Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 361-376

PRODUCTS OF WEAK TOPOLOGIES

YOSHIO TANAKA

Abstract. We consider conditions for weak topologies to be productive. We give relations between products of weak topologies and products of sequential spaces (k-spaces, quasi-k-spaces, resp.), and we also consider products of these spaces.

1. Introduction

We assume that all spaces are regular, T_1, and all maps are continuous and onto.

Let X be a space. Then X *is determined by* a cover $\mathcal{P}$ [7], if X has the weak topology with respect to $\mathcal{P}$; that is, $G \subset X$ is open in X if $G \cap P$ is open in P for each $P \in \mathcal{P}$. Here, it is possible to replace "open" with "closed."

Every space is determined by an open cover or a hereditarily closure preserving (HCP) closed cover. Let us recall the following elementary facts which will be used often in this paper. These facts are routinely shown (see, e.g., [20] and [21]).

Fact 1.1. *Let X be determined by a cover $\mathcal{C}$. If $\mathcal{C}$ is a refinement of a cover $\mathcal{P}$ (i.e., each $C \in \mathcal{C}$ is contained in some $P \in \mathcal{P}$), then X is determined by $\mathcal{P}$.*

2000 *Mathematics Subject Classification.* Primary 54B10, 54B15; Secondary 54D50, 54D99.

Key words and phrases. k-spaces, quasi-k-spaces, sequential spaces, weak topologies.

The author was supported by Grant-in-Aid for Scientific Research (C) 14540066 (during 2002-2004).

Fact 1.2. *Let X be determined by a cover $\{P_\alpha : \alpha\}$. If each P_α is determined by a cover $\mathcal{P}_\alpha$, X is determined by a cover $\bigcup\{\mathcal{P}_\alpha : \alpha\}$.*

A space X is a *sequential space* (*k-space*, *quasi-k-space*, resp.) if X is determined by the cover of all compact metric subsets (compact subsets, countably compact subsets, resp.). Here, sequential spaces (quasi-k-spaces, resp.) are introduced by S. P. Franklin [5] (Jun-iti Nagata [14], resp.). Sequential spaces are k-spaces, and k-spaces are quasi-k-spaces, but the converses of these do not hold. However, quasi-k-spaces in which every point is a G_δ-set are sequential, and paracompact quasi-k-spaces are k-spaces [10]. As is well-known, every sequential space, k-space, and quasi-k-space is characterized as a quotient image of a metric space, locally compact (paracompact) space, M-space, respectively; see, e.g., [10] and [14]. Here, a space is an M-space iff it is a quasi-perfect inverse image of a metric space.

For a topological property (P) on a space X, X is *locally* (P) if each point of X has an nbd whose closure has the property (P). The following basic result holds by means of Fact 1.1 and Fact 1.2.

Proposition 1.3. *If X is determined by a cover of sequential spaces (k-spaces, quasi-k-spaces, resp.) then X is a sequential space (k-space, quasi-k-space, resp.). In particular, every locally sequential space (locally k-space, locally quasi-k-space, resp.) is a sequential space (k-space, quasi-k-space, resp.).*

Now, as is well-known, weak topologies are preserved by closed or open subsets and quotient maps. Namely, the following holds.

For space X determined by a cover $\mathcal{P}$, every open or closed subset S of X is determined by $\{P \cap S : P \in \mathcal{P}\}$, and every image of X under a quotient map f is determined by $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$.

However, weak topologies need not be preserved by products in view of [3, Example 5, p. 132] due to C. H. Dowker [2], or by Example 1.5 below. So, let us consider the following classic question, and then we will give some relations between products of weak topologies and products of sequential spaces, k-spaces, or quasi-k-spaces. Also, as their applications, we will consider products of these spaces.

Question 1.4. For each i = 1, 2, let X_i be a space determined by a cover $\mathcal{P}_i$. What is a (necessary and sufficient) condition for $X_1 \times X_2$ to be determined by $\mathcal{P}_1 \times \mathcal{P}_2 = \{P_1 \times P_2 : P_i \in \mathcal{P}_i\}$?

Example 1.5. (1) There exist paracompact σ-spaces X_i $(i = 1, 2)$ determined by an HCP cover $\mathcal{P}_i$ of compact metric subsets, and (a) or (b) below holds, but $X_1 \times X_2$ is not determined by $\mathcal{P}_1 \times \mathcal{P}_2$.

(a) $|\mathcal{P}_1| = \omega$ and $|\mathcal{P}_2| = 2^\omega$;
(b) $X_1 = X_2$, and $|\mathcal{P}_i| = \omega_1$.

(2) There exist both a countable space X_1 determined by a countable HCP cover $\mathcal{P}_1$ of compact metric subsets and a countable metric space X_2 determined by a cover $\mathcal{P}_2$ of compact metric subsets (or a cover $\mathcal{P}_2 = \{X_2\}$), but $X_1 \times X_2$ is not determined by $\mathcal{P}_1 \times \mathcal{P}_2$.

(3) In (1) and (2), it is possible to replace "HCP" with "point-finite" for both $\mathcal{P}_1$ and $\mathcal{P}_2$, but use $\mathcal{P}_2 = \{X_2\}$ in (2).

No $X_1 \times X_2$ in (1), (2), or (3) is a quasi-k-space or is determined by $\mathcal{P}_1 \times \mathcal{P}_2$ for any cover $\mathcal{P}_1$ of X_1 and any cover $\mathcal{P}_2$ of X_2 by locally compact subsets (or vice versa).

