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DENSE ARC COMPONENTS IN WEAKENED TOPOLOGICAL GROUPS

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Abstract. We will show how to construct metrizable group topologies on the subgroups and quotient groups of $\mathbb{R}^\omega$ that are weaker than the standard topologies. Even though these groups can be very complicated, we will show, by constructing a local isometry between many of the groups, that they are locally the "same." We give special attention to "unusual" topological groups in which the arc component of the identity is dense.

1. Introduction

Understanding the structure of various classes of topological groups is a major goal of those interested in the field. Locally compact, compactly generated abelian groups have, for example, a nice and well understood structure. If G is a locally compact, compactly generated abelian group, then it is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$ where a and b are nonnegative integers and F is a compact abelian group [2, Theorem 9.8]. We know, however, that great diversity and complexity exist even within classes of topological groups with "nice" properties. In particular, little is known about the structure of an arbitrary, metrizable topological group,

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and so we must place additional constraints on a class of metrizable groups to understand what it “looks like.” That is one focus of this paper. In particular, we will construct classes of metrizable topological groups that can exhibit many unusual, and sometimes pathologic, characteristics. All of these topological groups will have topologies that are weaker than their standard topology, and they will be either subgroups or quotient groups of $\mathbb{R}^\omega$. We will obtain the weakened groups by requiring certain types of sequences to converge at a controlled “rate” to the identity of the group. Even though these topological groups are extremely complicated, we will show that understanding the local structure of one allows us to understand the local structure of many. This will be accomplished by constructing a local isometry between the topological groups. We will then extend this local isometry to the completions of the groups. Knowing that a local isometry exists gives us a powerful tool to use to understand the local structure (dimensionality, local connectedness, etc.) of the topological groups. If we can somehow understand the local structure of one of the groups, then we can describe the local structure of many. This idea is explored more fully, though in a restricted case, in [8].

The construction of the weakened, metrizable topological groups in this paper is related to, but goes beyond, the construction given in [8], where we obtain similar results on the subgroups and quotient groups of $\mathbb{R}^m$ where m is a positive integer. In fact, the topological groups arising in [8] and [10] can be viewed as subgroups or quotients of the topological groups described in this paper. Even though they are interesting on their own, it is shown in [9] and [10] that these types of topological groups also arise in the study of weakened analytic groups and immersions of Lie groups.

As we know, the arc component of the identity plays a special role in certain types of topological groups. In particular, each nonempty open subset in a compact connected group contains a point on a one-parameter subgroup or, equivalently, contains a point in the arc component of the identity [3, Corollary 8.5]. The groups constructed in this paper are, in general, not compact (or complete for that matter). Thus, we do not have as much structure as the compact connected case, but it is still interesting to see how the arc component of the identity “sits” in these groups. That brings us to the other focus of the paper, which is explored beginning in section

5. There we illustrate how our construction can be used to obtain topological groups with an interesting property. In particular, we force the arc component of the identity to be dense. This gives interesting examples of Hausdorff, topological group structures on, for example, $\mathbb{R}^\omega \times \mathbb{Z}$ (and its completion) and $\mathbb{R}^\omega \times (\mathbb{Z}_2)^\omega$ (and its completion).

2. Notation and terminology

We will use the notation presented in this section throughout the paper. Other notation will be introduced as needed.

We denote the topological space G with topology $\mathcal{T}$ as $(G, \mathcal{T})$. If d is a metric on a topological space X, or a norm that induces a metric, we will blur the distinction between d and the topology it induces and denote the space with the metric topology by (X, d). The closure of (X, d) will be denoted $\overline{(X, d)}$. If $(G, \mathcal{T})$ is a topological space and A is a subset of G, then the subspace topology A inherits from $\mathcal{T}$ will be denoted $\mathcal{T}_A$, and the quotient topology induced on G/A by $\mathcal{T}$ will be denoted by $\mathcal{T}/A$.

All topological groups in this paper are abelian and we will denote the identity element by 0. Since the groups are abelian, a completion in the category of topological groups exists [1, Theorem 2, p. 249], and we will denote the completion of the topological group $(G, \mathcal{T})$ by $\mathcal{C}(G, \mathcal{T})$.

$\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$ will denote, respectively, the natural numbers, the integers, and the real numbers. The standard topology on $\mathbb{R}^j$ will be denoted τ^j and if $j = 1$ as τ. If $x \in \mathbb{R}^m$, for $m \in \mathbb{N}$, then $\|x\|$ will be the standard norm of x. If $m = 1$, $\|x\|$ will be abbreviated $|x|$. If $x \in \mathbb{R}$, then $\lfloor x \rceil$ will denote the greatest integer less than or equal to $x - 1$.

All sequences in a topological space will be indexed on $\mathbb{N}$. Let $\{x_j\}$ and $\{y_j\}$ be sequences in $\mathbb{R}^m$ and $\mathbb{R}^\omega$, respectively. We will often have cause to construct a sequence in $\mathbb{R}^\omega$ from $\{x_j\}$ and $\{y_j\}$. To see how this is accomplished, let $x_j = (x_{j1}, x_{j2}, \ldots, x_{jm})$ and $y_j = (y_{j1}, y_{j2}, y_{j3}, \ldots)$. Then

$$\{(x_{j1}, x_{j2}, \ldots, x_{jm}, y_{j1}, y_{j2}, y_{j3}, \ldots) : j \in \mathbb{N}\}$$

will be a sequence in $\mathbb{R}^\omega$. We will denote this sequence by $\{(x_j, y_j)\}$.

