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δ-SEMIOPEN SETS IN TOPOLOGY

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Abstract. It is the object of this paper to study further the notion of δ-Λ_s-semiclosed sets which is defined as the intersection of a δ-Λ_s-set and a δ-semiclosed set. Moreover, we introduce some low separation axioms using the above notions. Also, we present and study the notions of δ-Λ_s-continuous functions, Λ_s-compact spaces and δ-Λ_s-connected spaces.

1. Introduction

In 1963, N. Levine [2] offered a new notion in the field of topology by introducing the notion of semi-open sets. He defined this new class of sets by using the known concept of the closure of an open set, i.e. a subset A of a space X is *semi-open* if it is a subset of the closure of its interior. This new class of sets properly contains the class of all open sets. Since the advent of the notion of semi-open sets, several other generalized open sets came to existence, such as preopen sets [[4], [10]], α-open sets [11], β-open sets [[1], [2]], δ-open sets [5] and etc. The study of topological properties via these generalized open sets has gained significant importance in General Topology and its applications (see for example [6], [7], [5]).

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In what follows (X,τ) and (Y,σ) (or X and Y) are topological spaces and A is a subset of X. By $Int(A)$ and $Cl(A)$ we denote the interior and the closure of A, respectively. In 1968, N. V. Velićko [5] introduced the notion of δ-open sets in order to investigate the characterization of H-closed spaces, i.e. a Hausdorff space is H-closed if its image is closed in every Hausdorff space in which it can be embedded. The *δ-interior* of a subset A of a space (X,τ), denoted by $Int_\delta(A)$, is the union of all regular open sets of (X,τ) contained in A. Recall that a subset A of X is called *regular open* if $A = Int(Cl(A))$. A subset A of a space (X,τ) is *δ-open* if $A = Int_\delta(A)$. This means that a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called *δ-closed*. Alternatively, a subset A of a space (X,τ) is called δ-closed if $A = Cl_\delta(A)$, where $Cl_\delta(A) = \{x \in X \mid Int(Cl(U)) \cap A \neq \emptyset, U \in \tau \quad \text{and} \quad x \in U\}$. It is worth to be noticed that the family of all δ-open sets form a topology on X. Moreover, the notion of δ-open sets found its way in digital topology in what it is called the Khalimsky line or the so-called digital line (see [7]). Such line is the set $\mathcal{Z}$ of integers equipped with the topology $\mathcal{K}$ having $\{\{2n-1, 2n, 2n+1\} \mid n \in \mathcal{Z}\}$ as a subbase. This kind of space is of great importance in the study of applications of point-set topology to computer graphics.

A subset A of (X,τ) is said to be δ-semiopen set [12] if there exists a δ-open set U of X such that $U \subset A \subset Cl(U)$. The complement of a δ-semiopen set is called a δ-semiclosed set. A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \emptyset$ for every δ-semiopen set U of X containing x. The set of all δ-semicluster points of A is called the δ-semiclosure of A, denoted by $sCl_\delta(A)$. We denote the collection of all δ-semiopen (resp. δ-semiclosed) sets by $\delta\mathrm{SO}(X,\tau)$ (resp. $\delta\mathrm{SC}(X,\tau)$). We say that a subset U of X is a δ-semineighborhood of a point $x \in X$ if U contains a δ-semiopen set to which x belongs.

Lemma 1.1. *(see* [3]*) The following statements are true:*

(1) *Intersection of arbitrary collection of δ-semiclosed sets in (X,τ) is δ-semiclosed*
(2) *Let A be a subset of a space (X,τ). Then*
$$sCl_\delta(A) = \cap\{F \in \delta SC(X,\tau) \mid A \subset F\}.$$
(3) *$sCl_\delta(A)$ is δ-semiclosed, that is $sCl_\delta(sCl_\delta(A)) = sCl_\delta(A)$.*

Lemma 1.2. *(see* [3]*) For subsets A and A_i $(i \in I)$ of a space X, the following hold:*

(1) $A \subset sCl_\delta(A)$.
(2) *If* $A \subset B$, *then* $sCl_\delta(A) \subset sCl_\delta(B)$.
(3) $sCl_\delta(\cap\{A_i : i \in I\}) \subset \cap\{sCl_\delta(A_i) : i \in I\}$.
(4) $sCl_\delta(\cup\{A_i : i \in I\} = \cup\{sCl_\delta(A_i) : i \in I\}$.
(5) *A is δ-semiclosed if and only if* $A = sCl_\delta(A)$.

