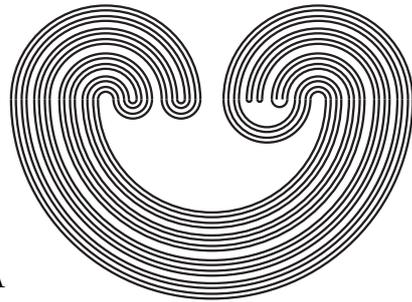
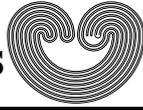


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**CAT(0) BOUNDARIES OF TRUNCATED
HYPERBOLIC SPACE**

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ABSTRACT. We prove that the CAT(0) boundary of a truncated hyperbolic space is homeomorphic to a sphere with disks removed. In dimensions $n \neq 5$, we show that if G acts properly discontinuously by isometries on \mathbb{H}^n with finite volume quotient, then G has a CAT(0) boundary that is homeomorphic to the Sierpiński curve of dimension $(n - 2)$. This uses work of G. T. Whyburn (“Topological characterization of the Sierpiński curve”) for $n = 3$ and of J. W. Cannon (“A positional characterization of the $(n - 1)$ -dimensional Sierpiński curve in S^n ($n \neq 4$)”) for all $n \neq 4$. Then, using work of G. Christopher Hruska (“Geometric invariants of spaces with isolated flats”), we deduce that any CAT(0) boundary for G is a Sierpiński curve.

INTRODUCTION

In this paper, we are interested in studying the space obtained by removing the interiors of a collection of disjoint closed horoballs from \mathbb{H}^n . This is often called *truncated* or *neutered* hyperbolic space. Such spaces arise as the universal covers of finite volume hyperbolic manifolds whose cusps have been removed. It is shown in [1] that with the induced path metric, this is a complete CAT(0) space. As such, we can try to determine the homeomorphism type of the visual boundary of this space. Recall that \mathbb{H}^n is also a CAT(0) space and its visual boundary is homeomorphic to S^{n-1} .

It is important to note that this is different from studying the limit set of a geometrically finite group action on \mathbb{H}^n . In particular, the limit set of a finite covolume group acting on \mathbb{H}^n is all of $\partial_\infty \mathbb{H}^n$,

whereas the CAT(0) boundary of the truncated space is certainly not a sphere.

The ideas in this paper came about while the author was trying to study groups, such as the fundamental group of the figure eight knot complement (or more generally, any finite volume hyperbolic 3-manifold). Such a group acts properly discontinuously, by isometries, and cocompactly on a truncated \mathbb{H}^3 . We call this type of group action a *geometric* action. The author was trying to understand the CAT(0) boundary of this space as it relates to the group. In that setting, one can actually show that the visual boundary is homeomorphic to the Sierpiński carpet, using the work of G. T. Whyburn [10]. The work of Whyburn that we use here was generalized to all dimensions other than $n = 4$ by J. W. Cannon in [3], so we can conclude an analogous result in higher dimensions. We will discuss this further in the third section of the paper.

I would like to give many thanks to the referee for pointing out an error in the original version of this paper and for his/her patience while the revisions were made (that took much longer than they should have!). I would also like to thank Mladen Bestvina for enlightening me about Cannon's work on the higher dimensional Sierpiński spaces.

1. CAT(0) SPACES AND BOUNDARIES

Let (X, d) be a metric space. Then X is *proper* if closed metric balls are compact. A (*unit speed*) *geodesic* from x to y for $x, y \in X$ is a map $c : [0, D] \rightarrow X$ such that $c(0) = x$, $c(D) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, D]$. (X, d) is called a *geodesic metric space* if every pair of points are joined by a geodesic.

Definition 1.1. Let (X, d) be a proper complete geodesic metric space. If Δabc is a geodesic triangle in X , then we consider $\Delta \bar{a}\bar{b}\bar{c}$ in \mathbb{E}^2 , a triangle with the same side lengths, and call this a *comparison triangle*. We say X satisfies the CAT(0) *inequality* if given Δabc in X , then for any comparison triangle and any two points p, q on Δabc , the corresponding points \bar{p}, \bar{q} on the comparison triangle satisfy

$$d(p, q) \leq d(\bar{p}, \bar{q}).$$

If (X, d) is a CAT(0) space, then the following basic properties hold:

- (1) The distance function $d: X \times X \rightarrow \mathbb{R}$ is convex.
- (2) X has unique geodesic segments between points.
- (3) X is contractible.