3. The Weakening on $\mathbb{R}^\omega$

In this section, we show how to obtain weakened, metrizable group topologies on $\mathbb{R}^\omega$ and its quotients. Examples are given to illustrate the process. The main result is Theorem 3.4. In the next section, we explore the structure of these groups. The theorems and proof in this section resemble those given in [8], but in a more general setting. We include certain proofs to illustrate the important differences in changing from a finite product of groups (addressed in [8]) to an infinite product. If there is no important change (as in 4.1), we will outline the proof and refer the reader to the appropriate result in [8].

We need a standard theorem before proceeding.

Theorem 3.1 ([4, Theorem 9.5]). *Let $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in \mathbb{R}^\omega$, and define $D : \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}$ by*

$$D(x, y) = \sup\left\{\frac{\min\{|x_i - y_i|, 1\}}{i} : i \in \mathbb{N}\right\}.$$

D is a metric that induces the product topology on $\mathbb{R}^\omega$.

The next lemma is a technical lemma which says that certain vectors in $\mathbb{R}^n$, written as a linear combination of the v_j's, must have a large norm. From now on, we assume that all sums are finite unless otherwise specified.

Lemma 3.2 ([10, p. 103]). *Let $\{p_j\}$ be a sequence of positive real numbers which converges to zero in τ, and let $\{v_j\}$ be a sequence of nonzero elements of $\mathbb{R}^n$ such that $\{\|v_j\|\}$ is nondecreasing and the sequence*

$$\{p_{j+1}\|v_{j+1}\|/\|v_j\|\}$$

has a positive lower bound. For every $k > 0$, there is a $d > 0$ such that, if $\sum |c_j|p_j < d$ and $c_j \in \mathbb{Z}$, then either $c_j = 0$ for all j or $\|\sum c_j v_j\| > k$.

Definition 3.3. If $\{p_j\}$ and $\{v_j\}$ are as in the previous lemma, then we will say that $\{p_j\}$ and $\{v_j\}$ form a sequential norming pair (SNP) and denote this norming pair as $(\{v_j\}, \{p_j\})$.

We are now able to state the result that allows us to use an (SNP) for $\mathbb{R}^n$ to weaken the topology on $\mathbb{R}^\omega$.

Theorem 3.4. *Let $(\{v_j\}, \{p_j\})$ be an (SNP) for $\mathbb{R}^n$ and let $\{x_j\}$ be any sequence in $\mathbb{R}^\omega$. Let $\{w_j\}$ be the sequence in $\mathbb{R}^\omega$ defined by $\{w_j\} = \{(v_j, x_j)\}$. Then, for $y \in \mathbb{R}^\omega$ define*

$$\mu(y) = \inf\left\{\sum |c_j|p_j + D\left(\sum c_j w_j, y\right) : c_j \in \mathbb{Z}\right\}.$$

In this situation, μ is a group norm on $\mathbb{R}^\omega$ such that $\mu(y) \le D(y, 0)$ and $\mu(w_j) \le p_j$. μ gives rise to a metrizable group topology on $\mathbb{R}^\omega$, weaker than the standard topology, in which $w_j \to 0$.

Note that a group norm is similar to a norm on a linear space, except that it does not necessarily respect scalar multiplication. For completeness, the formal definition is given below.

Definition 3.5. A group norm on a group G is a function $\nu : G \to \mathbb{R}$ satisfying, for all $x, y \in G$,

(i) $\nu(x) = \nu(x^{-1})$;
(ii) $\nu(x) \ge 0$;
(iii) $\nu(x \cdot y) \le \nu(x) + \nu(y)$;
(iv) $\nu(x) = 0$ if and only if $x = 0$.

A group norm induces a metrizable group topology $\mathcal{T}$ on G by taking as a basis $\{x \cdot U_\epsilon \,|\, x \in G, \epsilon \in \mathbb{R}, \epsilon > 0\}$ where $U_\epsilon = \{x \in G \,|\, \nu(x) < \epsilon\}$.

Proof: Everything is clear except the fact that μ gives rise to a true metric, or, equivalently, that $\mu(y) = 0 \Leftrightarrow y = 0$.

For notational convenience, let $w_j = (v_{j1}, \ldots, v_{jn}, x_{j(n+1)}, x_{j(n+2)}, \ldots)$, and denote elements $y \in \mathbb{R}^\omega$ as $y = (y_1, y_2, \ldots)$. Now suppose that $\mu(y) = 0$. Let $k = 1 + \|(y_1, y_2, \ldots, y_n)\|$. Since $(\{v_j\}, \{p_j\})$ forms an (SNP), we can apply 3.2 to get a d corresponding to k such that if $\sum |c_j|p_j < d$, then $c_j = 0$ for all j, or $\|\sum c_j v_j\| > k$. Choose ϵ such that $0 < \epsilon \le \min(d, 1)$. Since $\mu(y) = 0$, there are c_j such that $\sum |c_j|p_j + D(\sum c_j w_j, y) < \frac{\epsilon}{n^2}$, and clearly this implies that $D(\sum c_j w_j, y) < \frac{\epsilon}{n^2}$. Now by the definition of D we see that

$$\left(\frac{|y_i - \sum c_j v_{ji}|}{i}\right)^2 < \frac{\epsilon^2}{n^4}$$

for $i = 1, \ldots, n$. Thus,

$$\begin{aligned}\frac{\epsilon^2}{n^3} > \sum_{i=1}^{n}\left(\frac{|y_i - \sum c_j v_{ji}|}{i}\right)^2 &\geq \sum_{i=1}^{n}\left(\frac{|y_i - \sum c_j v_{ji}|}{n}\right)^2 \\ &= \sum_{i=1}^{n}\frac{(|y_i - \sum c_j v_{ji}|)^2}{n^2} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}|y_i - \sum c_j v_{ji}|^2.\end{aligned}$$

From this we can conclude that $\sum_{i=1}^{n}|y_i - \sum c_j v_{ji}|^2 < \epsilon^2$, and, of course,

$$\sum_{i=1}^{n}|y_i - \sum c_j v_{ji}|^2 < \epsilon^2 \Rightarrow \sqrt{\sum_{i=1}^{n}|y_i - \sum c_j v_{ji}|^2} < \epsilon.$$

Hence, $\|(y_1, \ldots, y_n) - \sum c_j v_j\| < \epsilon$ so that

$$\|\sum c_j v_j\| < \|(y_1, \ldots, y_n)\| + \epsilon \leq \|(y_1, \ldots, y_n)\| + 1 = k.$$

Since $\sum |c_j| p_j < d$, we are in the situation of 3.2 and can conclude that $c_j = 0$ for all j. Therefore, $D(y, 0) < \epsilon$, and since this argument holds for any ϵ, and D is a metric, $y = 0$.

