetilde{\gamma_n})$ is a loop based at b and contained in $S^2 - P_F$. This modification can be done so that the length of γ_n is still at most, say, $2B$, and the argument proceeds as above. □

4.5. Restrictions

Definition 4.6 (Restriction)**.** Let F be a Thurston map and (b, S, L) a set of defining data. Let $g \in \mathrm{IMG}(F)$ and $v \in L^*$. The *restriction* $g|_v$ is the bijection of L^* defined by $g|_v = (l_{v^g})^{-1} \circ g \circ l_v$. (Meaning, apply first l_v, then g, then the inverse of l_{v^g}.)

We emphasize that the definition of restriction depends on L but not on S.

Proposition 4.7 (Properties of restrictions)**.** *For all $g, h \in \mathrm{IMG}(F)$ and all $u, v \in L^*$,*

(1) *if w_g is a word in S representing g, then $g|_v = (l_{v^g})^{-1} \circ w_g \circ l_v$. If representative loops and arcs for S and L are chosen, so that the projection $p : \Sigma \to S^2$ is defined, then the action of $g|_v$ on L^* coincides with the monodromy action of the loop $p(e)$ where $e = l_v * w_g * l_{v^g}^{-1}$ is the edge-path in Σ based at b which follows l_v from b, then w_g, then l_{v^g} backwards back to b.*

In particular, $g|_v \in \mathrm{IMG}(F)$.

(2) $(uv)^g = u^g v^{g|_u}$.

(3) $g|_{uv} = (g|_u)|_v$.

(4) $(gh)|_v = (g|_v)(h|_{v^g})$ *(written as right action).*

Proof: (1) The first assertion is obvious from the definition of restriction. To see the second, just note that from the definitions, the loop $p(l_v * w_g * (l_{v^g})^{-1})$ is a loop in $S^2 - P_F$ based at b whose monodromy action on L^* is given by $(l_{v^g})^{-1} \circ w_g \circ l_v$. Hence, $g|_v \in IMG(F)$.

(2) Let w_g be a word in S representing g. Write $(uv)^g = u'v'$ where $|u| = |u'|$ and $|v| = |v'| = n$. Consider the edge-path e starting at v given by the word

$$l_u * w_g * (l_{u'})^{-1}.$$

This runs up from v via l_u to uv, over via w_g to $u'v'$, then back down to v' via $(l_{u'})^{-1}$. Since F_Σ acts as the right shift (Proposition 4.2(2)), and preserves the labels of edges, the image $F_\Sigma^{\circ n}(e)$ is the edge-path which runs from the basepoint up via l_u to u, over via w_g to u', and back down via $(l_{u'})^{-1}$ to the basepoint. By the definition of restriction, $F_\Sigma^{\circ n}(e)$ represents $g|_u$. Thus, $u' = u^g$ and $v' = v^{g|_u}$.

(3) We have

$$\begin{array}{rcll} g|_{uv} & = & (l_{(uv)^g})^{-1} \circ g \circ l_{uv} & \text{by definition} \\ & = & (l_{u^g v^{g|_u}})^{-1} \circ g \circ l_{uv} & \text{by (2)} \\ & = & (l_{u^g} \circ l_{v^{g|_u}})^{-1} \circ g \circ l_u \circ l_v & \text{by eq. (4.2)} \\ & = & (l_{v^{g|_u}})^{-1} \circ \left((l_{u^g})^{-1} \circ g \circ l_u\right) \circ l_v & \\ & = & (g|_u)|_v & \text{by definition.} \end{array}$$

(4) Since our action of monodromy is a right action, we have

$$\begin{array}{rcll}(gh)|_v & = & (l_{v^{gh}})^{-1} \circ hg \circ l_v & \text{by definition} \\ & = & ((l_{v^{gh}})^{-1} \circ h \circ l_{v^g}) \circ (l_{v^g})^{-1} \circ g \circ l_v & \\ & = & (h|_{v^g}) \circ (g|_v) & \text{by definition} \\ & = & (g|_v)(h|_{v^g}) & \text{writing as right action.}\end{array}$$

□

Remark. (1) and (2) imply that the monodromy action of $\pi_1(S^2 - P_F, b)$ on the set L^* of finite words in L is *self-similar* in the sense of [1].