2. Λ^s_δ-SETS

Definition 1. Let A be a subset of a space X. By $\Lambda^s_\delta(A)$ we denote the set

$$\cap\{O \in \delta SO(X, \tau) : A \subset O\}.$$

The subset A of X is called Λ^s_δ-set if $A = \Lambda^s_\delta(A)$.

Lemma 2.1. *For subsets A and A_i $(i \in I)$ of a space X, the following hold:*

(1) $A \subset \Lambda^s_\delta(A)$.
(2) $\Lambda^s_\delta(\Lambda^s_\delta(A)) = \Lambda^s_\delta(A)$.
(3) *If* $A \subset B$, *then* $\Lambda^s_\delta(A) \subset \Lambda^s_\delta(B)$.
(4) $\Lambda^s_\delta(\cap\{A_i : i \in I\}) \subset \cap\{\Lambda^s_\delta(A_i) : i \in I\}$.
(5) $\Lambda^s_\delta(\cup\{A_i : i \in I\}) = \cup\{\Lambda^s_\delta(A_i) : i \in I\}$.
(6) $\Lambda^s_\delta(A)$ *is a* Λ^s_δ*-set.*
(7) *If A is δ-semiopen, then A is a* Λ^s_δ*-set.*
(8) *If A_i is* Λ^s_δ*-set for each $i \in I$, then* $\cap\{A_i : i \in I\}$ *is a* Λ^s_δ*-set.*
(9) *If A_i is* Λ^s_δ*-set for each $i \in I$, then* $\cup\{A_i : i \in I\}$ *is a* Λ^s_δ*-set.*

Definition 2. Let A be a subset of a space X. By $\Lambda^{s\,*}_\delta(A)$ we denote the set

$$\cup\{B \in \delta SC(X, \tau) : B \subset A\}.$$

The subset A of X is called $\Lambda^{s\,*}_\delta$-set if $A = \Lambda^{s\,*}_\delta(A)$.

We obtain the following lemma which is similar to Lemma 2.1.

Lemma 2.2. *For subsets A, B and A_i $(i \in I)$ of a space X the following properties hold:*

(1) $\Lambda^{s\,*}_\delta(A) \subseteq A$.
(2) *If* $A \subseteq B$, *then* $\Lambda^{s\,*}_\delta(A) \subseteq \Lambda^{s\,*}_\delta(B)$.
(3) *If A is δ-closed, then* $\Lambda^{s\,*}_\delta(A) = A$.
(4) $\Lambda^{s\,*}_\delta(\cap\{A_i : i \in I\}) = \cap\{\Lambda^{s\,*}_\delta(A_i) : i \in I\}$.
(5) $\cup\{\Lambda^{s\,*}_\delta(A_i) : i \in I\} \subseteq \Lambda^{s\,*}_\delta(\cup\{A_i : i \in I\})$.

(6) $\Lambda^s_\delta(X-A) = X - \Lambda^{s\ *}_\delta(A)$ *and* $\Lambda^{s\ *}_\delta(X-A) = X - \Lambda^s_\delta(A)$.
(7) $\Lambda^{s\ *}_\delta(A)$ *is a* $\Lambda^{s\ *}_\delta$*-set.*
(8) *If* A *is a* δ*-closed set, then* A *is a* $\Lambda^{s\ *}_\delta$*-set.*
(9) *If* A_i *is a* $\Lambda^{s\ *}_\delta$*-set for each* $i \in I$*, then* $\cup\{A_i : i \in I\}$ *and* $\cap\{A_i : i \in I\}$ *are* $\Lambda^{s\ *}_\delta$*-sets.*