It is clear from the definition of μ that $\mu(0) = 0$. Thus, $\mu(y) = 0 \Leftrightarrow y = 0$. □

Clearly, this proposition also gives a way to weaken the topology on any subgroup of $\mathbb{R}^\omega$. Using the next proposition, we can weaken the topology on the quotients of $\mathbb{R}^\omega$. These group topologies will also be metrizable and Hausdorff.

Definition 3.6. Suppose that $(\{v_j\}, \{p_j\})$ is an (SNP) on $\mathbb{R}^n$. If the topology on $\mathbb{R}^\omega$ is weakened as described in 3.4, then $(\{w_j\}, \{p_j\})$ will be called an extended-norming pair and denoted (ENP). $(\{w_j\}, \{p_j\}, \mu)$ will be called an extended-norming triple and denoted (ENT).

Proposition 3.7. *Let the hypotheses be as in 3.4. Then* $\{0\}^n \times \mathbb{R}^\omega$ *inherits the standard topology from* $(\mathbb{R}^\omega, \mu)$*, and* $\{0\}^n \times \mathbb{R}^\omega$ *is* μ*-closed.*

Proof: Let $(\{0\}^n, z) \in \{0\}^n \times \mathbb{R}^\omega$. We will show that the identity map from

$$(\{0\}^n \times \mathbb{R}^\omega, \mu_{\{0\}^n \times \mathbb{R}^\omega}) \to (\{0\}^n \times \mathbb{R}^\omega, {\tau^\omega}_{\{0\}^n \times \mathbb{R}^\omega})$$

is uniformly continuous. To do this we need to show that for any ϵ such that $0 < \epsilon < 1$ there is a $\delta > 0$ such that

$$\mu((\{0\}^n, z)) < \delta \Rightarrow D((\{0\}^n, z), (\{0\}^n, 0)) = D(0, z) < \epsilon.$$

Apply 3.2 to the norming pair $(\{v_j\}, \{p_j\})$ to get a d corresponding to ϵ, and then choose $\delta = \min\left(d, \frac{\epsilon}{n^2}\right)$. If $\mu((\{0\}^n, z)) < \delta$ then there exists c_j such that $D\left(\sum c_j w_j, z\right) < \delta < \frac{\epsilon}{n^2}$. We can then argue as in the proof of 3.4 that $\|\sum c_j v_j\| < \epsilon$ and that $\sum |c_j| p_j < d$. So, once again, 3.2 implies that all $c_j = 0$ so that $D\left(\sum c_j w_j, z\right) = D\left(0, z\right) < \epsilon$. With this we have that the identity is uniformly continuous. To see that $\{0\}^n \times \mathbb{R}^\omega$ is μ-closed, we note that $\{0\}^n \times \mathbb{R}^\omega$ is complete as a subspace of $(\mathbb{R}^\omega, \tau^\omega)$ and this implies, along with the uniform continuity of the identity map, that $\{0\}^n \times \mathbb{R}^\omega$ is complete as a subspace of $(\mathbb{R}^\omega, \mu)$. Since every complete subspace of a Hausdorff space is closed [1, p. 186], we have the desired statement. □

One use of this proposition is to identify μ-closed subgroups of $(\mathbb{R}^\omega, \mu)$. If A is such a group, then $\mathbb{R}^\omega / A$ is a Hausdorff topological group with topology weaker than the standard topology on $\mathbb{R}^\omega / A$. This is a good place to consider an example, illustrating the use of 3.7 to get a weakened group topology on a quotient group of $\mathbb{R}^\omega$.

Example 3.8. Let $v_j = 3^{j!} + 1$ and $p_j = \frac{1}{j}$. Then $(\{v_j\}, \{p_j\})$ forms an (SNP) on $\mathbb{R}$. Let $\{x_j\} = \{(2j+1, 2j+3, 2j+5, \ldots)\}$ be a sequence in $\mathbb{R}^\omega$ and define $w_j = (v_j, x_j) = (3^{j!} + 1, 2j+1, 2j+3, 2j+5, \ldots)$. Then by 3.4, $(\{w_j\}, \{p_j\}, \mu)$ is an (ENT) on $\mathbb{R}^\omega$. In particular, the sequence $\{w_j\}$ μ-converges to zero and $\mu(w_j) \leq 1/j$. Since the sequence is actually a sequence in $\mathbb{R} \times \mathbb{Z}^\omega$, we will restrict our attention to this subgroup of $\mathbb{R}^\omega$. We know that $\{0\} \times (2\mathbb{Z})^\omega$ is closed in $(\mathbb{R} \times \mathbb{Z}^\omega, {\tau^\omega}_{\mathbb{R} \times \mathbb{Z}^\omega})$. Hence, 3.7 implies that it is closed in $(\mathbb{R} \times \mathbb{Z}^\omega, \mu_{\mathbb{R} \times \mathbb{Z}^\omega})$. Thus,

$$((\mathbb{R} \times \mathbb{Z}^\omega)/(\{0\} \times (2\mathbb{Z})^\omega), \mu/(\{0\} \times (2\mathbb{Z})^\omega))$$

is a Hausdorff topological group with a countable basis at zero and thus is metrizable. (See, for example, [2, section 8].) Since $(\mathbb{R} \times \mathbb{Z}^\omega)/(\{0\} \times (2\mathbb{Z})^\omega) \cong \mathbb{R} \times {\mathbb{Z}_2}^\omega$ as a group, we have a weakened group topology on $\mathbb{R} \times \mathbb{Z}_2{}^{\omega}$. In this topology, the sequence $\{q_j\} = \{(3^{j!} + 1, 1, 1, 1, \ldots)\}$ will converge to zero.