The following observation is the key ingredient in showing hyperbolicity of Σ. The conclusion says that the action of the IMG of any expanding Thurston map is *contracting* in the sense of [1].

Proposition 4.8. *Let (b, S, L) be a set of defining data and F, an expanding Thurston map. Then there is a unique nonempty smallest set $\mathcal{N} \subset \mathrm{IMG}(F)$, called the* nucleus, *such that for all $g \in \mathrm{IMG}(F)$, there is a "magic level" $m(g)$ such that for all vertices v satisfying $|v| \geq m(g)$, the restriction $g|_v$ lies in $\mathcal{N}$. Furthermore, $\mathcal{N}$ is closed under restrictions.*

Proof: For $g \in \mathrm{IMG}(F)$ set

$$\mathcal{R}(g, \geq n) = \{g|_w \ : \ |w| \geq n\}$$

$$\mathcal{N}(g) = \bigcap_{m \geq 0} \mathcal{R}(g, \geq m)$$

and set

$$\mathcal{N} = \bigcup_{g \in \mathrm{IMG}(F)} \mathcal{N}(g).$$

Choose representatives for S and L, so that the projection p is defined. Let C, ρ be the constants as in the definition of expansion, and let M be as in Proposition 4.4. Choose $\delta > 0$ arbitrary and set

$$B = 2C^{-1}M(1 - 1/\rho)^{-1} + \delta$$

Note that the length on the sphere of the image under p of any infinite ray from b in the coding tree is then strictly less than B.

Fix $g \in G$, write g as a word w_g in S, and let $v \in L^*$. By Proposition 4.7(1), the action of the restriction $g|_v$ on L^* coincides with the monodromy action of the loop on the sphere given by

$$\gamma_v = p(l_v * w_g * (l_{v^g})^{-1}) = p(l_v) * p(w_g) * p((l_{v^g})^{-1}).$$

By Proposition 4.4,

$$l_{||\cdot||}(\gamma_v) \leq C^{-1}M(1-1/\rho)^{-1} + C^{-1}M\rho^{-|v|} + C^{-1}M(1-1/\rho)^{-1}.$$

Hence, if $m'(g)$ is chosen so that $C^{-1}M\rho^{-m'(g)} < \delta$, then

$$|v| \geq m'(g) \implies l_{||\cdot||}(\gamma_v) < B \implies g|_v \in B(G) \subset \mathrm{IMG}(F).$$

By Lemma 4.5, $G(B)$ is finite.

The argument in the previous paragraph shows that for each $n \geq m'(g)$, $\mathcal{R}(g, \geq n)$ is a nonempty subset of $G(B)$. Since $G(B)$ is finite, it follows that the nested intersection $\mathcal{N}(g)$ is also contained in $G(B)$ and is finite and nonempty. Let $m(g)$ be the smallest integer such that the nested intersection stabilizes, i.e., such that for all $n \geq m(g)$, we have $\mathcal{R}(g, \geq m(g)) = \mathcal{R}(g, n)$. Since $g|_{uv} = (g|_u)|_v$ by Proposition 4.7, $\mathcal{N}$ is closed under restriction. □

Lemma 4.9. *Let F be a $(1,\rho)$-expanding Thurston map and (b, S, L) given defining data for the self-similarity complex Σ. Then there exists a finite subset $S^{good} \subset \mathrm{IMG}(F)$ containing S such that $\Sigma(F; b, S^{good}, L)$ is an augmented rooted tree without arbitrarily large geodesic squares.*

Proof: Let

$$S^{good} = \bigcup_{s\in S} \mathcal{R}(s, \geq 0).$$

By propositions 4.8 and 4.7, respectively, S^{good} is finite and closed under restrictions.

$\mathbf{\Sigma(F; b, S^{good}, L)}$ **is an augmented tree.** Suppose $v = u^s$ with $s \in S^{good}$, i.e., u, v are joined by a horizontal edge. Write $u = lw$. Then since S^{good} is closed under restrictions,

$$v = u^s = (lw)^s = l^s w^{s|_l} = l^s w^t, \;\; t = s|_l \in S^{good}.$$

This means that $u^{[-1]} = w$ is joined by a horizontal edge labeled t to $v^{[-1]} = w^t$.