Proof: We show that the latter part holds. For (1), (2), and (3), $X_1 \times X_2$ is not sequential by [6, Theorem 1]; by [5, Example 1.11]; and by [19, Fact C(2) and Lemma 1(2)], respectively. Since each point of $X_1 \times X_2$ is a G_δ-set, $X_1 \times X_2$ is not even a quasi-k-space, by [10, Theorem 7.3]. Next, suppose that if $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$, then it is also determined by $\mathcal{P} = \{X_1 \times P_2 : P_2 \in \mathcal{P}_2\}$ from Fact 1.1. But, as is well-known, every product of a k-space with a locally compact space is a k-space. Thus, the elements of $\mathcal{P}$ are k-spaces, so $X_1 \times X_2$ is a k-space by Proposition 1.3, a contradiction. Therefore, $X_1 \times X_2$ is not determined by $\mathcal{P}_1 \times \mathcal{P}_2$. □

2. Results

Lemma 2.1. *Let X be determined by a cover $\mathcal{P}$. Then $X \times Y$ is determined by $\{P \times Y : P \in \mathcal{P}\}$ if either (a) or (b) holds.*
(a) Y is locally compact.
(b) X is sequential, and Y is locally countably compact.

Proof: For $G \subset X \times Y$, let $G \cap (P \times Y)$ be open in $P \times Y$ for every $P \in \mathcal{P}$. To show G is open in $X \times Y$, let $(a, b) \in G$, and let $a \in P_0 \in \mathcal{P}$. Put $H = \{y \in Y : (a, y) \in G\}$. Then

$H = \{y \in Y : (a, y) \in G \cap (P_0 \times Y)\}$; thus, H is an nbd of b in Y. For (a) ((b), resp.), there exists an nbd V of b such that $\overline{V}$ is compact (countably compact, resp.) with $\overline{V} \subset H$. Let $W = \{x : x \times \overline{V} \subset G\}$. Then $a \in W$. For (a), $W \cap P$ is obviously open in P for every $P \in \mathcal{P}$; hence, W is open in X. For (b), to see W is also open in X, suppose not. Then, since X is sequential, there exists a sequence $L = \{p_n : n \in N\}$ converging to some point $p \in W$ with $L \cap W = \emptyset$. Since X is determined by $\mathcal{P}$, but L is not closed in X, then $P^* \in \mathcal{P}$ frequently contains the point p and L. So, we can assume that $L \subset P^*$ without loss of generality. Then, there exists a sequence $S = \{(p_n, q_n) : n \in N\}$ in $X \times Y$ such that $(p_n, q_n) \in p_n \times \overline{V}$, but $S \cap G = \emptyset$. Since $\overline{V}$ is countably compact, some subsequence T of S has an accumulation point $(p, q) \in (p \times \overline{V})$, so $(p, q) \in G$. But $G \cap (P^* \times Y)$ is open in $P^* \times Y$ and contains the point (p, q). Since $P^* \times Y$ contains the sequence T, the sequence T is contained eventually in G, a contradiction. Hence, $W \times V$ is an nbd of (a, b) in $X \times Y$ such that $W \times V \subset G$. Thus, G is open in $X \times Y$. Then the result for (a) or (b) holds. □

Corollary 2.2. *For each $i = 1, 2$, let X_i be a space determined by a cover $\mathcal{P}_i$. Then $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$ if the following (a), (b), or (c) holds. The result for (b) is due to* [11].

(a) $\mathcal{P}_1$ is an open cover or a locally finite closed cover, as is $\mathcal{P}_2$.

(b) $\mathcal{P}_2$ is a locally finite closed cover of locally compact subsets.

(c) X_1 is sequential, and $\mathcal{P}_2$ is a locally finite closed cover of locally countably compact subsets.

Proof: We show that the result for (b) holds since the result for (a) or (c) is similarly shown. Obviously, $X_1 \times X_2$ is determined by a locally finite closed cover $\{X_1 \times P_2 : P_2 \in \mathcal{P}_2\}$. But, by Lemma 2.1, each element $X_1 \times P_2$ is determined by a cover $\{P_1 \times P_2 : P_1 \in \mathcal{P}_1\}$. Thus, the result holds by Fact 1.2. □

Lemma 2.3. *For each $i = 1, 2$, let X_i be a space determined by a cover $\mathcal{P}_i$. Then $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$ if the following (a) or (b) holds.*

(a) The elements of $\mathcal{P}_1$ are k-spaces, and $X_1 \times X_2$ is determined by $\{X_1 \times L : L \in \mathcal{L}\}$, where $\mathcal{L}$ is a cover of X_2 by locally compact subsets.

(b) X_1 is sequential, and $X_1 \times X_2$ is determined by $\{X_1 \times L : L \in \mathcal{L}\}$, where $\mathcal{L}$ is a cover of X_2 by locally countably compact subsets.

