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EXTENSIONAL DIMENSION AND COMPLETION OF MAPS

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ABSTRACT. We prove the following completion theorem for closed maps between metrizable spaces: Let $f\colon X \to Y$ be a closed surjection between metrizable spaces with e-dim$f \leq K$, e-dim$X \leq L_X$, and e-dim$Y \leq L_Y$ for some countable CW-complexes K, L_X, and L_Y. Then there exist completions $\widetilde{X}$ and $\widetilde{Y}$ of X and Y, respectively, and a closed surjection $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ extending f such that e-dim$\widetilde{f} \leq K$, e-dim$\widetilde{X} \leq L_X$, and e-dim$\widetilde{Y} \leq L_Y$. We also establish a parametric version of a result of Miroslav Katetov characterizing the covering dimension of metrizable spaces in terms of uniformly 0-dimensional maps into finite-dimensional cubes.

1. Introduction

Miroslav Katetov [5] and Kiiti Morita [8] proved that every finite-dimensional metrizable space has a metrizable completion of the same dimension. A completion theorem for extensional dimension with respect to countable CW-complexes was established by Wojciech Olszewski [10] in the class of separable metrizable spaces and recently by Michael Levin [7] in the class of all metrizable spaces.

Concerning completions of maps with the same dimension, James E. Keesling [6] proved that if $f\colon X \to Y$ is a closed surjective

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map between metrizable finite-dimensional spaces, then there are completions $\widetilde{X}$ and $\widetilde{Y}$ of X and Y, respectively, and an extension $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ of f such that $\widetilde{f}$ is closed, $\dim \widetilde{f} = \dim f$, $\dim \widetilde{X} = \dim X$, and $\dim \widetilde{Y} = \dim Y$. In the present note we extend this result for extensional dimension with respect to countable CW-complexes. We also establish an analogue (see Theorem 3.1) of a result of Katetov [5] characterizing the dimension dim of metrizable spaces in terms of uniformly 0-dimensional maps into finite-dimensional cubes.

Recall that $e - dimX \leq K$ if and only if every continuous map $g\colon A \to K$, where $A \subset X$ is closed, can be extended to a map $\bar{g}\colon X \to K$; see [3]. For a map $f\colon X \to Y$, we write $e - dimf \leq K$ provided $e - dimf^{-1}(y) \leq K$ for every $y \in Y$. Unless indicated otherwise, all spaces are assumed to be metrizable and all maps continuous. By a CW-complex we always mean a countable CW-complex.

2. Completion of maps

We begin with the following lemma.

Lemma 2.1. *Let $f\colon X \to Y$ be a perfect map between metrizable spaces and K a CW-complex. Then $B_K = \{y \in Y : \text{e-dim} f^{-1}(y) \leq K\}$ is a G_δ-subset of Y.*

Proof: By [11], there exists a map g from X into the Hilbert cube Q such that $f \times g\colon X \to Y \times Q$ is an embedding. Let $\{W_i\}_{i\in\mathbb{N}}$ be a countable finitely-additive base for Q. For every i we choose a sequence of mappings $h_{ij}\colon \overline{W_i} \to K$, representing all the homotopy classes of mappings from $\overline{W_i}$ to K. (This is possible because K is a countable CW-complex and all $\overline{W_i}$ are metrizable compacta.) For any i, j let U_{ij} be the set of all $y \in Y$ having the following property:

> the map $h_{ij} \circ g\colon g^{-1}(\overline{W_i}) \to K$ can be continuously extended to a map over the set $g^{-1}(\overline{W_i}) \cup f^{-1}(y)$.

Let every U_{ij} be open in Y. Indeed, if $y_0 \in U_{ij}$, then there exists a map $h\colon g^{-1}(\overline{W_i}) \cup f^{-1}(y_0) \to K$ extending $h_{ij} \circ g$. Since K is an absolute extensor for metrizable spaces, we can extend h to a map $\overline{h}\colon V \to K$, where $V \subset X$ is open and contains $g^{-1}(\overline{W_i}) \cup f^{-1}(y_0)$. Because f is closed, there exists a neighborhood G of y_0

in Y with $f^{-1}(G) \subset V$. Then, for every $y \in G$, the restriction of $\overline{h}$ on $g^{-1}(\overline{W_i}) \cup f^{-1}(y)$ is an extension of $h_{ij} \circ g$. Hence, $G \subset U_{ij}$.