For details, see [1].

Let (X, d) be a proper CAT(0) space. First, define the boundary ∂X as a set as follows:

Definition 1.2. Two geodesic rays $c, c': [0, \infty) \rightarrow X$ are said to be *asymptotic* if there exists a constant K such that $d(c(t), c'(t)) \leq K$ for all $t > 0$ —this is an equivalence relation. The boundary of X , denoted ∂X , is then the set of equivalence classes of geodesic rays. The union $X \cup \partial X$ will be denoted \overline{X} . The equivalence class of a ray c is denoted by $c(\infty)$.

We will now give a topology on \overline{X} so that the induced topology on X is the original metric topology.

There is a natural neighborhood basis for a point in \overline{X} . Let c be a geodesic emanating from x_0 and $r > 0, \epsilon > 0$. Also, let $S(x_0, r)$ denote the sphere of radius r centered at x_0 with $p_r : X \rightarrow S(x_0, r)$ denoting projection. Define

$$U(c, r, \epsilon) = \{x \in \overline{X} \mid d(x, x_0) > r, d(p_r(x), c(r)) < \epsilon\}.$$

This consists of all points in \overline{X} such that when projected back to $S(x_0, r)$, this projection is not more than ϵ away from the intersection of that sphere with c . These sets along with the metric balls about x_0 form a basis for the *cone topology* on \overline{X} . The induced topology on ∂X is also called the cone topology on ∂X . The set ∂X with the cone topology is often called the *visual boundary* and will be denoted $\partial_\infty X$.

As one expects, the visual boundary of \mathbb{E}^n is S^{n-1} , as is the visual boundary of \mathbb{H}^n . Thus, the visual boundary does not capture the difference between these two CAT(0) spaces. Another topology one can put on ∂X , called the Tits topology, distinguishes these two CAT(0) spaces. We will briefly describe this topology in the last section of the paper.

2. TRUNCATED \mathbb{H}^n

Many of the preliminary facts used here concerning truncated hyperbolic space can be found in Part II of [1]. We quote the necessary results without proof. The first is [1, Theorem 11.27, p. 362].

Theorem 2.1. *Let $X \subset \mathbb{H}^n$ be a subspace obtained by deleting a family of disjoint open horoballs. When endowed with the induced length metric, X is a complete CAT(0) space.*

Remark 2.2. The theorem does not extend to more general rank one symmetric spaces because the stabilizers of horospheres are nilpotent and not virtually abelian. This would violate the Solvable Subgroup Theorem [1, p. 248] which says every solvable subgroup of a group acting geometrically on a CAT(0) space is virtually abelian.

Remark 2.3. It is an easy exercise in hyperbolic geometry to show that any disjoint collection of closed horoballs is locally finite. This means for any compact set C in \mathbb{H}^n , only finitely many horoballs intersect C . For instance, see [1, p. 363].

Recall that a group action on a proper geodesic metric space is called *geometric* if it is properly discontinuous, isometric, and cocompact. Suppose Γ is a *lattice* in $SO(n, 1)$ – this simply means Γ acts properly discontinuously by isometries on \mathbb{H}^n . If Γ is cocompact, then Γ acts geometrically on \mathbb{H}^n , a CAT(0) space. If Γ is not cocompact, then Γ acts geometrically on a truncated \mathbb{H}^n . Thus, the following is direct corollary of Theorem 2.1.

Corollary 2.4. *Every lattice $\Gamma \subset SO(n, 1)$ acts geometrically on a CAT(0) space X .*

The main theorem of this paper discusses the homeomorphism type of the visual boundary of this space. For the proof we will use the upper half space model of \mathbb{H}^n . Thus, $\mathbb{H}^n = \{(x_1, \dots, x_n) \mid x_n > 0\}$ with the Riemannian metric $\frac{ds}{x_n}$. An open *horoball* in \mathbb{H}^n is a translate of $H_c = \{(x_1, \dots, x_n) \mid x_n > c\}$ where $c > 0$ is a constant, by an element of $Isom(\mathbb{H}^n)$. A *horosphere* is a translate of the set $S_c = \{(x_1, \dots, x_n) \mid x_n = c\}$. The horoball H_c is said to be *centered* at the point at infinity in $\partial_\infty \mathbb{H}^n$ corresponding to the x_n axis.