Proof: For (a), let $\mathcal{P}_1 = \{P_\alpha : \alpha\}$, where each P_α is a k-space. Then each P_α is determined by a cover $\mathcal{P}_\alpha$ of compact subsets. Then, by Fact 1.2, X_1 is determined by a cover $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha\}$ of compact subsets, and $\mathcal{P}$ is a refinement of $\mathcal{P}_1$. Let $\mathcal{L} = \{L_\alpha : \alpha\}$. Then $X_1 \times X_2$ is determined by $\{X_1 \times L_\alpha : \alpha\}$, and each element $X_1 \times L_\alpha$ is determined by a cover $\mathcal{P} \times \{L_\alpha\}$ from Lemma 2.1, for L_α is locally compact. Thus, by Fact 1.2, $X_1 \times X_2$ is determined by a cover $\mathcal{P} \times \mathcal{L}$ which is a refinement of $\mathcal{C} = \{P \times X_2 : P \in \mathcal{P}\}$. Then, by Fact 1.1, $X_1 \times X_2$ is determined by $\mathcal{C}$. But, by Lemma 2.1, each $P \times X_2 \in \mathcal{C}$ is determined by a cover $P \times \mathcal{P}_2$, for P is compact. Thus, by Fact 1.2, $X_1 \times X_2$ is determined by $\mathcal{P} \times \mathcal{P}_2$. Then, by Fact 1.1, $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$. For (b), X_1 is sequential, then X_1 is determined by a cover $\mathcal{F}$ of compact metric subsets. But each $F \in \mathcal{F}$ is closed in X_1, so F is determined by a cover $\{F \cap P : P \in \mathcal{P}_1\}$ of metric subsets. Then, by Fact 1.2, X_1 is determined by a cover $\mathcal{M} = \{F \cap P : F \in \mathcal{F}, P \in \mathcal{P}_1\}$ of metric subsets. But each $M \in \mathcal{M}$ is metric, so M is determined by a cover of compact subsets. Thus, by Fact 1.2, X_1 is determined by a cover $\mathcal{P}^*$ of compact subsets, and $\mathcal{P}^*$ is a refinement of $\mathcal{P}_1$. Then the result for (b) is similarly shown in (a) by means of Lemma 2.1(2), but use $\mathcal{P}^*$ instead of the cover $\mathcal{P}$ in (a). □

Theorem 2.4. *For each $i = 1, 2$, let X_i be a space determined by a cover $\mathcal{P}_i$. Then $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$ if the following (a) or (b) holds.*

(a) $X_1 \times X_2$ is a k-space, and the elements of $\mathcal{P}_1$ are k-spaces.

(b) X_1 is sequential, and $X_1 \times X_2$ is a quasi-k-space.

Proof: For (b), since $X_1 \times X_2$ is a quasi-k-space, $X_1 \times X_2$ is determined by a cover $\mathcal{C}$ of countably compact subsets. Since $\mathcal{C}$ is a refinement of $\mathcal{P} = \{X_1 \times C : C$ is countably compact in $X_2\}$, by Fact 1.1, $X_1 \times X_2$ is determined by $\mathcal{P}$. Then the result holds by Lemma 2.3. Similarly, (a) holds. □

Reviewing E. A. Michael [10], let us recall some matters around k-spaces, and certain quotient spaces. A space X is *a space of pointwise countable type* (a *q-space*, resp.) if each $x \in X$ has a sequence $\{V_n : n \in N\}$ of nbd's such that $C = \bigcap\{V_n : n \in N\}$ is a compact set (countably compact closed set, resp.), and each nbd of C contains some V_n. First-countable spaces, locally compact spaces, or paracompact M-spaces are of pointwise countable type. Locally countably compact spaces, M-spaces, or spaces of pointwise countable type are q-spaces.

The implications in (A) and (B) below hold, and for each $i = 1, 2, 3, 4, 5$, (i) in (A) implies (i) in (B). For these implications and definitions of related spaces, and for five kinds of quotient spaces below and their related maps, see, e.g., [10].

(A) (1) *spaces of pointwise countable type* $\to$ (2) *bi-k-spaces* $\to$ (3) *countably bi-k-spaces* $\to$ (4) *singly bi-k-spaces* $\to$ (5) *k-spaces.*

(B) (1) *q-spaces* $\to$ (2) *bi-quasi-k-spaces* $\to$ (3) *countably bi-quasi-k-spaces* $\to$ (4) *singly bi-quasi-k-spaces* $\to$ (5) *quasi-k-spaces.*

The spaces in (1), (2), (3), (4), and (5) in (A) ((B), resp.) are characterized, respectively, as open, bi-quotient, countably bi-quotient, hereditarily quotient, and quotient spaces of paracompact M-spaces (M-spaces, resp.). Here, a map $f : X \to Y$ is *bi-quotient* (*countably bi-quotient*, resp.) if, whenever $y \in Y$ and $\mathcal{U}$ is a cover (countable cover, resp.) of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$, with $U \in \mathcal{U}$, covers an nbd of y in Y, and f is *hereditarily quotient* (or *pseudo-open*) if $f|f^{-1}(S)$ is quotient for every $S \subset Y$. (Equivalently, for any nbd U of $f^{-1}(y)$ in X, $f(U)$ contains an nbd of y in Y.)

Corollary 2.5. *For each i = 1, 2, let X_i be a space determined by a cover $\mathcal{P}_i$. Then $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$ if either (a) or (b) holds.*

(a) $X_1 \times X_2$ is a k-space, and each element of $\mathcal{P}_1$ is a closed set or a union of G_δ-sets in X_1.

(b) $X_1 \times X_2$ is sequential.

Proof: The result for (b) holds by Theorem 2.4. For (a), $X_1 \times X_2$ is determined by a cover $\mathcal{C} = \{C_1 \times X_2 : C_1 \text{ is compact in } X_1\}$. Let $(C_1 \times X_2) \in \mathcal{C}$. Then the closed subset C_1 of X_1 is determined by a cover $\mathcal{P} = \{C_1 \cap P_1 : P_1 \in \mathcal{P}_1\}$. Let $C_1 \cap P_1 \in \mathcal{P}$. When P_1 is a closed set in X_1, $C_1 \cap P_1$ is a k-space. When P_1 is a union of G_δ-sets in X_1, $C_1 \cap P_1$ is a union of G_δ-sets in C_1; thus, $C_1 \cap P_1$ is a space of pointwise countable type by [10, Remark, p. 103], and therefore, it is also a k-space. Thus, the set $C_1 \cap P_1$ is a k-space. However, each element $C_1 \times X_2$ of $\mathcal{C}$ is a k-space, for it is closed in the k-space $X_1 \times X_2$. Thus, by Theorem 2.4, each $C_1 \times X_2$ is determined by a cover $\{(C_1 \cap P_1) \times P_2 : P_i \in \mathcal{P}_i\}$. Hence, by Fact 1.1 and Fact 1.2, $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$. $\square$

The author asks the following question in view of Theorem 2.4 and Corollary 2.5.