By choosing our sequences carefully, we can use the procedure given in the above example to arrive at weakened group topologies on $\mathbb{Z}$. For example, weaken the topology on $\mathbb{Z}^{\omega}$ using the (ENP) $(\{n!, 0, 0, 0, \ldots\}, \{1/n\})$. This topology is just the product of a weakened topology on $\mathbb{Z}$ and the standard topology on $\mathbb{Z}^{\omega}$. In the weakened topology on $\mathbb{Z}$, if d is its metric, $d(n!, 0) \leq 1/n$; i.e., $\{n!\}$ converges to zero. In particular, all of the interesting action is taking place on the first factor. Weakened topologies of this sort on $\mathbb{Z}$, and their completions, were studied in detail by J. W. Nienhuys in [6] and [7]. We can similarly obtain the weakened topologies on $\mathbb{R}^n$ discussed in [8]. Thus, our procedure is more general than both of these situations.

It should be noted that the weakened topologies we are obtaining on the subgroups and quotient groups of $\mathbb{R}^{\omega}$ are typically *nonproduct topologies.* In particular, projections onto every factor $\{0\}^n \times \mathbb{R} \times \{0\}^{\omega}$, for n a positive integer, of $\mathbb{R}^{\omega}$ will not be continuous. In fact, we can guarantee the topologies are nonproduct, and that the projection onto any factor is not continuous, by picking the sequence $\{x_j\}$ from 3.4 so that its coordinate sequences do not converge to zero in $(\mathbb{R}, \tau)$. This follows from 3.7, by noting that $\{0\}^n \times \mathbb{R} \times \{0\}^{\omega}$ has its standard topology as a subspace of the weakened topology. The sequence $\{x_j\}$ given in the previous example will induce a nonproduct topology since the coordinate sequences are $\{2j + 1\}, \ \{2j + 3\}, \ldots$.

4. Local Structure of Weakened Topologies

We saw in the previous section how to obtain weakened metrizable group topologies on $\mathbb{R}^{\omega}$ and its quotients. Since this procedure used sequences to weaken the topology, a natural question is, "How do the topological groups relate if different sequences are chosen?" We might guess that no "relation" is possible since there are infinitely many different sequences that we could choose. That is not, however, the case if we place small restrictions on the sequences chosen. The next theorem gives the specifics of this notion. Recall that for $x \in \mathbb{R}$, $\lfloor x \rceil$ represents the greatest integer less than or equal to $x - 1$.

Theorem 4.1. *Let* $(\{v_i\}, \{p_i\}, \nu)$ *and* $(\{n_i\}, \{p_i\}, \mu)$ *be (SNT)s on* $\mathbb{R}^m$ *such that*

$$p_i \left\lfloor \frac{\|v_{i+1}\|}{\|v_i\|} \right\rceil, p_i \left\lfloor \frac{\|n_{i+1}\|}{\|n_i\|} \right\rceil \geq 1 \tag{4.1}$$

for all but finitely many i. *Let* $(\{v_i{}'\}, \{p_i\}, \nu')$ *and* $(\{n_i{}'\}, \{p_i\}, \mu')$ *be two (ENT)s corresponding to* $(\{v_i\}, \{p_i\}, \nu)$ *and* $(\{n_i\}, \{p_i\}, \mu)$, *respectively. Then*

(i) $(\mathbb{R}^\omega, \nu')$ *is locally isometric to* $(\mathbb{R}^\omega, \mu')$.
(ii) $\mathcal{C}(\mathbb{R}^\omega, \nu')$ *is locally isometric to* $\mathcal{C}(\mathbb{R}^\omega, \mu')$.

Notice that the norming triples $(\{v_i\}, \{p_i\}, \nu)$ and $(\{n_i\}, \{p_i\}, \mu)$ have the same sequence $\{p_i\}$ but that $\{v_i\}$ is not necessarily equal to $\{n_i\}$. Since the sequence $\{p_i\}$ controls the rate of convergence, it somehow restricts the "types" of topologies possible. We should note that Nienhuys considered issues similar to the above theorem in [7] and [6]. His sequences, however, consisted of increasing integers, and for his strongest results, consecutive terms of the sequences would divide each other. Our sequences can be much more complicated, but, as might be expected, our results are of a different flavor.

Theorem 4.1 is similar to [8, Theorem 8], which is the main result of that paper. The proof of [8, Theorem 8] requires many pages of work. However, with the results given in §3, the reader can supply the details of the proof of 4.1 using [8, Theorem 8] as a guide. Thus, we will not exhibit the proof of 4.1 in detail, choosing instead to give an outline of the argument.