The following can be found in [1] in Part II as Lemma 11.32.

Lemma 2.5. *Let X be as in Theorem 2.1. The bounding horospheres are convex subspaces of X and with the induced path metric, each is isometric to \mathbb{E}^{n-1} .*

We use the notation \mathbb{E}^{n-1} when viewing Euclidean space metrically, and we use \mathbb{R}^{n-1} when viewing it only as a topological space.

The next result characterizes exactly which paths in X are geodesics. This can be found as Corollary 11.34 in Part II of [1].

Lemma 2.6. *Let X be as in Theorem 2.1. A path $c : [a, b] \rightarrow X$ parameterized by arclength is a geodesic in X if and only if it can be expressed as a concatenation of non-trivial paths c_1, c_2, \dots, c_n parameterized by arclength, such that*

- (1) *each of the paths c_i is either a hyperbolic geodesic or else its image is contained in one of the horospheres bounding X and in that horosphere it is a Euclidean geodesic;*
- (2) *if c_i is a hyperbolic geodesic, then the image of c_{i+1} is contained in a horosphere and vice versa;*
- (3) *when viewed as a map $[a, b] \rightarrow \mathbb{H}^n$, the path c is C^1 .*

By Lemma 2.5, a truncated hyperbolic space X is a CAT(0) space that does contain $(n-1)$ -flats. By an n -flat, we mean an isometric embedding of \mathbb{E}^n into X . It can be shown that these are in fact, the only flats in X . In order to understand the arrangement of flats in X , we give the following definition due to [8].

Definition 2.7 (Isolated Flats Property). A CAT(0) space X has the *Isolated Flats Property* (IFP) if it contains a family of flats \mathcal{F} with the following two properties.

- (1) There is a constant C so that every flat in X lies in the C -neighborhood of some flat $F \in \mathcal{F}$.
- (2) There is a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any two distinct flats $F_1, F_2 \in \mathcal{F}$ and for any positive number r , the intersection

$$\mathcal{N}_r(F_1) \cap \mathcal{N}_r(F_2)$$

of Hausdorff neighborhoods of F_1 and F_2 has diameter at most $\psi(r)$.

If we consider two maximal flats to be equivalent when their Hausdorff distance is finite, then the family \mathcal{F} in the preceding definition consists of one maximal flat from each equivalence class.

Remark 2.8. It is immediate from the definition of IFP that any two maximal flats are either parallel, or *disjoint at infinity*, meaning that their corresponding boundary spheres are disjoint.

G. Christopher Hruska [7, Proposition 9.1] shows that if X is a truncated hyperbolic space, then X has IFP. We give the argument here for completeness:

Lemma 2.9. *Let X be a subspace of \mathbb{H}^n obtained by removing a collection of disjoint open horoballs. Then X endowed with the induced metric, is a CAT(0) space with IFP.*

Proof: We already know X is CAT(0) by Theorem 2.1. Since X is negatively curved away from the bounding horospheres, it is immediate that these are the only flats in X . We know a uniform neighborhood of any horoball in \mathbb{H}^n is a slightly larger horoball. Then since any two horoballs have a bounded intersection, the flats in X are clearly isolated. \square

Intuitively, X has the IFP if, given any two maximal flats in X which are not parallel, the two flats diverge from each other in all directions. More precisely, we have the following lemma which will be useful in our applications to dimension three.

Lemma 2.10. *Suppose X is a CAT(0) space with IFP and fix a basepoint $x_0 \in X$. For each flat F in X let $\pi_F(x_0)$ be the unique projection of x_0 onto F . Then there exists a constant $C > 0$, independent of F such that the following is true: if γ is a geodesic ray in X from x_0 to a point in $\partial_\infty F$, then the image of γ must intersect the ball of radius C about $\pi_F(x_0)$.*

Remark 2.11. The proof of this lemma follows easily from the structure of the *asymptotic cone* of a CAT(0) space with IFP. We will not need this notion here so we omit the details. See [8] or [5] for details.