Question 2.6. For each i = 1, 2, let X_i be a space determined by a cover $\mathcal{P}_i$. If $X_1 \times X_2$ is a k-space, then is $X_1 \times X_2$ determined by $\mathcal{P}_1 \times \mathcal{P}_2$?

We consider equivalent relations between products of weak topologies and products of sequential spaces, k-spaces, or quasi-k-spaces.

Let X be a space determined by a cover $\mathcal{P}$. Then $\mathcal{P}$ is a *weak k-system* [24] if the elements of $\overline{\mathcal{P}} = \{\overline{P} : P \in \mathcal{P}\}$ are compact. When the cover $\mathcal{P}$ is closed, $\mathcal{P}$ is called a *k-system* [1]. Similarly, let us say that $\mathcal{P}$ is a *weak lk-system* by replacing "compact" with "locally compact," and that $\mathcal{P}$ is an *lk-system* when the cover $\mathcal{P}$ is closed. A space with a countable k-system is called a *k_ω-space* [9] (or *space of class $\mathcal{S}'$* [13]). A space with a countable lk-system is called an *lk_ω-space* [19] (or *space of class $\mathcal{T}'$* [17]). Obviously, a space is a k-space iff it has a weak lk-system (or k-system) by Proposition 1.3. Point-countable k-systems are considered in [18] or [24].

Remark 2.7. (1) A space has a countable k-system iff it is a quotient image of a locally compact Lindelöf space, by [13, Theorem 5]. Similarly, a space has a point-countable k-system iff it is a quotient, s-image (i.e., each point-inverse is separable) of a locally compact paracompact space. Analogously, a space X has a point-countable weak k-system $\mathcal{P}$ iff X is an image of a locally compact paracompact space L under a map f such that for some $S \subset L$, $f|S$ is a quotient s-map onto X. Here, it is possible to replace "paracompact" with "meta-Lindelöf" (i.e., every open cover has a point-countable open refinement). (In fact, for the "only if" part, let L be the topological sum of $\overline{\mathcal{P}}$ and S be the union of elements of $\mathcal{P}$ in L, and let f be the obvious map from L. For the "if" part, L is determined by a point-countable open cover $\mathcal{G}$ such that the elements of $\overline{\mathcal{G}}$ are compact. Since S is determined by a point-countable open cover $\mathcal{V} = \{S \cap G : G \in \mathcal{G}\}$, X has a point-countable weak k-system $f(\mathcal{V})$.)

(2) Every countably bi-quasi-k-space (bi-quasi-k-space, resp.) X with a point-countable lk-system (weak lk-system, resp.) is locally compact by [23, Theorem 3.16(2)] (in view of the proof of [24, Theorem 3.8(1)], resp.). For the parenthetic part, by [24, Lemma 2.3], we can add the prefix "countably" before bi-quasi-k-space if X is sequential, or generally, $t(X) \leq \omega$ (i.e., for any $x \in \overline{A}$, there is a countable $C \subset A$ with $x \in \overline{C}$).

Proposition 2.8. *For each $i = 1, 2$, let X_i have a cover $\mathcal{P}_i$. Then $X_1 \times X_2$ is a k-space determined by a cover $\overline{\mathcal{P}}_1 \times \mathcal{P}_2$ if each of $(a_1) \sim (a_4)$ below holds. When X_1 is sequential, or the elements of $\mathcal{P}_1$ are k-spaces (such as (a_3)), $X_1 \times X_2$ is determined by a cover $\mathcal{P}_1 \times \mathcal{P}_2$.*

(a_1) $\mathcal{P}_i$ are countable weak lk-systems.

(a_2) X_i are locally separable singly bi-quasi-k-spaces, and $\mathcal{P}_i$ are point-countable weak lk-systems.

(a_3) X_i are singly bi-quasi-k-spaces, and $\mathcal{P}_i$ are point-countable lk-systems.

(a_4) $\mathcal{P}_1$ is a countable weak lk-system, and X_2 has the same properties as in (a_3).

Proof: The result for (a_1), (a_3), or (a_4) holds in view of the proof of [20, Theorem 6]; here, for the space X_i having a countable weak lk-system $\mathcal{P}_i$, X_i has the countable lk-system $\overline{\mathcal{P}}_i$ which is assumed to be increasing. For (a_2), each point of X_1 has an nbd V whose closure is separable, while $\overline{V}$ has a point-countable weak lk-system $\mathcal{V} = \{\overline{V} \cap P : P \in \mathcal{P}_1\}$. But $\overline{V}$ is a separable singly bi-quasi-k-space. Thus, by [23, Lemma 2.8], $\overline{V}$ has a countable lk-system $\overline{\mathcal{W}}$ for some countable $\mathcal{W} \subset \mathcal{V}$. This implies that each point in $X_1 \times X_2$ has an nbd whose closure is a k-space in view of (a_1). Hence, $X_1 \times X_2$ is a k-space by Proposition 1.3. Thus, the result holds by Theorem 2.4. □

Similar modifications of Proposition 2.8 would be obtained from changing the combinations: for example, the same result for (a_4) holds if we replace "(a_3)" with "(a_2)" in (a_4).