We first focus on the subgroups of $\mathbb{R}^\omega$ generated by $\{v_i{}'\}$ and $\{n_i{}'\}$ and denoted $< v_i{}' >$ and $< n_i{}' >$, respectively. If we choose balls small enough, say B_r and B_s, in ν' and μ', respectively, then the function $f : B_r \to B_s$ defined by

$$f\left(\sum c_i v_i{}'\right) = \sum c_i n_i{}'$$

is a local isometry. Note that choosing the balls "small enough" is imperative because f is not, in general, well defined outside of small balls. When, however, r and s are sufficiently small, $x \in B_r$ can be expressed uniquely as $\sum c_i v_i{}'$ so that $\sum |c_i| p_i < r$. It is, in fact, quite complicated to show this uniqueness. Again the reader is referred to [8]. The next step is to determine what happens to an element in $\mathbb{R}^\omega$ that is not in the subgroup $< v_i' >$. In this case, we once again reduce the radius of our balls. With small enough balls, we show that the chosen element in $\mathbb{R}^\omega$ is close (in the standard topology) to an element $x \in < v_j' >$. This allows us to define another function g, using f and x, which is a local isometry as claimed. The proofs for the completions follow easily at this stage.

Theorem 4.1 gives a powerful tool to use in the understanding of the local structure of the topological groups constructed in 3.4. We illustrate this with an example.

Example 4.2. Let $v_j = 3^{j!} + 1$ and $p_j = 1/j$. Then $(\{v_j\}, \{p_j\})$ forms an (SNP) on $\mathbb{R}$. We now take any two sequences in $\mathbb{R}^\omega$. For the sake of this example we will choose sequences of a very different nature. Let $\{r_j\} = \{(e^j, j+1, e^{j+2}, j+3, e^{j+4}, j+5, \ldots\}$ and $\{s_j\} = \{(1/e^j, 1/(j+1), 1/e^{(j+2)}, 1/(j+3), \ldots)\}$. Let $\{w_j\} = \{(v_j, r_j)\}$ and $\{x_j\} = \{(v_j, s_j)\}$. We can now use 3.4 to obtain two metrizable topological groups on $\mathbb{R}^\omega$, denoted $(\mathbb{R}^\omega, \mathcal{W})$ and $(\mathbb{R}^\omega, \mathcal{X})$, respectively. $\{w_j\}$ will converge to zero in $(\mathbb{R}^\omega, \mathcal{W})$, and $\{x_j\}$ will converge to zero in $(\mathbb{R}^\omega, \mathcal{X})$. Based upon the very different natures of the sequences $\{w_j\}$ and $\{x_j\}$, which determine the weakened topologies, we might assume that the topologies are very different. Theorem 4.1 says, however, that these topological groups are locally isometric.

In the above example, we used one norming pair for $\mathbb{R}$ to obtain our two weakened topologies on $\mathbb{R}^\omega$. This was done primarily to introduce the reader to the idea, but it is not necessary. Consider the example below.

Example 4.3. Consider the two norming pairs for $\mathbb{R}$ given by $(\{3^{j!} + 1\}, \{1/j\})$ and $(\{j! + 1\}, \{1/j\})$. We can now obtain two extended norming pairs for $\mathbb{R}^\omega$ by using the sequences $\{r_j\}$ and $\{s_j\}$ from above, respectively. Here $w_j = (3^{j!}+1, r_j)$ and $x_j = (j!+1, s_j)$. Theorem 4.1 says that the two topological groups determined by these extended norming pairs are locally isometric.

The reader is cautioned about drawing conclusions concerning the local structure of two quotient groups obtained from two locally isometric groups. We will show that the quotient groups of two locally isometric groups can have different topologies.

5. A dense arc component of the identity

In this section, we show how a judicious choice of the sequences in the extended norming pair $(\{w_j\}, \{p_j\})$ will guarantee that the arc component of the identity is dense in the weakened topological group (and its completion). This construction gives connected, but not necessarily arcwise connected, group topologies on the subgroups and quotient groups of $\mathbb{R}^\omega$. In the remainder, we will focus our examples on $\mathbb{R} \times \mathbb{Z}^\omega$ and $\mathbb{R} \times (\mathbb{Z}_2)^\omega$. These examples are easily generalizable to $\mathbb{R}^n \times \mathbb{Z}^\omega$ and $\mathbb{R}^n \times (\mathbb{Z}_2)^\omega$, but to simplify notation, we consider the case where $n = 1$. We also show that there are two topological group structures for $\mathbb{R} \times (\mathbb{Z}_2)^\omega$, obtained from locally isometric group structures on $\mathbb{R} \times \mathbb{Z}^\omega$, that are not homeomorphic or locally homeomorphic under the identity mapping.

Suppose that $(\{v_j\}, \{p_j\})$ is an (SNP) for $\mathbb{R}^n$. Now weaken the topology on the subgroups and quotient groups of $\mathbb{R}^\omega$ according to the methods given in §3. We thus can view the weakened group as, algebraically, $\mathbb{R}^n \times L$ where L is a subgroup, quotient group, or quotient group of a subgroup of $\mathbb{R}^\omega$ (see Example 3.8).

Proposition 5.1. *Suppose that we have weakened the topology $\mathbb{R}^n \times L$ according the methods in 3.4 and 3.7. Let $\mathcal{U}$ be this weakened topology, $\Gamma = \{U \in \mathcal{U} : 0 \in U\}$, and $p : \mathbb{R}^n \times L \to L$ be the projection mapping*[1]. *Define*

$$A = \left\{ y \in L : y \in \bigcap_{U \in \Gamma} p(U) \right\}.$$

Then A is a subgroup of L, and if $\overline{A}$ is the closure of A in the standard topology on L, then $\mathbb{R}^n \times \overline{A}$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n \times \{0\}^\omega$.

Proof: We first show that A is a subgroup of L. $A \neq \emptyset$ since it contains the identity element. Now let $x, y \in A$ and let $U \in \mathcal{U}$ be a neighborhood of the identity in $\mathbb{R}^n \times L$. Choose a neighborhood V of the identity such that $V + V \subset U$ and $V = -V$. Since V is a neighborhood of the identity and $x, y \in A$, $x, y \in p(V)$. Thus, there exist $r_1, r_2 \in \mathbb{R}^n$ such that (r_1, x) and (r_2, y) are elements of V. Now $(r_1 + r_2, x + y) = (r_1, x) + (r_2, y) \in V + V \subseteq U$. Hence,

[1]Recall that the projection is in general not continuous.