3. THE TOPOLOGY OF THE BOUNDARY

For simplicity, we first consider the case that X is obtained by removing only one horoball from \mathbb{H}^n . The main theorem will follow from this result. Recall that $\partial_\infty \mathbb{H}^n$ is homeomorphic to S^{n-1} . When working in the upper half space model, we view this boundary as the union of \mathbb{R}^{n-1} (corresponding to hyperplane $x_n = 0$) and

the point at infinity p_0 corresponding to the positive x_{n+1} axis. The author would like to thank Phil Bowers for recollecting the following elementary fact from hyperbolic geometry.

Lemma 3.1. *Given a point x_0 in \mathbb{H}^n that is not in $\overline{H_c} = H_c \cup S_c$, there exists an $R(x_0) = R > 0$ such that endpoints of the geodesic rays in \mathbb{H}^n that begin at x_0 and end in \mathbb{R}^{n-1} with image disjoint from H_c are contained in a closed (Euclidean) ball of radius R in \mathbb{R}^{n-1} .*

Proof: Suppose γ is such a ray. Then γ must meet \mathbb{R}^{n-1} orthogonally. Either the image of γ does not meet S_c or else it meets S_c in exactly one point (tangentially). Those rays which meet S_c tangentially (along with the choice of x_0) will determine the radius R . Note that R will depend on the choice of x_0 .

We indicate the proof in \mathbb{H}^2 , and the general case follows the same idea. We know S_c is at Euclidean height c and the point x_0 is below the horoball at Euclidean height b , so $0 < b < c$. The extreme line, the one through the point at height b and tangent to S_c , is a semi-circle intersecting the boundary \mathbb{R}^1 orthogonally. Let the Euclidean center of that circle be a units away from the orthogonal projection onto \mathbb{R}^1 of the point x_0 . Thus, $R = a + c$ and by Pythagorean, $a^2 + b^2 = c^2$. Thus, $R = c + \sqrt{c^2 - b^2}$. \square

Remark 3.2. Notice that we can start with any horoball in \mathbb{H}^n and any basepoint outside the corresponding closed horoball in the above lemma and obtain a corresponding disk in $\partial_\infty \mathbb{H}^n$. We simply use $\text{Isom}(\mathbb{H}^n)$ to translate the situation to above situation.

Theorem 3.3. *Suppose $X = \mathbb{H}^n - H_c$. Then X is a CAT(0) space whose visual boundary is homeomorphic to $S^{n-1} \setminus D^{n-1}$.*

Proof: We know X is CAT(0) by Theorem 2.1. To compute the visual boundary $\partial_\infty X$, we describe three different types of geodesic rays in X . Fix a basepoint $x_0 \in X$ so that x_0 is not in S_c . It is clear from Lemma 2.6 that if γ is a geodesic ray in X with $\gamma(0) = x_0$, then γ satisfies one of the following:

- (1) γ is a hyperbolic geodesic, i.e., γ is a geodesic ray in \mathbb{H}^n , whose image is disjoint from H_c . In this case, either the image of γ does not intersect S_c or else it intersects S_c in exactly one point (tangentially).

- (2) There exists points $x, y \in S_c$ so that γ can be expressed as the concatenation of the unique hyperbolic geodesic between x_0 and x , followed by the Euclidean segment between x and y , followed by the hyperbolic segment from y to $\gamma(\infty)$. In this case, the hyperbolic geodesic $[x_0, x]$ and the hyperbolic geodesic ray $[y, \gamma(\infty)]$ meet S_c tangentially and, of course, the hyperbolic geodesic ray $[y, \gamma(\infty)]$ meets the boundary \mathbb{R}^{n-1} orthogonally.
- (3) There exists a point $x \in S_c$ such that γ is the concatenation of the unique hyperbolic geodesic from x_0 to x followed by the Euclidean geodesic ray from x to $\gamma(\infty)$. In this case, the hyperbolic geodesic $[x_0, x]$ meets S_c tangentially. Clearly, here $\gamma(\infty)$ lies in $\partial_\infty S_c$.

Thus, as a set, we can decompose $\partial_\infty X$ into three disjoint pieces corresponding to the three cases above. To see what these sets are, look in the upper half space model. The horoball H_c used here is the horoball centered at the point at infinity at Euclidean height c for some $c > 1$.