Now, every product of k-spaces (or quasi-k-spaces) need not be a quasi-k-space, by either Example 1.5 or Example 2.9.

Example 2.9. (1) There exist countably compact spaces X and Y, but $X \times Y$ is not a quasi-k-space [10, Example 10.7].

(2) ($2^{\omega_0} < 2^{\omega_1}$). There exist Fréchet countably bi-k-spaces X and Y, but $X \times Y$ is not a quasi-k-space [15, Example 6.6].

The following holds in view of the proof for (a) and (b) in [20, Theorem 6], using Fact 1.1.

Lemma 2.10. *Let X be a bi-k-space (sequential countably bi-quasi-k-space, resp.), and let Y be a countably bi-quasi-k-space (bi-quasi-k-space, resp.). Then $X \times Y$ is determined by a cover $\{C \times Y : C$ is compact in $X\}$.*

Proposition 2.11. *(1) $X \times Y$ is a k-space if the following (b_1), (b_2), or (b_3) holds.*

(b_1) X is a k-space and Y is locally compact.

(b_2) X is a bi-k-space, and Y is a k-space which is countably bi-quasi-k.

(b_3) X is a sequential countably bi-quasi-k-space, and Y is a k-space which is bi-quasi-k.

(2) $X \times Y$ is a quasi-k-space if each of the following (c_1) $\sim$ (c_4) holds.

(c_1) X is a quasi-k-space, and Y is locally compact.

(c_2) X is a sequential space, and Y is locally countably compact.

(c_3) X is a bi-k-space, and Y is a countably bi-quasi-k.

(c_4) X is a sequential countably bi-quasi-k-space, and Y is a bi-quasi-k-space.

Proof: The result for (b_1) is well-known, and the one for (c_2) is shown in [16]. But the result for (b_1), (c_1), or (c_2) holds by Proposition 1.3 and Corollary 2.2. For example, for (c_1), $X \times Y$ is determined by a cover of locally countably compact subsets, taking $\mathcal{P}_1$ as the cover of all countably compact subsets, and $\mathcal{P}_2 = \{Y\}$ in Corollary 2.2(b). Hence, $X \times Y$ is a quasi-k-space by Proposition 1.3. The result for (b_2) or (b_3) is shown in [20]. But the result for (b_2), (b_3), (c_3), or (c_4) holds by use of Lemma 2.10. For example, for (c_3), $X \times Y$ is determined by a cover $\{C \times Y : C$ is compact in $X\}$, but the elements are quasi-k-spaces by (c_1). Thus, $X \times Y$ is a quasi-k-space by Proposition 1.3. $\square$

By Proposition 2.11 and Theorem 2.4, we have the following partial answer to Question 3.18 in [24] of whether X^2 has a point-countable weak k-system if X is a locally compact space X with a point-countable weak k-system.

Corollary 2.12. *Let X be a space with a weak k-system (weak lk-system, resp.) $\mathcal{P}$. Then X^2 has a weak k-system (weak lk-system, resp.) $\mathcal{P}^2$ if X is a sequential bi-quasi-k-space, or X is a bi-k-space with the elements of $\mathcal{P}$ k-spaces.*

The following holds in view of the proof of Example 1.5, but use Proposition 2.11(2) for the parenthetic part.

Lemma 2.13. *Let X_1 be a k-space (quasi-k-space, resp.) determined by a cover $\mathcal{P}_1$, and let X_2 be a space determined by a cover $\mathcal{P}_2$ by locally compact subsets. If $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$, then $X_1 \times X_2$ is a k-space (quasi-k-space, resp.). For the parenthetic part, when X_1 is sequential, it is possible to replace "locally compact subsets" with "locally countably compact subsets."*

Theorem 2.14. *For each i = 1, 2, let X_i be a space determined by a cover $\mathcal{P}_i$.*

(1) Let X_i be k-spaces. Suppose that each of (d_1) $\sim$ (d_4) below holds, for example. Then $X_1 \times X_2$ is a k-space iff $X \times Y$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$.

(d_1) The elements of $\mathcal{P}_2$ are locally compact.

(d_2) The elements of $\mathcal{P}_1$ are bi-k-spaces, and the elements of $\mathcal{P}_2$ are countably bi-quasi-k-spaces.

(d_3) The elements of $\mathcal{P}_i$ are (locally) lk_ω-spaces.

(d_4) The elements of $\mathcal{P}_i$ are singly bi-quasi-k-spaces with a point-countable lk-system.

(2) Let X_1 be a sequential space. Suppose that (e_1), (e_2), or (e_3) below holds. Then $X_1 \times X_2$ is a quasi-k-space iff $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$.

(e_1) The elements of $\mathcal{P}_2$ are locally countably compact.

(e_2) Same as (d_2).

(e_3) The elements of $\mathcal{P}_1$ are countably bi-quasi-k-spaces, and the elements of $\mathcal{P}_2$ are bi-quasi-k-spaces.