$x + y \in p(V + V) \subseteq p(U)$. Now since $x \in p(V)$ and $V = -V$, $-x \in p(V) \subseteq p(U)$. Since U was arbitrarily chosen, A is a subgroup of L.

Now let $(r, a) \in \mathbb{R}^n \times A$ and let U be a $\mathcal{U}$-neighborhood of (r, a). We then have that $-(r, a) + U = (-r, -a) + U$ is a neighborhood of the identity. Since A is a group, $-a \in A$, and thus, $-a \in p((-r, -a) + U)$. Hence, there exists $r' \in \mathbb{R}^n$ such that $(r', -a) \in (-r, -a) + U$. This gives that $(r' + r, 0) \in U$ and, in particular, that U intersects $\mathbb{R}^n \times \{0\}^\omega$. We have shown that $\mathbb{R}^n \times A$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n \times \{0\}^\omega$. Since $\mathcal{U}$ is weaker than the standard topology on $\mathbb{R}^n \times L$, we get that if $\overline{A}$ is the closure of A in the standard topology on L, then $\mathbb{R}^n \times \overline{A}$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n \times \{0\}^\omega$. □

Recall that we intend to make the arc component of the identity of the group dense in the group. This proposition is a first step and says that if the closure of A is "large," then the closure of $\mathbb{R}^n \times \{0\}^\omega$, which is contained in the arc component of the identity of $(\mathbb{R}^n \times L, \mathcal{U})$, must also be "large."

From now on, L will be a subgroup of $\mathbb{R}^w$ unless otherwise specified. Let us examine a procedure that will guarantee that the closure of A is dense in L. This procedure will be dependent upon the fact that any subgroup of $(\mathbb{R}^\omega, \tau^\omega)$ contains a countable dense subset. Let $(\{v_j\}, \{p_j\})$ be an (SNP) on $\mathbb{R}^n$. Since L is a subgroup of $\mathbb{R}^\omega$, it contains a countable, dense (in the usual topology) subset, which we will denote $L^\diamond$. Since $L^\diamond$ is countable, we can list its elements. Without loss of generality, let

$$L^\diamond = \{l_1, l_2, l_3, \ldots\}.$$

Before proceeding with the construction, we have a need to consider sequences exhibiting a special property.

Definition 5.2. The sequence $\{x_j\}$ is cascading if for any $N \in \mathbb{N}$, $\{x_j : j > N\}$ contains every element of $\{x_j : j \in \mathbb{N}\}$.

There are many simple examples of cascading sequences. Of course, constant sequences are cascading. A nontrivial example (and one we will use later on) is $\{1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots\}$.

Now we construct a cascading sequence $\{b_j\}$ in $\mathbb{R}^\omega$ using all of the elements of $L^\diamond$, and let

$$\{w_j\} = \{(v_j, b_j)\}$$

be a sequence in $\mathbb{R}^\omega$. Use this as part of the (ENP) $(\{w_j\}, \{p_j\})$ to weaken the topology on $\mathbb{R}^n \times L$ as described in §3.

Lemma 5.3. *With the topology on $\mathbb{R}^n \times L$ weakened as in the above paragraph, we have that $L^\diamond \subseteq A$ and that L is a subset of the standard closure of A.*

Proof: Let $p : \mathbb{R}^n \times L \to L$ be the projection mapping, $\mathcal{U}$ denote the weakened topology on $\mathbb{R}^n \times L$, and $V \in \mathcal{U}$ contain the identity. Since $\{w_j\} = \{(v_j, b_j)\}$ $\mathcal{U}$-converges to the identity, there exists a natural number N such that $w_j \in V$ for all $j \geq N$. Since $\{b_j\}$ is a cascading sequence of the elements of $L^\diamond$, $L^\diamond \subseteq p(V)$. Since V is arbitrary, we can conclude that $L^\diamond \subseteq A$. $L^\diamond$ is dense in L, and thus, the closure of A contains L. □

Although we have not obtained our main result at this point, we introduce an example to illustrate the ideas of 5.1 and 5.3.

Example 5.4. As we have seen, $(\{v_j\}, \{p_j\}) = (\{3^{j!} + 1\}, \{1/j\})$ is an (SNP) for $\mathbb{R}$. Let $L = \mathbb{Z}^\omega$. L has a countable dense subset. One countable dense subset is easily recognized as the subgroup consisting of all tuples in $\mathbb{Z}^\omega$ with only finitely many nonzero terms. Denote this subgroup by $L^\diamond$, and list its elements as $\{l_1, l_2, l_3, \ldots\}$. $\{l_1, l_2, l_1, l_2, l_3, l_1, l_2, l_3, l_4, \ldots\}$ is a cascading sequence using all of the elements of $L^\diamond$. We use this sequence to construct the sequence $\{w_j\}$ in $\mathbb{R} \times \mathbb{Z}^\omega$ as shown below. Note that we could use any cascading sequence that uses every element of $L^\diamond$.

$$
\begin{aligned}
w_1 &= (3^{1!} + 1, l_1) \\
w_2 &= (3^{2!} + 1, l_2) \\
w_3 &= (3^{3!} + 1, l_1) \\
w_4 &= (3^{4!} + 1, l_2) \\
w_5 &= (3^{5!} + 1, l_3) \\
w_6 &= (3^{6!} + 1, l_1) \\
w_7 &= (3^{7!} + 1, l_2) \\
w_8 &= (3^{8!} + 1, l_3) \\
w_9 &= (3^{9!} + 1, l_4) \\
&\vdots
\end{aligned}
$$

$(\{w_j\},\{p_j\})$ is an (ENP) on $\mathbb{R}\times\mathbb{Z}^\omega$, and $\{w_j\}$ $\mathcal{U}$-converges to 0. Denote the weakened topology by $\mathcal{U}$. By 5.1, $\mathbb{R}\times\overline{A}$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n\times\{0\}^\omega$. The closure of A is dense in L by the construction of the sequence $\{w_j\}$. In fact, we have the following general result.