For (1), we can view this piece as part of $\partial_\infty \mathbb{H}^n$ since these rays are already rays in \mathbb{H}^n . These points, when considered in $\partial_\infty \mathbb{H}^n$, form a subset of \mathbb{R}^{n-1} . This subset will be homeomorphic to a closed $n - 1$ disk D in \mathbb{R}^{n-1} , by Lemma 3.1.

In (2), the endpoints of these rays are also in \mathbb{R}^{n-1} , but they are not hyperbolic geodesics. However, there is clearly a one-to-one correspondence between the points in this set and the points in $\mathbb{R}^{n-1} \setminus D$. Thus together, these two cases give all the points of $\partial_\infty \mathbb{H}^n$ except p_0 .

In (3), the endpoints lie in ∂S_c . Since S_c is a convex subset of X , we know S_c is a CAT(0) subspace and $\partial_\infty S_c$ embeds in $\partial_\infty X$. Since S_c is isometric to \mathbb{E}^{n-1} , we obtain that $\partial_\infty S_c$ is homeomorphic to S^{n-2} .

Thus, we can see that $\partial_\infty X$ is the disjoint union of D , $\mathbb{R}^{n-1} \setminus D$ and $\partial_\infty S_c$ (as a point set). Equivalently, we can view this as $(\partial_\infty \mathbb{H}^n \setminus \{p_0\}) \cup \partial_\infty S_c$ which is the same as $\mathbb{R}^{n-1} \cup S^{n-2}$. How does the topology work?

The rays in (2) each have a finite length Euclidean subsegment lying in S_c . To obtain rays whose endpoints are “close to infinity” in \mathbb{R}^{n-1} , we simply use longer and longer finite segments in S_c . The

resulting limit ray is a geodesic ray from (3). The rays in (1) can be seen as the limiting rays from rays in (2) in the other direction. Explicitly, if the finite pieces along S_c are allowed to be arbitrarily small, then the limiting rays will have a finite piece of size zero, i.e., they will meet S_c in exactly one point.

In summary, each point $z \in \partial_\infty S_c$ can be viewed as a limit of rays from (2) and likewise, each point in the boundary of the disk D coming from (1), can be viewed as a limit of rays from (2).

So we now see topologically $\partial_\infty X$ is the union of \mathbb{R}^{n-1} and $\partial_\infty S_c$. This boundary is not obtained as a subset of the sphere boundary of \mathbb{H}^n ; however, $\partial_\infty X$ is obtained from $\partial_\infty \mathbb{H}^n$ as follows:

- (1) Remove p_0 to obtain a copy of \mathbb{R}^{n-1} (a punctured S^{n-1}).
- (2) Compactify this \mathbb{R}^{n-1} with an S^{n-2} coming from $\partial_\infty S_c$ to obtain a closed D^{n-1} .

The resulting topological space is simply a closed disk; however, we wish to view it as a topological sphere minus an open disk which we can do via the above description. The topology is that we remove the point p_0 from \mathbb{H}^n , a topological sphere, which punctures this sphere. Then, we stretch this puncture out to make an open disk which is finally compactified with a S^{n-2} to close it up. \square

To obtain the main theorem from Theorem 3.3, we must remove more than one horoball and compute the boundary of the resulting CAT(0) space.

Theorem 3.4. *Suppose X is obtained by removing the interiors of a disjoint collection of closed horoballs \mathcal{H} from \mathbb{H}^n endowed with the induced length metric. For each horoball H in \mathcal{H} , there is a corresponding open disk D_H in a topological sphere S^{n-1} such that $\partial_\infty X$ is homeomorphic to S^{n-1} minus the disjoint collection of open disks $\{D_H \mid H \in \mathcal{H}\}$. Furthermore, the closures of these disks are disjoint.*

Proof: Choose a basepoint x_0 in X that is not in any of the bounding horospheres. For each $H \in \mathcal{H}$, let S_H denote the bounding horosphere for H and let $p_H \in \partial_\infty \mathbb{H}^n$ be the center of H . Again, we give a description of the geodesic rays γ in X based at x_0 . By Lemma 2.6, a typical geodesic γ in $\partial_\infty X$ has the following form: γ begins with a hyperbolic geodesic segment from x_0 that ends tangent to a horosphere S_{H_1} followed by a Euclidean straight

line segment in S_{H_1} , followed by a hyperbolic geodesic that begins tangent to S_{H_1} and ends tangent to another horosphere S_{H_2} , followed by a Euclidean segment in S_{H_2} , and so on. There are three possible outcomes for γ that are analogous to the three outcomes in Theorem 3.3:

- (1) γ is a hyperbolic geodesic. This means the image of γ as a ray in \mathbb{H}^n avoids the interior of H for all $H \in \mathcal{H}$. If γ intersects S_H for some $H \in \mathcal{H}$, then it does so tangentially.
- (2) γ jumps infinitely often from horosphere to horosphere.
- (3) γ eventually enters a horosphere S_H and never leaves S_H , in which case $\gamma(\infty)$ lies in $\partial_\infty S_H$.

Clearly, the rays from (1) and (2) correspond to points in $\partial_\infty \mathbb{H}^n \setminus \cup \{p_H | H \in \mathcal{H}\}$. The rays in (3) correspond to points in $\partial_\infty S_H$ for some $H \in \mathcal{H}$. As in Theorem 3.3, the rays in (1) and (3) can be viewed as limits of the rays in (2).

We now describe the topology of $\partial_\infty X$:

- (1) For each $H \in \mathcal{H}$, there is a corresponding point p_H in $\partial_\infty \mathbb{H}^n$ which is removed from this topological sphere.
- (2) Each of the points is replaced by an open disk D_H .
- (3) Each of the disks D_H is closed by an S^{n-2} coming from $\partial_\infty S_H$ as in Theorem 3.3.

By construction, the open disks $\{D_H | H \in \mathcal{H}\}$ are disjoint. Indeed, for $H \neq K \in \mathcal{H}$, we have $p_H \neq p_K$ in $\partial_\infty \mathbb{H}^n$. The topological boundary of a disk D_H for $H \in \mathcal{H}$ corresponds to the boundary of a flat in X . By Remark 2.8, no two of these can intersect because X has IFP. \square

4. GROUP THEORY AND SIERPIŃSKI CURVES

As stated in the introduction, the original motivation for the main result was to understand groups, such as the fundamental group of the figure eight knot complement or more generally, the fundamental group of a finite volume (non-compact) hyperbolic 3-manifold.

Begin with a group G that acts properly discontinuously by isometries on \mathbb{H}^3 with finite volume quotient. These groups have been studied in many places in the literature. For our purposes, [9] and [2] contain all the necessary background. The main facts

needed here about these discrete, finite volume groups are as follows:

- (1) The *limit set* of G is all of $\partial_\infty \mathbb{H}^n$ (cf. [9, Lemma 12.1.15]). Recall that the *limit set* of G is the set of all limit points of G . A point $p \in \partial_\infty \mathbb{H}^n$ is a *limit point* of G if there exists $x \in \mathbb{H}^n$ and a sequence of elements $\{g_i\}$ of G such that the sequence $\{g_i \cdot x\}$ converges to p in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$.
- (2) G acts cocompactly (and thus geometrically) on a truncated \mathbb{H}^n X , and thus, G is a CAT(0) group by Theorem 2.1. The truncated space is obtained by removing the interiors of a G -equivariant disjoint collection of closed horoballs centered around the parabolic fixed points. The collection of horoballs is locally finite

We know from Theorem 3.4 that $\partial_\infty X$ is homeomorphic to S^{n-1} minus a disjoint collection of open disks. In this setting, we can actually say a bit more than this.

Theorem 4.1. *Suppose G acts properly discontinuously by isometries on \mathbb{H}^n with finite volume quotient. Let X denote the corresponding truncated space on which G acts geometrically and let $\{D_H \mid H \in \mathcal{H}\}$ be the collection of disks as in the conclusion of Theorem 3.4. Then the following are true:*

- (1) *The union $\cup_{H \in \mathcal{H}} \overline{D_H}$ is dense in the sphere S^{n-1} ;*
- (2) *the collection of closed disks $\{\overline{D_H} \mid h \in \mathcal{H}\}$ in S^{n-1} is disjoint and shrinking, i.e., for any given diameter $D > 0$, there are only finitely many $\overline{D_H}$ with diameter greater than D (use any metric on S^{n-1}).*

Proof: The first part follows easily from the fact that the limit set of G is all of $\partial_\infty \mathbb{H}^n$. The fact that the closed disks $\{\overline{D_H} \mid H \in \mathcal{H}\}$ are disjoint follows from Remark 2.8.