Proof. We prove only the relations (6) and (7).
(6) We have:

$$\begin{aligned} X - \Lambda^{s\ *}_\delta(A) &= X - \cup\{B \in \delta SC(X,\tau) : B \subset A\} \\ &= \cap\{B^c \in \delta SO(X,\tau) : A^c \subset B^c\} \\ &= \Lambda^s_\delta(A^c) = \Lambda^s_\delta(X-A) \end{aligned}$$

Similarly, we have: $\Lambda^{s\ *}_\delta(X-A) = X - \Lambda^s_\delta(A)$.
(7) We have

$$\begin{aligned} \Lambda^{s\ *}_\delta(\Lambda^{s\ *}_\delta(A)) &= \Lambda^{s\ *}_\delta(X - \Lambda^s_\delta(X-A)) \\ &= X - \Lambda^s_\delta(\Lambda^s_\delta(X-A)) \\ &= X - \Lambda^s_\delta(X-A) \\ &= X - (X - \Lambda^{s\ *}_\delta(A)) = \Lambda^{s\ *}_\delta(A) \end{aligned}$$

Thus, the set $\Lambda^{s\ *}_\delta(A)$ is a $\Lambda^{s\ *}_\delta$-set. □

Proposition 2.3. *Let* (X,τ) *be a space,*

$$\tau^{\Lambda^s_\delta} = \{A : A \text{ is a } \Lambda^s_\delta - \text{set of } X\}$$

and

$$\tau^{\Lambda^{s\ *}_\delta} = \{A : A \text{ is a } \Lambda^{s\ *}_\delta\text{-set of } X\}.$$

Then, the pairs $(X, \tau^{\Lambda^s_\delta})$ *and* $(X, \tau^{\Lambda^{s\ *}_\delta})$ *are Alexandroff spaces.*

3. δ-Λ_s-SEMICLOSED SETS

Definition 3. A subset A of a space X is called δ-Λ_s-semiclosed, denoted by $(\Lambda, s\delta)$-closed, if $A = T \cap C$, where T is a Λ^s_δ-set and C is a δ-semiclosed set.

Example 3.1. ([3], Example 5) Let (X,τ) be a space such that $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Clearly, the family of all δ-open sets is the family τ and

$$\delta SO(X,\tau) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,c\}\}.$$

Also, the family of all $(\Lambda, s\delta)$-closed sets is the family

$$\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}.$$

The set $A = \{c\}$ is Λ^s_δ-set and the set $B = \{a, b\}$ is $(\Lambda, s\delta)$-closed but are not δ-semiopen sets.

Proposition 3.2. *Let A be a subset of a space X. The following conditions are equivalent:*

(1) *A is $(\Lambda, s\delta)$-closed,*
(2) *$A = T \cap sCl_\delta(A)$, where T is a Λ^s_δ-set,*
(3) *$A = \Lambda^s_\delta(A) \cap sCl_\delta(A)$.*

Proof. (1) $\Rightarrow$ (2): Let $A = T \cap C$, where T is a Λ^s_δ-set and C is a δ-semiclosed set. Since $A \subset C$, we have $sCl_\delta(A) \subset C$ and

$$A = T \cap C \supset T \cap sCl_\delta(A) \supset A.$$

Therefore, we obtain $A = T \cap sCl_\delta(A)$.

(2) $\Rightarrow$ (3): Let $A = T \cap sCl_\delta(A)$, where T is a Λ^s_δ-set. Since $A \subset T$, we have $\Lambda^s_\delta(A) \subset \Lambda^s_\delta(T) = T$ and hence

$$A \subset \Lambda_\delta(A) \cap Cl_\delta(A) \subset T \cap sCl_\delta(A) = A.$$

Therefore, we obtain $A = \Lambda^s_\delta(A) \cap sCl_\delta(A)$.