Theorem 5.5. *Let the topology on $\mathbb{R}^n\times L$ be weakened as in 5.3 and let $\mathcal{U}$ be this weakened topology. Then $\mathbb{R}^n\times\{0\}^\omega$ is dense in $(\mathbb{R}^n\times L,\mathcal{U})$ and in $\mathcal{C}((\mathbb{R}^n\times L,\mathcal{U}))$. In particular, the arc component of the identity is dense in $(\mathbb{R}^n\times L,\mathcal{U})$ and $\mathcal{C}((\mathbb{R}^n\times L,\mathcal{U}))$.*

Proof: By 5.3, L is a subset of the standard closure of A, and by 5.1, $\mathbb{R}^n\times\overline{A}$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n\times\{0\}^\omega$. Therefore, we have that $\mathbb{R}^n\times L$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n\times\{0\}^\omega$. Since our space under consideration is $\mathbb{R}^n\times L$, we can conclude that $\mathbb{R}^n\times L$ equals the $\mathcal{U}$-closure of $\mathbb{R}^n\times\{0\}^\omega$. Thus, the arc component of the identity of $(\mathbb{R}^n\times L,\mathcal{U})$, which contains $\mathbb{R}^n\times\{0\}^\omega$, is dense. It is clear from this that the arc component of the identity in the completion must also be dense. □

Continuing Example 5.4 and applying 5.5, we see that the topology on $\mathbb{R}\times\mathbb{Z}^\omega$, weakened using the norming pair $(\{w_j\},\{p_j\})$, has a dense arc component of the identity. This, of course, implies that the topological group (and its completion) is connected.

There is another interesting idea associated with this example. By Proposition 3.7, we can use this weakened topology on $\mathbb{R}\times\mathbb{Z}^\omega$ to obtain a weakened, Hausdorff topology on $\mathbb{R}\times(\mathbb{Z}_2)^\omega$. Recall this weakened topology is denoted $\mathcal{U}/(\{0\}\times(2\mathbb{Z})^\omega)$. Clearly, $\mathbb{R}\times(\mathbb{Z}_2)^\omega$ is a connected topological group. But also, since the quotient map q is continuous and $\mathbb{R}\times\{0\}^\omega$ is dense in $\mathbb{R}\times\mathbb{Z}^\omega$, we see that

$$\mathbb{R}\times{\mathbb{Z}_2}^\omega=q(\overline{\mathbb{R}\times\mathbb{Z}^\omega})=q(\overline{\mathbb{R}\times\{0\}^\omega})=\overline{q(\mathbb{R}\times\{0\}^\omega)}.$$

Thus, $\mathbb{R}\times(\mathbb{Z}_2)^\omega$ also has a dense arc component of the identity. This weakened topology clearly uses the "compact" factor in an essential way.

We are now in a position to show that the quotient groups of our locally isometric groups may not be homeomorphic. First, consider the groups constructed in the above argument. There we used the (ENP) $(\{w_j\},\{p_j\})$ where $w_j=(3^{j!}+1,l_j)$, and $\{l_j\}$ was a cascading sequence formed using all of the elements of the dense subgroup of $\mathbb{Z}^\omega$ consisting of all tuples with all but finitely many nonzero terms. We forced the arc component of the identity to be dense in $\mathbb{R} \times \mathbb{Z}^w$ and in $\mathbb{R} \times (\mathbb{Z}_2)^\omega$. Recall that the topologies were represented by $\mathcal{U}$ and $\mathcal{U}/(\mathbb{R} \times \mathbb{Z}^\omega)$, respectively. Now consider the weakened topology on $\mathbb{R} \times \mathbb{Z}^\omega$ obtained from the (ENP) (x_j, p_j) where $x_j = (3^{j!}+1, 0, 0, 0, \ldots)$. Let the weakened topology obtained from this norming pair be denoted by $\mathcal{V}$. All of the interesting action is taking place in the first factor of $(\mathbb{R} \times \mathbb{Z}^\omega, \mathcal{V})$. In fact, $(\mathbb{R} \times \mathbb{Z}^\omega, \mathcal{V})$ is the product topology of a weakened topology on $\mathbb{R}$ and the standard topology on $\mathbb{Z}^\omega$. By 4.1, $(\mathbb{R} \times \mathbb{Z}^\omega, \mathcal{V})$ is locally isometric to $(\mathbb{R} \times \mathbb{Z}^\omega, \mathcal{U})$. Now form the quotient group $(\mathbb{R} \times (\mathbb{Z}_2)^\omega, \mathcal{V}/(\mathbb{R} \times \mathbb{Z}^\omega))$. This group is clearly not homeomorphic to the connected group $(\mathbb{R} \times (\mathbb{Z}_2)^\omega, \mathcal{U}/(\mathbb{R} \times \mathbb{Z}^\omega))$, since it is disconnected, being the product of a weakened topology on $\mathbb{R}$ and the standard topology on the compact group $(\mathbb{Z}_2)^\omega$. Most of our results have dealt with the local nature of topological groups. In showing that the two topological groups above are not homeomorphic, the local difference is not readily apparent. The following argument addresses that issue by showing that the topologies are not locally homeomorphic under the identity mapping.