It is clear that B_K is contained in every U_{ij}. It remains only to show that $\cap_{i,j=1}^{\infty} U_{ij} \subset B_K$. Take $y \in \cap_{i,j=1}^{\infty} U_{ij}$ and a map $h\colon A \to K$, where A is a closed subset of $f^{-1}(y)$. Because the map $g_y = g|f^{-1}(y)$ is a homeomorphism, $h' = h \circ g_y^{-1}\colon g(A) \to K$ is well defined. Next, extend h' to a map from a neighborhood W of $g(A)$ in Q (recall that $f^{-1}(y)$ is compact, so $g(A) \subset Q$ is closed) into K and find W_k with $g(A) \subset W_k \subset \overline{W_k} \subset W$. Therefore, there exists a map $h''\colon \overline{W_k} \to K$ extending h'. Then h'' is homotopy equivalent to some h_{kj}, so are $h'' \circ g$ and $h_{kj} \circ g$ (considered as maps from $g^{-1}(\overline{W_k})$ into K). Since $y \in U_{kj}$, $h_{kj} \circ g$ can be extended to a map from $g^{-1}(\overline{W_k}) \cup f^{-1}(y)$ into K. Then, by the Homotopy Extension Theorem, there exists a map $\bar{h}\colon g^{-1}(\overline{W_k}) \cup f^{-1}(y) \to K$ extending $h'' \circ g$. Obviously, $\bar{h}|f^{-1}(y)$ extends h. Hence, e-dim$f^{-1}(y) \leq K$. □

The next lemma, though not explicitly stated in this form, was actually proved by Levin [7].

Lemma 2.2. *Let X be a subset of the metrizable space Y with* e-dim$X \leq K$ *for some CW-complex K. Then there exists a G_δ-subset $\widetilde{X}$ of Y containing X such that* e-dim$\widetilde{X} \leq K$.

Theorem 2.3. *Let $f\colon X \to Y$ be a closed surjective map between metrizable spaces such that* e-dim$f \leq K$, e-dim$X \leq L_X$, *and* e-dim$Y \leq L_Y$, *where K, L_X, and L_Y are CW-complexes. Then there exist completions $\widetilde{X}$ and $\widetilde{Y}$ of X and Y, respectively, and a closed surjection $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ extending f with* e-dim$\widetilde{f} \leq K$, e-dim$\widetilde{X} \leq L_X$, *and* e-dim$\widetilde{Y} \leq L_Y$.

Proof: Since f is closed, $Frf^{-1}(y) = \emptyset$ if and only if y is a discrete point in Y, where $Frf^{-1}(y)$ denotes the boundary of $f^{-1}(y)$ in X. On the other hand, it is easily seen that the validity of the theorem for any metrizable Y without discrete points implies its validity for any metrizable Y. Therefore, we can assume that Y doesn't have any discrete points, or equivalently, $Frf^{-1}(y) \neq \emptyset$ for every $y \in Y$. According to the classical result of I. A. Vaĭnsteĭn [13] (see also [6]), there are completions X_1 and Y_1 of X and Y, respectively, and a closed surjection $f_1\colon X_1 \to Y_1$ which extends f. For any $y \in Y_1$

we denote by $Frf_1^{-1}(y)$ the boundary of $f_1^{-1}(y)$ in X_1. Then, the following two facts occur:

(1) $Frf^{-1}(y)$ coincides with $Frf_1^{-1}(y)$ provided $y \in Y$;

(2) $f_1^{-1}(y) = Frf_1^{-1}(y)$ provided $y \in Y_1 \backslash Y$.

Therefore, $Frf_1^{-1}(y) \neq \emptyset$ for all $y \in Y_1$. Moreover, $f_1|H\colon H \to Y_1$ is a perfect surjection (see [13]), where $H = \bigcup\{Frf_1^{-1}(y) : y \in Y_1\}$. Obviously, H is closed in X_1, so e-dim$(H \cap X) \leq L_X$. Then, by Lemma 2.2, there exists a G_δ-subset P of H with $H \cap X \subset P$ and

(3) e-dim$P \leq L_X$.

It follows from (1) that $(f_1|H)^{-1}(Y) \subset P$. Therefore, $f_1(H \backslash P)$ does not meet Y. Since $f_1|H$ is a closed surjection onto Y_1 and $H \backslash P$ is F_σ in H, $f_1(H \backslash P)$ is F_σ in Y_1. So, $Y_2 = Y_1 \backslash f_1(H \backslash P)$ is a G_δ-set in Y_1 containing Y such that

(4) $(f_1|H)^{-1}(Y_2) \subset P$.

Condition (1) also implies that every fiber $(f_1|H)^{-1}(y)$ is of extensional dimension $\leq K$ provided $y \in Y$. Hence, applying Lemma 2.1 and then Lemma 2.2, we can find a G_δ-subset $\widetilde{Y}$ of Y_2 such that e-dim$\widetilde{Y} \leq L_Y$ and

(5) e-dim$Frf_1^{-1}(y) \leq K$ for all $y \in \widetilde{Y}$.