Proof: The "only if" parts of (1) and (2) hold by Theorem 2.4, because the elements of $\mathcal{P}_1$ (or $\mathcal{P}_2$) are k-spaces for (1), and X_1 is sequential for (2). Their "if" parts hold, using propositions 2.8 and 2.11. Indeed, let $X_1 \times X_2$ be determined by $\mathcal{P}_1 \times \mathcal{P}_2$, and let us consider ($d_1$), ($d_2$), and ($e_3$), for example. For ($d_1$), the result holds by Lemma 2.13. For (d_2), by Lemma 2.10, each element $P_1 \times P_2$ of $\mathcal{P}_1 \times \mathcal{P}_2$ is determined by a cover $\{C \times P_2 : C$ is compact in

$P_1\}$. Thus, by Fact 1.1 and Fact 1.2, $X_1 \times X_2$ is determined by a cover $\{C \times X_2 : C$ is compact in $X_1\}$ of k-spaces. Then $X_1 \times X_2$ is a k-space by Proposition 1.3. For (e_3), the elements of $\mathcal{P}_1$ are quasi-k-spaces; hence, they are determined by a cover of countably compact subsets. Since X_1 is sequential, these countably compact subsets are closed in X_1, hence sequential. Thus, the elements of $\mathcal{P}_1$ are sequential by Proposition 1.3. Then $X_1 \times X_2$ is a quasi-k-space by Proposition 2.11(2). $\square$

Theorem 2.15. *For each $i = 1, 2$, let X_i be a space determined by a cover $\mathcal{P}_i$. Let X_1 be a sequential space, and X_2 be a k-space (sequential space, resp.). Then (a) $\Leftrightarrow$ (b) below, and (b) $\Rightarrow$ (c) hold. When each of (e_i) in Theorem 2.14 holds, (a), (b), and (c) are equivalent.*

(a) $X_1 \times X_2$ is a k-space (sequential space, resp.).
(b) $X_1 \times X_2$ is a quasi-k-space.
(c) $X_1 \times X_2$ is determined by $\mathcal{P}_1 \times \mathcal{P}_2$.

Proof: (a) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (c) holds by Theorem 2.4. Also, (b) $\Rightarrow$ (a) holds, by taking $\mathcal{P}_1$ as a cover of compact metric subsets, and $\mathcal{P}_2$ as a cover of compact (compact metric, resp.) subsets in (c). For the latter part, (c) $\Rightarrow$ (b) holds by Theorem 2.14(2). $\square$

To consider other sufficient conditions on X for a quasi-k-space $X \times Y$ to be a k-space, we introduce the following condition (C):

(C) For each countably compact subset K of X, there exists a locally paracompact subsets S of X such that $S \supset K$.

Every locally compact (generally, locally paracompact) space satisfies (C). Let us consider other spaces satisfying (C).

Let X be a space, and let $\mathcal{P}$ be a closed cover of X. Then, X *is dominated by* $\mathcal{P}$ [8], or $\mathcal{P}$ *dominates* X (= X has the *weak topology with respect to* $\mathcal{P}$ in the sense of Morita [11]), if the union of elements of any subcollection $\mathcal{Q}$ of $\mathcal{P}$ is closed in X, and the union is determined by $\mathcal{Q}$. Every space is dominated by an HCP closed cover. Obviously, if X is dominated by $\mathcal{P}$, then X is determined by $\mathcal{P}$, but the converse doesn't hold. As is well known, every CW-complex is dominated by a cover of compact metric subsets, and every space dominated by paracompact subsets is paracompact [8], [12].

Lemma 2.16. *Let $f : X \to Y$ be a closed map such that (a) or (b) below holds. Then each countably compact subset K of Y is contained in a finite union of elements of $f(\mathcal{P})$.*

(a) X is determined by a point-countable cover $\mathcal{P}$.

(b) X is dominated by a closed cover $\mathcal{P}$.

Proof: Suppose that the result doesn't hold for some countably compact subset K of Y. For (a), let $\{P \in \mathcal{P} : x \in P\} = \{P_n(x) : n \in N\}$ for each $x \in X$. Then, there exists $D = \{x_n : n \in N\}$ such that $x_n \notin P_j(x_i)$ for $i, j < n$, and $f(D) \subset K$ with $f(D)$ infinite. Then $D \cap P$ is finite for each $P \in \mathcal{P}$, so D is closed discrete in X. Then $f(D)$ is discrete closed in the countably compact set K, a contradiction. For (b), the result is shown by the same way, replacing " $x_n \notin P_j(x_i)$ for $i, j < n$ " with " $x_n \in P_n - \bigcup\{P_i : i < n\}$ for some $\{P_n : n \in N\} \subset \mathcal{P}$." □

Proposition 2.17. *A space X satisfies the condition (C) if each of the following $(f_1) \sim (f_4)$ holds.*

(f_1) X is determined by a point-countable cover $\mathcal{P}$ such that the elements of $\overline{\mathcal{P}}$ are locally paracompact subsets; in particular, X has a point-countable weak lk-system.

(f_2) X is dominated by a closed cover of locally paracompact subsets.

(f_3) X is a closed image of a space having the same property in (f_1) or (f_2), but the locally paracompact subsets are moreover meta-Lindelöf.

(f_4) X is a quotient s-image of a locally Lindelöf (or locally separable) meta-Lindelöf space.

Proof: The result for (f_1) or (f_2) holds by Lemma 2.16. For (f_3), let $f : S \to X$ be a closed map. Let S be determined by a point-countable cover $\mathcal{P}$ such that the elements of $\overline{\mathcal{P}}$ are locally paracompact meta-Lindelöf. Let K be a countably compact subset of X. Then, by Lemma 2.16, K is contained in some $F = \bigcup\{f(\overline{P}_i) : i = 1, 2, \cdots, k\}$, where $P_i \in \mathcal{P}$. Let T be the topological sum of these finitely many closed sets $\overline{P}_i$ in S. Then F is a closed image of the locally paracompact meta-Lindelöf space T, and F contains K. Then we can assume that S is locally paracompact meta-Lindelöf. Thus, S is determined by a point-countable open cover $\mathcal{G}$ such that every element of $\overline{\mathcal{G}}$ is paracompact. Hence, by Lemma 2.16, K

is contained in some $f(\overline{H})$, where $\overline{H}$ is paracompact, so $f(\overline{H})$ is paracompact, as is well known. Thus, X satisfies condition (C). Similarly, the result for (f_2) in (f_3) also holds.