(3) $\Rightarrow$ (1): Since $\Lambda^s_\delta(A)$ is a Λ^s_δ-set, $sCl_\delta(A)$ is δ-semiclosed and $A = \Lambda^s_\delta(A) \cap sCl_\delta(A)$. □

Proposition 3.3. *The following statements are true:*

(1) *Every Λ^s_δ-set (resp. δ-semiclosed set) is $(\Lambda, s\delta)$-closed.*
(2) *If A_i is $(\Lambda, s\delta)$-closed for each $i \in I$, then $\cap\{A_i : i \in I\}$ is $(\Lambda, s\delta)$-closed.*

Proof. (1) It is obvious. (2) Suppose that A_i is $(\Lambda, s\delta)$-closed for each $i \in I$. Then, for each $i \in I$ there exist a Λ^s_δ-set T_i and a semi-δ-closed set C_i such that $A_i = T_i \cap C_i$. Now

$$\begin{aligned}\bigcap\{A_i : i \in I\} &= \bigcap\{T_i \cap C_i : i \in I\}\\ &= \bigcap\{T_i : i \in I\} \cap \bigcap\{C_i : i \in I\}\end{aligned}$$

By Lemma 2.1, $\bigcap\{T_i : i \in I\}$ is a Λ^s_δ-set and $\bigcap\{C_i : i \in I\}$ is δ-semiclosed. This shows that $\bigcap\{A_i : i \in I\}$ is $(\Lambda, s\delta)$-closed. □

Definition 4. A subset A of a space X is said to be $(s\delta, s\delta)$-generalized closed if $sCl_\delta(A) \subseteq G$ holds whenever $A \subseteq G$ and $G \in \delta SO(X, \tau)$.

Proposition 3.4. *The following statements are true:*

(1) *A subset A of a space X is $(s\delta, s\delta)$-generalized closed if and only if $sCl_\delta(A) \subseteq \Lambda^s_\delta(A)$.*
(2) *A subset A of a space X is δ-semiclosed if and only if A is $(s\delta, s\delta)$-generalized closed and $(\Lambda, s\delta)$-closed.*

Proof. (1) *Necessity.* Let $x \in X$ such that $x \notin \Lambda^s_\delta(A)$. So there exists a δ-semiopen subset O such that $A \subseteq O$ with $x \notin O$. This means that $x \notin sCl_\delta(A)$ since A is $(s\delta, s\delta)$-generalized closed.

Sufficiency. Obvious.

(2) *Necessity.* Obvious

Sufficiency. Since A is $(s\delta, s\delta)$-generalized closed, then $sCl_\delta(A) \subseteq \Lambda^s_\delta(A)$ (Proposition 3.4(1)). Now $A = \Lambda^s_\delta(A) \cap sCl_\delta(A) = sCl_\delta(A)$. □

Definition 5. A subset A of a space X is called $(\Lambda, s\delta)$-open if $X \setminus A$ is $(\Lambda, s\delta)$-closed.

Proposition 3.5. *The following statements are true:*

(1) *The union of any family of $(\Lambda, s\delta)$-open sets is a $(\Lambda, s\delta)$-open set.*
(2) *Every $\Lambda^s_\delta{}^*$-set is $(\Lambda, s\delta)$-open.*

Proof. (1) The proof of this theorem follows by the fact that the intersection of a family of $(\Lambda, s\delta)$-closed sets is $(\Lambda, s\delta)$-closed (Proposition 3.3).

(2) Take $A = A \cup \emptyset$, where A is a $\Lambda^s_\delta{}^*$-set, X is δ-semiclosed and $\emptyset = X \setminus X$. □

Proposition 3.6. *The following statements are equivalent for a subset A of a space X:*

(1) *A is $(\Lambda, s\delta)$-open*
(2) *$A = T \cup C$, where T is a $\Lambda^s_\delta{}^*$-set and C is a δ-semiopen set.*

Proof. Straightforward. □

Definition 6. Let X be a space and $A \subseteq X$. A point $x \in X$ is called $(\Lambda, s\delta)$-cluster point of A if for every $(\Lambda, s\delta)$-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all $(\Lambda, s\delta)$-cluster points is called the $(\Lambda, s\delta)$-closure of A and is denoted by $A^{(\Lambda, s\delta)}$.