Consider the set $W = X_1 \backslash H$. It is open in X_1, so $W \cap X$ is open in X. Moreover, $f^{-1}(y) \cap W$ is the interior of $f^{-1}(y)$ in X, $y \in Y$. Therefore, e-dim$\big(f^{-1}(y) \cap W\big) \leq K$ for every $y \in Y$. Consequently, e-dim$\big(W \cap X\big) \leq K$. On the other hand, $W \cap X$ is a subset of X, so e-dim$\big(W \cap X\big) \leq L_X$. Since the property of metrizable spaces to have extensional dimension less than or equal to a given countable CW-complex is hereditary (see, for example [2]), we can apply Lemma 2.2 twice to obtain a G_δ-subset U of W which contains $W \cap X$ such that

(6) e-dim$U \leq K$ and e-dim$U \leq L_X$.

Finally, let $\widetilde{X} = f_1^{-1}(\widetilde{Y}) \cap (U \cup P)$ and $\widetilde{f} = f_1|\widetilde{X}$. Obviously, $\widetilde{X} \cap U$ and $\widetilde{X} \cap P$ are disjoint, respectively, open and closed subsets of $\widetilde{X}$. Since e-dim$\big(\widetilde{X} \cap U\big) \leq$ e-dim$U \leq L_X$ and e-dim$\big(\widetilde{X} \cap P\big) \leq$ e-dim$P \leq L_X$, $\widetilde{X}$ can be represented as the union of countably many of its closed subsets F_i with e-dim$F_i \leq L_X$ for each i. Then, by the countable sum theorem, e-dim$\widetilde{X} \leq L_X$. It follows from our construction that $\widetilde{f}$ maps $\widetilde{X}$ onto $\widetilde{Y}$ and each $\widetilde{f}^{-1}(y)$, $y \in \widetilde{Y}$, is the union of the disjoint sets $Frf_1^{-1}(y)$ and $\widetilde{f}^{-1}(y) \cap U$ which are, respectively, closed and open in $\widetilde{f}^{-1}(y)$. By (5) and (6), both $Frf_1^{-1}(y)$ and $\widetilde{f}^{-1}(y) \cap U$ are of extensional dimension $\leq K$. Hence, e-dim$f_1^{-1}(y) \leq K$ for each $y \in \widetilde{Y}$.

It remains only to show that $\widetilde{f}$ is a closed map. To this end, let $A \subset \widetilde{X}$ be closed and $y_n = \widetilde{f}(x_n)$ converges to y_0, where $\{x_n\}$ is a sequence of points from A. Suppose that $y_0 \notin \widetilde{f}(A)$. Then, by (1), (2), and (4), $Frf_1^{-1}(y_0) \subset \widetilde{X}$ and it does not meet A (as a subset of $\widetilde{f}^{-1}(y_0)$). Being compact, $Frf_1^{-1}(y_0)$ is closed in $\widetilde{X}$. Consequently, there is an open $V \subset X_1$ containing $Frf_1^{-1}(y_0)$ such that $V \cap A = \emptyset$. Let V_1 be the union of V and the interior of $f_1^{-1}(y_0)$ in X_1. Obviously, V_1 is open in X_1, contains $f_1^{-1}(y_0)$, and does not meet A. Since f_1 is a closed map, there exists a neighborhood $O(y_0)$ of y_0 in Y_1 such that $f_1^{-1}(y) \subset V_1$ for all $y \in O(y_0)$. Therefore, $f_1^{-1}(y_m) \subset V_1$ for some m. The last inclusion implies $x_m \in V_1 \cap A$, which is a contradiction. Therefore, $y_0 \in \widetilde{f}(A)$; i.e., $\widetilde{f}$ is closed. □

3. σ-uniformly 0-dimensional maps

A map $f \colon X \to Y$ is called uniformly 0-dimensional [5] if there exists a metric on X generating its topology such that for every $\epsilon > 0$ every point of $f(X)$ has a neighborhood U in Y with $f^{-1}(U)$ being the union of disjoint open subsets of X each of diameter $< \epsilon$. Uniformly 0-dimensional maps are called in [1] completely 0-dimensional. It is well known that if $f \colon X \to Y$ is uniformly 0-dimensional and $\dim Y \leq n$, then $\dim X \leq n$ (see, for example, [5], [1], or [7]).

We say that a map $g \colon X \to Y$ is σ-uniformly 0-dimensional if X can be represented as the union of countably many of its closed subsets X_i such that each restriction $g|X_i$ is uniformly 0-dimensional. Katetov [5] (see also [9]) proved that a space X is at most n-dimensional if and only if for each metrization of X there exists a uniformly 0-dimensional map of X into $\mathbb{I}^n$. Moreover, the space $C(X, \mathbb{I}^n)$ with the uniform convergence topology contains a dense G_δ-subset consisting of uniformly 0-dimensional maps. The next theorem can be considered as a parametric version of Katetov's result; (see [4] for the definition of C-spaces).