For (f_4), let $g : L \to X$ be a quotient s-map, and let L be locally Lindelöf meta-Lindelöf. Then L is determined by a point-countable open cover $\mathcal{V}$ such that the elements of $\overline{\mathcal{V}}$ are Lindelöf. Since g is a quotient s-map, X is determined by a point-countable cover $g(\mathcal{V})$. Thus, by Lemma 2.16, each countably compact subset K of X is contained in a finite union of elements $g(V)$ of $g(\mathcal{V})$. Since the $g(\overline{V})$ are Lindelöf, K is contained in a Lindelöf space. Thus, X satisfies condition (C). The parenthetic part also holds, because every separable meta-Lindelöf space is Lindelöf. □

Theorem 2.18. *For each $i = 1, 2$, let X_i be a space determined by a cover $\mathcal{P}_i$, and let X_2 be a k-space. For relations among the following (a) $\sim$ (d), (1), (2), and (3) below hold. Obviously, (a) $\Rightarrow$ (b), and (d) $\Rightarrow$ (c) hold.*

(a) $X_1 \times X_2$ is a k-space.
(b) $X_1 \times X_2$ is a quasi-k-space.
(c) $X_1 \times X_2$ is determined by a cover $\overline{\mathcal{P}}_1 \times \mathcal{P}_2$.
(d) $X_1 \times X_2$ is determined by a cover $\mathcal{P}_1 \times \mathcal{P}_2$.

(1) Suppose that $\mathcal{P}_1$ is a point-countable cover (a closed cover dominating X_1, resp.). Then (a) $\Rightarrow$ (c) ((a) $\Rightarrow$ (d), resp.) holds.

(2) Suppose that X_1 satisfies the condition (C). Then (a) and (b) are equivalent. When $\mathcal{P}_1$ is a point-countable weak lk-system (an lk-system dominating X_1, resp.), then (a), (b), and (c) ((a), (b), (c), and (d), resp.) are equivalent.

(3) Suppose that X_1 is sequential (the elements of $\mathcal{P}_1$ are k-spaces, resp.). If $\mathcal{P}_1$ is a point-countable weak lk-system, or an lk-system dominating X_1, then (a), (b), (c), and (d) ((a), (c), and (d), resp.) are equivalent.

Proof: For (1), to show (a) $\Rightarrow$ (c) holds, let $X_1 \times X_2$ be a k-space. Since X_1 is a k-space, it is determined by a cover $\mathcal{K}$ of compact subsets. Then, by Theorem 2.4, $X_1 \times X_2$ is determined by $\mathcal{K} \times \mathcal{P}_2$. But each compact subset of X_1 is contained in a finite union of elements of $\overline{\mathcal{P}}_1$ by Lemma 2.16. Hence, by Fact 1.1, $X_1 \times X_2$ is determined by a cover $\{F \times P_2 : F$ is a finite union of elements of $\overline{\mathcal{P}}_1$, and $P_2 \in \mathcal{P}_2\}$. But each element $F \times P_2$ is determined by some finite subcover of $\{P \times P_2 : P \in \overline{\mathcal{P}}_1\}$, because each $P \in \overline{\mathcal{P}}_1$ is closed in X_1. Thus, by Fact 1.2, $X_1 \times X_2$ is determined by $\overline{\mathcal{P}}_1 \times \mathcal{P}_2$. Similarly, the parenthetic part holds.

For (2), to show (b) $\Rightarrow$ (a) holds, let $X_1 \times X_2$ be a quasi-k-space. Since $X_1 \times X_2$ is determined by a cover of countably compact subsets, it is determined by $\mathcal{C} = \{C \times X_2 : C$ is countably compact in $X_1\}$. Let $C \times X_2 \in \mathcal{C}$. Then, by the condition (C), the countably compact subset C of X_1 is contained in a locally paracompact subset S of X_1. Thus, C is covered by a collection $\mathcal{V}$ of open subsets in S such that the elements of $\overline{\mathcal{V}}$ are paracompact. Here, the "closure" is taken in S. Then $C \times X_2$ is determined by an open cover $\mathcal{V}_C = \{(C \cap V) \times X_2 : V \in \mathcal{V}\}$. Let $(C \cap V) \times X_2 \in \mathcal{V}_C$, and let $K = C \cap \overline{V}$. We show that $L_C = \overline{K} \times X_2$ is a k-space. Since $\overline{V} \supset \overline{K}$, $\overline{K} = cl_{\overline{V}} K$. Since K is countably compact and $\overline{V}$ is normal in S, $\overline{K}$ is countably compact; thus, it is compact. Hence, by Proposition 2.11(1), L_C is a k-space which contains $(C \cap V) \times X_2$. Therefore, each element of $\mathcal{V}_C$ is contained in a k-space L_C of $X_1 \times X_2$, and, by Fact 1.1, $X_1 \times X_2$ is determined by a cover $\bigcup\{\mathcal{V}_C : C \times X_2 \in \mathcal{C}\}$. Thus, by Fact 1.1, $X_1 \times X_2$ is determined by a cover of k-spaces. Hence, $X_1 \times X_2$ is a k-space by Proposition 1.3. Let $\mathcal{P}_1$ be a weak lk-system. Then (c) $\Rightarrow$ (a) holds, since $X_1 \times X_2$ is determined by a cover $\{\overline{P} \times X_2 : P \in \mathcal{P}_1\}$ of k-spaces. Thus, (a), (b), and (c) are equivalent.