Proposition 3.7. *Let A and B be subsets of a space X. Then, the following properties hold.*

(1) $A \subset A^{(\Lambda,s\delta)}$.
(2) $A^{(\Lambda,s\delta)} = \cap\{F \mid A \subset F$ *and* F *is* $(\Lambda, s\delta) - closed\}$.
(3) *If* $A \subset B$, *then* $A^{(\Lambda,s\delta)} \subset B^{(\Lambda,s\delta)}$.
(4) A *is* $(\Lambda, s\delta)$*-closed if and only if* $A = A^{(\Lambda,s\delta)}$.
(5) $A^{(\Lambda,s\delta)}$ *is* $(\Lambda, s\delta)$*-closed.*

Definition 7. A subset A of a space X is called a Λ^s_δ-D set if there are two $(\Lambda, s\delta)$-open sets U and V in X such that $U \neq X$ and $A = U - V$.

Clearly every $(\Lambda, s\delta)$-open set U different from X is a Λ^s_δ-D set.

Definition 8. A space X is called

(1) Λ^s_δ-D_0 if for any distinct pair of points x and y of X there exists a Λ^s_δ-D set of X containing x but not y or a Λ^s_δ-D set of X containing y but not x.
(2) Λ^s_δ-D_1 if for any distinct pair of points x and y of X there exist a Λ^s_δ-D set of X containing x but not y and a Λ^s_δ-D set of X containing y but not x.
(3) Λ^s_δ-D_2 if for any distinct pair of points x and y of X there exist disjoint Λ^s_δ-D sets G and E of X containing x and y, respectively.

Example 3.8. Let (X, τ) be a space such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{c\}, \{c, d\}, \{a, b\}, \{a, b, c\}\}$. Then

$$\delta SO(X, \tau) = \{\emptyset, X, \{a, b\}, \{c, d\}\}.$$

Also the family of all $(\Lambda, s\delta)$-closed and the family of all Λ^s_δ-D sets is the family $\{\emptyset, X, \{a, b\}, \{c, d\}\}$. Clearly, the space (X, τ) is not Λ^s_δ-D_2 and it is not Λ^s_δ-D_1.

Definition 9. A space X satisfies the $(\Lambda, s\delta)$*-property* if for any distinct pair of points in X, there is a $(\Lambda, s\delta)$-open set containing one of the points but not the other.

Proposition 3.9. *For a space X, the following statements are true:*

(1) X *is* Λ^s_δ-D_0 *if and only if it satisfies the* $(\Lambda, s\delta)$*-property.*
(2) X *is* Λ^s_δ-D_1 *if and only if it is* Λ^s_δ-D_2.

Proof. The sufficiency for (1) and (2) are obvious.

Necessity condition for (1). Let X be Λ_δ^s-D_0 so that for any distinct pair of points x and y of X at least one belongs to a Λ_δ^s-D set O. Therefore we choose $x \in O$ and $y \notin O$. Suppose $O = U - V$ for which $U \neq X$ and U and V are $(\Lambda, s\delta)$-open sets in X. This implies that $x \in U$. For the case that $y \notin O$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the space X satisfies the $(\Lambda, s\delta)$-property since $x \in U$ and $y \notin U$. For (ii), the space X also satisfies the $(\Lambda, s\delta)$-property since $y \in V$ but $x \notin V$.