Consider the subsequence of $\{w_j\}$ defined by $w_{j_k} = (3^{j_k!} + 1, 1, 0, 0, \ldots)$. Since $\{w_j\}$ $\mathcal{U}$-converges to the identity, so does $\{w_{j_k}\}$. Likewise, the sequence $\{x_{j_k}\}$ defined by $x_{j_k} = (3^{j_k!} + 1, 0, 0, 0, \ldots)$ $\mathcal{V}$-converges to 0. Since the quotient map is continuous, the image of each sequence will converge to the identity in the relative quotient spaces. Let ${w_{j_k}}' = w_{j_k} + [\{0\} + (2\mathbb{Z})^\omega]$ be the image of w_{j_k} under the quotient map, and similarly define ${x_{j_k}}' = x_{j_k} + [\{0\} + (2\mathbb{Z})^\omega]$ to be the image of x_{j_k} under the quotient map. If the identity map is a local homeomorphism, then the sequence defined by ${w_{j_k}}' - {x_{j_k}}' = (0, 1, 0, 0, 0, \ldots) + [\{0\} + (2\mathbb{Z})^\omega]$ would $\mathcal{V}/(\mathbb{R} \times \mathbb{Z}^\omega)$-converge to 0. This is clearly not the case, since it converges to $(0, 1, 0, 0, 0, \ldots) + [\{0\} + (2\mathbb{Z})^\omega]$, and $\mathcal{V}/(\mathbb{R} \times \mathbb{Z}^\omega)$ is a Hausdorff topology. Thus, the identity is not a local homeomorphism.

6. A Note On Finite Products

In this paper we focused on extending many of the results obtained on finite products in [8], to infinite products, as well as new results that forced unusual topological group structures on these products. Here we note that the arguments which allowed us to obtain dense arc components also give interesting results in the case of finite products. For example, if $(\{v_j\},\{p_j\})$ is a norming pair for $\mathbb{R}^n$ and we extend to a norming pair on $\mathbb{R}^{n+m}$ using the techniques in [8], we can weaken the topology on the subgroups and quotient groups of $\mathbb{R}^{m+n}$ and view the weakened group as algebraically $\mathbb{R}^n \times L$ where L is a subgroup, quotient group, or quotient group of a subgroup of $\mathbb{R}^m$ (see Example 3.8). Now Proposition 5.1 becomes

Proposition 6.1. *Suppose that we have weakened the topology $\mathbb{R}^n \times L$ as above. Let $\mathcal{U}$ be this weakened topology, $\Gamma = \{U \in \mathcal{U} : 0 \in U\}$, and $p : \mathbb{R}^n \times L \to L$ be the projection mapping. Define*

$$A = \left\{ y \in L : y \in \bigcap_{U \in \Gamma} p(U) \right\}.$$

Then A is a subgroup of L, and if $\overline{A}$ is the closure of A in the standard topology on L, then $\mathbb{R}^n \times \overline{A}$ is a subset of the $\mathcal{U}$-closure of $\mathbb{R}^n \times \{0\}^m$.

The proof is the same as for 5.1 since the infinite product does not manifest itself in the proof. The reason that this is interesting in the finite case is that, since A is a group, the subgroup generated by any element of A must be a subgroup of A. For example, take the (SNP) for $\mathbb{R}$ $(\{n!\},\{1/n\})$ and extend to an (ENP) for $\mathbb{R} \times \mathbb{Z}$ given by $(\{(n!,1)\},\{1/n\})$. Since $\{(n!,1),1/n)\}$ converges to zero, A contains 1. But A is a group, and thus it contains the subgroup generated by 1. Hence, A contains $\mathbb{Z}$, and 6.1 gives that $\mathbb{R} \times \{0\}$ is dense in $\mathbb{R} \times \mathbb{Z}$.

7. Concluding Remarks

We have shown that we can construct large classes of weakened Lie groups that can be very different globally but, with small restrictions, will be the same locally. We focused, in particular, on constructing groups in which the arc component of the identity is dense. There is a complementary question related to this construction: Can we force the arc component of the identity in the completion of a connected, weakened Lie group to be "small"? If somehow dense arc components of the identity are characteristic of the completion of these groups, then results by T. Christine Stevens can be viewed as structure theorems for these groups [9, §7]. We plan to explore this question in a future paper.

References

[1] N. Bourbaki, *General Topology, Chapters 1-4.* Elements of Mathematics. New York: Springer-Verlag, 1989.

[2] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis. Vol. I: Structure of Topological Groups, Integration Theory, Group Representations.* Die Grundlehren der mathematischen Wissenschaften, Vol. 115. Berlin: Springer-Verlag, 1963.

[3] Karl H. Hofmann and Sidney A. Morris, *The Structure of Compact Groups. A Primer for Students – A Handbook for the Expert.* de Gruyter Studies in Mathematics, 25. Berlin: Walter de Gruyter & Co., 1998.

[4] James R. Munkres, *Topology: A First Course.* Englewood Cliffs, NJ: Prentice Hall, 1975.

[5] J. W. Nienhuys, "Construction of group topologies on abelian groups," *Fund. Math.*, **75** (1972), 101–116.

[6] ———, "Not locally compact monothetic groups I," *Nederl. Akad. Wetensch. Proc. Ser. A 73 = Indag. Math.*, **32** (1970), 295–310.

[7] ———, "Not locally compact monothetic groups II," *Nederl. Akad. Wetensch. Proc. Ser. A 73 = Indag. Math.*, **32** (1970), 311–326.

[8] Jon W. Short and T. Christine Stevens, "Weakened Lie groups and their locally isometric completions," *Topology Appl.*, **135** (2004), 47–61.

[9] T. Christine Stevens, "Closures of weakened analytic groups," *Proc. Amer. Math. Soc.*, **119** (1993), no. 1, 291–297.

[10] ———, "Decisive subgroups of analytic groups," *Trans. Amer. Math. Soc.*, **274** (1982), no. 1, 101–108.

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