For (3), (b) $\Rightarrow$ (d) ((a) $\Rightarrow$ (d), resp.) holds by Theorem 2.4. $\square$

In the following corollary, (1) follows from Theorem 2.15, and (2) follows from Theorem 2.18(2) with Proposition 2.17.

Corollary 2.19. *(1) If X is a sequential space and Y is a k-space (sequential space, resp.), then $X \times Y$ is a k-space (sequential space, resp.) iff $X \times Y$ is a quasi-k-space* [16].

(2) If each of $(f_1) \sim (f_4)$ in Proposition 2.17 holds, and Y is a k-space, then $X \times Y$ is a k-space iff $X \times Y$ is a quasi-k-space.

The author doesn't know whether the condition on X in Corollary 2.19 (or the condition (C) in Theorem 2.18(2)) is essential.

Question 2.20. For each i = 1, 2, let X_i be a k-space. If $X_1 \times X_2$ is a quasi-k-space, then is $X_1 \times X_2$ a k-space? In particular, when the X_i's are countably compact (then $X_1 \times X_2$ is countably compact [4, Theorem 3.10.13], since X_1 is a k-space), is $X_1 \times X_2$ a k-space?

By theorems 2.15 and 2.18(1), and by Proposition 2.11(2), we have

Proposition 2.21. *(1) Suppose that X satisfies the condition (C). Then $X \times Y$ is a k-space if the following (a), (b), or (c) holds.*

(a) X is a quasi-k-space, and Y is locally compact.

(b) X is locally countably compact, and Y is sequential.

(c) X is a bi-quasi-k-space, and Y is a sequential countably bi-quasi-k-space.

(2) Suppose that X is a k-space in (b) (X is sequential in (b) or (c), resp.). Then $X \times Y$ is a k-space (sequential, resp.) [20].

References

[1] A. V. Arhangel'skii, "Factor mappings of metric spaces," *Soviet Math. Dokl.* **5** (1964), 368–371.

[2] C. H. Dowker, "Topology of metric complexes," *Amer. J. Math.* **74** (1952), 555–577.

[3] James Dugundji, *Topology.* Boston, Mass.: Allyn and Bacon, Inc., 1966.

[4] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[5] S. P. Franklin, "Spaces in which sequences suffice," *Fund. Math.* **57** (1965), 107–115.

[6] Gary Gruenhage, "k-spaces and products of closed images of metric spaces," *Proc. Amer. Math. Soc.* **80** (1980), no. 3, 478–482.

[7] G. Gruenhage, E. Michael, and Y. Tanaka, "Spaces determined by point-countable covers," *Pacific J. Math.* **113** (1984), no. 2, 303–332.

[8] Ernest Michael, "Continuous selections, I," *Ann. of Math. (2)* **63** (1956), no. 2, 361–382.

[9] ——, "Bi-quotient maps and Cartesian products of quotient maps," *Ann. Inst. Fourier (Grenoble)* **18** (1968), fasc. 2, 287–302 (1969).

[10] ——, "A quintuple quotient quest," *General Topology and Appl.* **2** (1972), 91–138.

[11] Kiiti Morita, "On spaces having the weak topology with respect to closed coverings," *Proc. Japan Acad.* **29** (1953), 537–543.

[12] ——, "On spaces having the weak topology with respect to closed coverings, II," *Proc. Japan Acad.* **30** (1954), 711–717.

[13] ——, "On decomposition spaces of locally compact spaces," *Proc. Japan Acad.* **32** (1956), 544–548.

[14] Jun-iti Nagata, "Quotient and bi-quotient spaces of M-spaces," *Proc. Japan Acad.* **45** (1969), 25–29.

[15] Roy C. Olson, "Bi-quotient maps, countably bi-sequential spaces, and related topics," *General Topology and Appl.* **4** (1974), 1–28.

[16] Yoshio Tanaka, "On quasi-k-spaces," *Proc. Japan Acad.* **46** (1970), 1074–1079.

[17] ______, "A characterization for the products of k-and-$\aleph$-spaces and related results," *Proc. Amer. Math. Soc.* **59** (1976), no. 1, 149–155.

[18] ______, "Point-countable k-systems and products of k-spaces," *Pacific J. Math.* **101** (1982), no. 1, 199–208.

[19] ______, "Products of k-Spaces Having Point-Countable k-Networks," *Topology Proc.* **22** (1997), 305–329.

[20] ______, "Products of k-spaces, and questions," *Comment. Math. Univ. Carolin.* **44** (2003), no. 2, 335–345.

[21] ______, "Quotient spaces and decompositions." *Encyclopedia of General Topology.* Ed. Klaas Pieter Hart, Jun-iti Nagata, and Jerry E. Vaughan. Amsterdam: North-Holland, 2004. 43–46.

[22] ______, "On products of k-spaces," *Topology Appl.* **146-147** (2005), 593–602.

[23] Yoshio Tanaka and Iwao Yoshioka, "Spaces determined by point-countable covers, and singly bi-quasi-k-spaces," *Questions Answers Gen. Topology* **23** (2005), no. 1, 35–54.

[24] Iwao Yoshioka, "On spaces with point-countable k-systems," *Comment. Math. Univ. Carolin.* **45** (2004), no. 4, 749–765.

Department of Mathematics; Tokyo Gakugei University; Tokyo 184-8501, Japan

E-mail address: `ytanaka@u-gakugei.ac.jp`