No big squares. Fix an integer $r > 1$. Set

$$m_r = \max\{m(g) | \; ||g||_{S^{good}} \leq r\}$$

and

$$k = \max\{||h||_{S^{good}} \mid h \in \mathcal{N}\}$$

where $\mathcal{N}$ is the nucleus of the action of $\mathrm{IMG}(F)$. Suppose w, w' are two vertices at level $> m_r$ such that $w' = w^g$ where $||g||_{S^{good}} \leq r$, i.e., w, w' are joined by a horizontal edge-path whose length is at most r. Write $w = uv$ where $|u| \geq m_r$. Then

$$w^g = (uv)^g = u^g v^{g|_u} = u^g v^h, \;\; h \in \mathcal{N}$$

since $|u| \geq m_r \geq m(g)$, the magic level of g. By the definition of the number k, the vertices v and v^h are joined by a horizontal edge-path of length at most k. But $v = w^{[-m_r]}$ and $v^h = w'^{[-m_r]}$.

Geometrically, this means that if two vertices are joined by a horizontal path of length at most r, then dropping down m_r steps yields a pair of vertices joined by a horizontal path of length at most k.

Now take $r = k+1$ in the previous paragraph. It follows that there cannot exist geodesic rectangles of horizontal size $k+1$ and vertical size m_{k+1}. Hence, there cannot exist geodesic squares of size $K = \max\{k+1, m_{k+1}\}$. □

Corollary 4.10 (of proof)**.** *If instead we set $S^{good} = \cup_{s \in S} \mathcal{R}(s, \geq 0) \cup \mathcal{N}$ where $\mathcal{N}$ is the nucleus of the action with respect to L, then the diameter k of the nucleus with respect to the word metric using S^{good} is 1. The proof above shows that in this case,*

(1) *if u, v are on the same level and joined by a horizontal path of length at most two, then $u^{[-m_2]}$ and $v^{[-m_2]}$ are joined by an edge labeled by an element $h \in \mathcal{N}$;*
(2) *there are at most $|\mathcal{N}|$ distinct rays representing a given point on the Gromov boundary, and the Hausdorff distance between any two such rays is exactly 1.*

4.6. Quasi-isometric invariance of $\mathbf{\Sigma}$

Proposition 4.11 (Quasi-isometric invariance of $\mathbf{\Sigma}$)**.** *Let F be any Thurston map.*

(1) *The quasi-isometry type of the self-similarity complex associated to a Thurston map F is independent of the choice of defining data (b, S, L).*
(2) *The self-similarity complexes for F and $F^{\circ N}$ are quasi-isometric.*

Proof: (1) Suppose that b' is a different basepoint. Let $V' = \cup_{n \geq 0} F^{-n}(b') \times \{n\}$. Choose an arc $\alpha : [0,1] \to S^2 - P_F$ joining b' to b and let $a : V' \to V$ denote the bijection obtained by lifting α. Set

$S' = \{s' = a^{-1} \circ s \circ a \mid s \in S\}$ and $L' = \{l' = a^{-1} \circ l \circ a \mid l \in L\}$. Then the map $\Sigma(F; b, S, L) \to \Sigma(F; b', S', L')$ given by $v' \mapsto a(v), s' \mapsto s, l' \mapsto l$ is a cellular isomorphism of 1-complexes and hence an isometry.

Consider the complexes $\Sigma = \Sigma(F; b, S, L)$ and $\Sigma' = \Sigma(F; b, S', L')$ defined by two different choices of defining data with the same basepoint. Let $|\cdot|$ denote the metric on Σ and $||\cdot||_S$ the word metric on $\pi_1(S^2 - P_F, b)$, and define similarly the metrics $|\cdot|', ||\cdot||_{S'}$. Let $M' = \max_{s \in S} ||s||_{S'}$. Then the $|\cdot|'$-distance between the endpoints of any edge in Σ labeled $s \in S$ is at most M'.