Theorem 3.1. *Let $f\colon X \to Y$ be a closed map of metrizable spaces with Y being a C-space. Then $\dim f \leq n$ if and only if there exists a map $g\colon X \to \mathbb{I}^n$ such that $f \times g$ is σ-uniformly 0-dimensional. Moreover, if $\dim f \leq n$, then the set of all such maps $g \in C(X, \mathbb{I}^n)$ is dense in $C(X, \mathbb{I}^n)$ with respect to the uniform convergence topology generated by the Euclidean metric on $\mathbb{I}^n$.*

Proof: All function spaces in this proof are equipped with the uniform convergence topology.

Suppose that $\dim f \leq n$. We represent X as the union $X = X_0 \cup (X \backslash X_0)$ such that X_0 is closed in X, $f_0 = f|X_0$ is a perfect map, and $\dim(X\backslash X_0) \leq n$. Let $X\backslash X_0 = \bigcup_{k=1}^{\infty} X_k$ such that each X_k is closed in X. Since $f_0\colon X_0 \to Y$ is perfect, the set C_0 of all $g\colon X \to \mathbb{I}^n$ with $(f \times g)|X_0$ being 0-dimensional is dense in $C(X, \mathbb{I}^n)$; (see for example, [12, Theorem 1.3]). It is easily seen that every perfect 0-dimensional map between metric spaces is uniformly 0-dimensional. Hence, all restrictions $(f \times g)|X_0$, $g \in C_0$, are uniformly 0-dimensional. For every $g \in C_0$ let $H(g) = \{h \in C(X, \mathbb{I}^n) : h|X_0 = g|X_0\}$. Each $H(g)$ is closed in $C(X, \mathbb{I}^n)$ and $C_0 = \cup\{H(g) : g \in C_0\}$. We also define the maps $p_k\colon C(X, \mathbb{I}^n) \to C(X_k, \mathbb{I}^n)$ by $p_k(h) = h|X_k$, $k = 1, 2, \ldots$, and let $p_{k,g}\colon H(g) \to C(X_k, \mathbb{I}^n)$ denote the restriction $p_k|H(g)$ for any $k \in \mathbb{N}$ and $g \in C_0$. Since X_0 and each X_k are disjoint closed sets in X, we can show that every $p_{k,g}$ is open and surjective. According to the Katetov result [5], there exists a dense and G_δ-subset C_k of $C(X_k, \mathbb{I}^n)$ consisting of uniformly 0-dimensional maps, $k = 1, 2, \ldots$. Consequently, for any $g \in C_0$, the sets $H_k(g) = p_{k,g}^{-1}(C_k)$ are dense and G_δ in $H(g)$. Since $H(g)$ has the Baire property (as a closed subset of $C(X, \mathbb{I}^n)$), $M(g) = \bigcap_{k=1}^{\infty} H_k(g)$ is also dense and G_δ in $H(g)$. Then $M = \cup\{M(g) : g \in C_0\}$ is dense in $C(X, \mathbb{I}^n)$. Moreover, it follows from the construction that, for any $g \in M$, the restrictions $(f \times g)|X_k$ are uniformly 0-dimensional, $k = 0, 1, 2, \ldots$. Therefore, M consists of σ-uniformly 0-dimensional maps.

To prove the other implication of Theorem 3.1, assume that there exists $g\colon X \to \mathbb{I}^n$ such that the map $f \times g\colon X \to Y \times \mathbb{I}^n$ is σ-uniformly 0-dimensional. Therefore, X can be represented as the union of countably many of its closed subsets A_i such that each $(f \times g)|A_i$ is uniformly 0-dimensional. The last condition implies that, for any $y \in Y$ and i, the map $g|(f^{-1}(y) \cap A_i)\colon f^{-1}(y) \cap A_i \to \mathbb{I}^n$ is uniformly 0-dimensional. Hence, $\dim\big(f^{-1}(y) \cap A_i\big) \leq n$. Since $f^{-1}(y) = \cup_{i=1}^{\infty} f^{-1}(y) \cap A_i$, by the countable sum theorem, $\dim f^{-1}(y) \leq n$ for each $y \in Y$. So, $\dim f \leq n$. □

Addendum. The referee of this paper suggested that Theorem 3.1 could remain true if the requirement $f \times g$ is σ-uniformly 0-dimensional is relaxed to $f \times g$ is uniformly 0-dimensional. In the case when f is a perfect map, this is really true, following from [12, Theorem 1.3] and the fact that any perfect 0-dimensional map is uniformly 0-dimensional. Unfortunately, we couldn't arrive at any conclusion in the general case.

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