Fix a basepoint x_0 in X which is not contained in any of the closed horoballs. To each disk $\overline{D_H}$, we would like to associate a size to $\overline{D_H}$. To do this, we will measure the size of the topological boundary sphere of $\overline{D_H}$ in $\partial_\infty X$ from the viewpoint of x_0 . Even though the size of a given D_H depends on x_0 , the fact that the collection is *shrinking* will not depend on the choice of basepoint. Recall that the topological boundary of the disk D_H is $\partial_\infty S_H$ for the corresponding horosphere in \mathbb{H}^n , and that S_H is isometric to

an $(n - 1)$ -flat F_H in X . Let π_H denote the unique projection of x_0 onto the flat F_H .

Again, we use the fact that X has IFP. Let $r_H = d(x_0, F_H) = d(x_0, \pi_H)$, and let C be the constant coming from Lemma 2.10. Denote by s_H the unique geodesic ray from x_0 to some point in $\partial_\infty F_H$. Then for any $z \in \partial_\infty F_H$ we have $z \in U(s_H, r_H, C) \cap \partial_\infty X$. This is an open set in $\partial_\infty X$ that contains the entire topological boundary of D_H .

Recall that $U(s_H, r_H, C)$ consists of all rays that travel C -close to s_H for at least r_H units of time. So if we have a number $d > r_H$, then the set $U(s_H, d, C)$ appears smaller than $U(s_H, r_H, C)$ from the viewpoint of x_0 . This smaller set may not contain all of the points in $\partial_\infty F_H$. If $d < r_H$, then we have an even larger open set that does contain all of $\partial_\infty F_H$. Thus, we can view r_H as measuring the size of the sphere $\partial_\infty F_H$ in $\partial_\infty X$.

In particular, if $H, K \in \mathcal{H}$ with $r_H < r_K$, then the set $U(s_H, r_H, C)$ is bigger than the set $U(s_K, r_H, C)$. It takes a bigger open set to contain $\partial_\infty F_H$ than it does for $\partial_\infty F_K$.

Since the collection of horoballs in \mathbb{H}^n is locally finite (see Remark 2.3), we also have that the collection of flats in X is locally finite. Thus, given a number $D > 0$, there are only finitely many $H \in \mathcal{H}$ with $r_H < D$. Thus, there are only finitely many disks D_H of size bigger than D . \square

In dimension three, we can identify the boundary using the main results of Whyburn [10]. Recall that one can construct a topological space called the *Sierpiński carpet* using a Cantor-like construction. Specifically, start with the unit square in the plane, subdivide it into nine equal subsquares, remove the middle open square, and then repeat this inductively on the remaining squares. This space can be characterized as a compact, 1-dimensional, planar, connected, locally connected space with no local cut points [10]. In the same article, another characterization is given in terms of removing a collection of disks from S^2 . The Sierpiński carpet is obtained from S^2 minus the interiors of a disjoint collection of closed disks whose union is dense in S^2 and whose diameters are shrinking as in Lemma 4.1. Moreover, any Sierpiński carpet embedded in S^2 is obtained this way. Thus, the following is a direct corollary of Whyburn's characterization [10] and Theorem 4.1

Corollary 4.2. *Suppose G acts properly discontinuously by isometries on \mathbb{H}^3 with finite volume quotient. Then we know G acts geometrically on an appropriately truncated \mathbb{H}^3 ; call this space X . Then X is a CAT(0) space with $\partial_\infty X$ homeomorphic to the Sierpiński carpet.*

In higher dimensions, we still have the results of Theorem 4.1. Bestvina pointed out to me that, in [3], Cannon generalizes the results of Whyburn [10] to all $n \neq 4$. Using the work there, we have the following corollary in higher dimensions.

Corollary 4.3. *Suppose G acts properly discontinuously by isometries on \mathbb{H}^n for $n \neq 5$ with finite volume quotient. Then G acts geometrically on an appropriately truncated \mathbb{H}^n ; call this space X . Then X is a CAT(0) space with $\partial_\infty X$ homeomorphic to the Sierpiński curve of dimension $(n - 2)$.*

5. FINAL REMARKS

Concerning these spaces and groups, most of the results can be found in [7] and [8] where the theory of CAT(0) spaces with IFP is developed more precisely.