Given $l \in L$ let $l' \in L'$ denote the unique element of L' such that both l and l' join b to the same point of $F^{-1}(b)$. The bijection $t_l = l \circ l'^{-1}$ of V to itself is then an element of $\mathrm{IMG}(F)$, since if representatives λ, λ' for l, l' are chosen then the concatenation $\lambda * \lambda'^{-1}$ is a loop based at b. Thus, for each $l \in L$, $l = l \circ l'^{-1} \circ l' = t_l \circ l'$. Let $T' = \max_{l \in L}\{||t_l||_{S'} + 1.\}$. Then the $|\cdot|'$-distance between the endpoints of any edge in Σ labeled $l \in L$ is at most T'.

Thus, the $|\cdot|'$ length of an edge path in Σ with h horizontal and v vertical edges is at most $hM' + vT' \leq (v + h)\max\{M', T'\}$. To get a bound in the other direction, define M, T similarly.

Let $\lambda = \max\{M', T', M, T\}$. Then the identity map is a $(\lambda, 0)$-quasi-isometry between the (non-geodesic) metric spaces $(V, |\cdot|)$, which is 1/2-cobounded in Σ and $(V, |\cdot|')$ which is 1/2-cobounded in Σ'. Hence, Σ and Σ' are quasi-isometric.

(2) Let $\Sigma = \Sigma(F; b, S, L)$ and $\Sigma' = \Sigma(F^{\circ N}; b, S, L^N)$ where L^N is the set of d^N bijections of the form $\prod_{i=1}^N l_i$, $l_i \in L$. The natural inclusion $V' \hookrightarrow V$ is obviously N-Lipschitz and has N-cobounded image.

We now show that this image cannot contract distances too much. Given $u \in V$, let $u' = u^{[-(|u| \mod N)]}$, i.e., drop down just enough steps to land in V'. If $|u - v| \leq 1$ and u, v are on different levels then clearly $|u' - v'| \leq N$. If $|u - v| \leq 1$ and u, v are on the same level, then $|u' - v'| \leq J^{N-1}$ where

$$J = \max_{l \in L}\{||\, s|_l \,||\}.$$

It follows that when $\lambda = \max\{N, J^{N-1}\}$, Σ' is (λ, N)-quasi-isometric to Σ. $\square$

References

[1] Laurent Bartholdi, Rostislav Grigorchuk, and Volodymyr Nekrashevych, "From Fractal Groups to Fractal Sets." arXiv:math. GR/0202001 v4 13 Sep. 2002. (See also Grabner and Woess 25-118).

[2] E. Ghys and P. de la Harpe, eds. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progress in Mathematics, 83. Boston, MA: Birkhäuser Boston, Inc., 1990.

[3] Peter Grabner and Wolfgang Woess, eds. *Fractals in Graz 2001: Analysis–Dynamics–Geometry–Stochastics* (Proceedings of the conference at Graz University of Technology, Graz, June 2001). Trends in Mathematics. Basel: Birkhäuser Verlag, 2003.

[4] Peter Haissinsky and Kevin M. Pilgrim. "Coarse Expanding Conformal Dynamical Systems." In progress.

[5] Juha Heinonen, *Lectures on Analysis on Metric Spaces*. Universitext. New York: Springer-Verlag, 2001.

[6] Vadim A. Kaimanovich, "Random Walks on Sierpiński Graphs: Hyperbolicity and Stochastic Homogenization." Grabner and Woess 145–183.

[7] Atushi Kameyama, "On Julia sets of postcritically finite branched coverings, I—coding of Julia sets," *J. Math. Soc. Japan* **55** (2003), no. 2, 439–454.

[8] Volodymyr Nekrashevych, "Limit Spaces of Self-similar Group Actions." Lecture. Random Walks and Geometry, Erwin Schroedinger Institute, Vienna. 22 June 2001. Preprint. Available at www.math.tamu.edu/˜ nekrash.

[9] ______, "Iterated Monodromy Groups," arXiv:math.DS/0312306 v1 16 Dec. 2003.

[10] ______, *Self-Similar Groups*. Mathematical Surveys and Monographs, vol. 117. Providence, RI: AMS, 2005.

[11] Feliks Przytycki, "Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map," *Invent. Math.* **80** (1985), no. 1, 161–179.

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