In the general setting of groups acting geometrically on CAT(0) spaces, one can ask whether such a group uniquely determines a topological space at infinity. More precisely, if a group G acts geometrically on two CAT(0) spaces X and Y , must their visual boundaries be homeomorphic? The answer to this question in general, is no. In [4], the authors give an example of a group G acting geometrically on two CAT(0) spaces that have non-homeomorphic visual boundaries. The following theorem of Hruska [7] shows that this does not happen if G is one of the groups we are dealing with in this section. More recently, this theorem has been generalized to include all groups acting on CAT(0) spaces with isolated flats in [8].

Theorem 5.1 ([7]). *Suppose G acts properly discontinuously by isometries on \mathbb{H}^n with finite quotient. Then G acts geometrically on a CAT(0) space X obtained by truncating \mathbb{H}^n . Now suppose G acts geometrically on any other CAT(0) space Y . Then $\partial_\infty X$ is homeomorphic to $\partial_\infty Y$. Thus, we can say G has unique boundary.*

As a final remark, we briefly discuss the Tits boundary for these spaces without giving formal definitions since, again, we will merely be quoting a result from [8]. For formal definitions, see Part II, section 9 of [1].

As mentioned in the introduction, the visual topology on ∂X is not always the most useful topology. For instance, it is the same for both \mathbb{E}^n and \mathbb{H}^n . The Tits topology is defined to be the length metric associated to the *angle metric*. The angle metric on ∂X is defined as one would expect: for $p, q \in \partial X$, we have $d(p, q)$ is the angle between geodesic rays for p and q . Of course, one has to be careful with basepoints. It certainly makes sense to talk about the angle between geodesics p and q beginning at a point x_0 using comparison triangles (see [1, Definition 1.12, p. 9]).

To define $\angle(p, q)$, one must take the supremum of these angles over all possible basepoints in X . This gives the angle metric on ∂X . For $X = \mathbb{E}^2$, the angle metric is the arclength metric on $\partial X = S^1$ (viewed as the unit sphere in \mathbb{E}^2). In $X = \mathbb{H}^2$, the angle metric is discrete since any two points $p, q \in \partial X$ can be joined by a geodesic line in X (i.e., pick a basepoint on that line to realize the supremum angle of π .)

In [8], the authors prove that if X is a CAT(0) space with the IFP, then the Tits boundary of X is a disjoint collection of points and spheres. Since our truncated spaces are spaces with this property, they also have this Tits structure.

REFERENCES

- [1] Martin R. Bridson and André Haefliger, *Metric Spaces of Non-positive Curvature*, Grundlehren der mathematischen Wissenschaften, 319. Berlin: Springer-Verlag, 1999.
- [2] B. H. Bowditch, “Boundaries of geometrically finite groups,” *Math Z.* **230** (1999), no. 3, 509–527.
- [3] J. W. Cannon, “A positional characterization of the $(n - 1)$ -dimensional Sierpiński curve in S^n for $n \neq 4$,” *Fund. Math.* **79** (1973), no. 2, 107–112.
- [4] Christopher B. Croke and Bruce Kleiner, “Spaces with nonpositive curvature and their ideal boundaries,” *Topology* **39** (2000), no. 3, 549–556.
- [5] Cornelia Druțu and Mark Sapir, “Tree-graded spaces and asymptotic cones of groups,” *Topology* **44** (2005), no. 5, 959–1058.
- [6] G. Christopher Hruska, *Nonpositively Curved Spaces with Isolated Flats*. Ph.D. dissertation, Cornell University, 2002.

- [7] ———, “Geometric invariants of CAT(0) spaces with isolated flats.” To appear in *Topology*.
- [8] G. Christopher Hruska and Bruce Kleiner, “Hadamard spaces with isolated flats,” *Geom. Topol.* **9** (2005), no. 9, 1501–1538.
- [9] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds*. Graduate Texts in Mathematics, 149. New York: Springer-Verlag, 1994.
- [10] G. T. Whyburn, “Topological characterization of the Sierpiński curve,” *Fund. Math.* **45** (1958), 320–324.

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