Necessity condition for (2). Suppose that X is Λ_δ^s-D_1. It follows from the definition that for any distinct points x and y in X there exist Λ_δ^s-D sets G and E such that G containing x but not y and E containing y but not x. Let $G = U - V$ and $E = W - D$, where U, V, W and D are $(\Lambda, s\delta)$-open sets in X. By the fact that $x \notin E$, we have two cases, i.e. either $x \notin W$ or both W and D contain x. If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or(ii) $y \in U$ and $y \in V$. If (i) is the case, then it follows from $x \in U - V$ that $x \in U - (V \cup W)$, and also it follows from $y \in W - D$ that $y \in W - (U \cup D)$. Thus we have $U - (V \cup W)$ and $W - (U \cup D)$ which are disjoint. If (ii) is the case, it follows that $x \in U - V$, $y \in V$ and $(U - V) \cap V = \emptyset$. If $x \in W$ and $x \in D$, we have $y \in W - D$, $x \in D$ and $(W - D) \cap D = \emptyset$. This shows that X is Λ_δ^s-D_2. □

Definition 10. Let X be a space. A point $x \in X$ which has only X as the $(\Lambda, s\delta)$-neighborhood is called a Λ_δ^s-neat point.

Proposition 3.10. *A space X which has the $(\Lambda, s\delta)$-property is Λ_δ^s-D_1 if and only if X has no Λ_δ^s-neat point.*

Proof. (1) ⇒ (2). Since X is Λ_δ^s-D_1, so each point x of X is contained in a Λ_δ^s-D set $O = U - V$ and thus in U. By definition $U \neq X$. This implies that x is not a Λ_δ^s-neat point.

(2) ⇒ (1). Since X satisfies the $(\Lambda, s\delta)$-property, then for each distinct pair of points $x, y \in X$, at least one of them, choose x for example has a $(\Lambda, s\delta)$-neighborhood U containing x and not y. Thus U which is different from X is a Λ_δ^s-D set. If X has no Λ_δ^s-neat point, then y is not a Λ_δ^s-neat point. This means that there exists a $(\Lambda, s\delta)$-neighborhood V of y such that $V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is a Λ_δ^s-D set. Hence X is Λ_δ^s-D_1. □

Remark 3.11. It is clear that a space X that satisfies the $(\Lambda, s\delta)$-property is not Λ_δ^s-D_1 if and only if there is a unique Λ_δ^s-neat point in X. It is unique because if x and y are both Λ_δ^s-neat point in X, then at least one of them say x has a $(\Lambda, s\delta)$-neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

Definition 11. A space X is called a $(\Lambda, s\delta)$-symmetric if for x and y in X, $x \in y^{(\Lambda,s\delta)}$ implies $y \in x^{(\Lambda,s\delta)}$.

Proposition 3.12. *A space X is $(\Lambda, s\delta)$-symmetric if and only if for $x \in X$, $x^{(\Lambda,s\delta)} \subseteq E$ whenever $x \in E$ and E is $(\Lambda, s\delta)$-open in X.*

Proof. Assume that $x \in y^{(\Lambda,s\delta)}$ but $y \notin x^{(\Lambda,s\delta)}$. This means that the complement of $x^{(\Lambda,s\delta)}$ contains y. Therefore the set $\{y\}$ is a subset of the complement of $x^{(\Lambda,s\delta)}$. This implies that $y^{(\Lambda,s\delta)}$ is a subset of the complement of $x^{(\Lambda,s\delta)}$. Now the complement of $x^{(\Lambda,s\delta)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E$ and E is $(\Lambda, s\delta)$-open in X but $x^{(\Lambda,s\delta)}$ is not a subset of E. This means that $x^{(\Lambda,p)}$ and the complement of E are not disjoint. Let y belong to their intersection. Now we have $x \in y^{(\Lambda,s\delta)}$ which is a subset of the complement of E and $x \notin E$. But this is a contradiction. □

Proposition 3.13. *For a $(\Lambda, s\delta)$-symmetric space X, the following are equivalent:*

(1) *X is Λ_δ^s-D_0.*
(2) *X is Λ_δ^s-D_1.*

Proof. $(2) \Rightarrow (1)$: Proposition 3.9.
$(1) \Rightarrow (2)$: Since the space X is Λ_δ^s-D_0 by Proposition 3.9 satisfies the $(\Lambda, s\delta)$-property. Let $x \neq y$. We may assume that $x \in E \subset \{y\}^c$ for some $(\Lambda, s\delta)$-open set E in (X, τ). Then $x \notin y^{(\Lambda,s\delta)}$ and hence $y \notin x^{(\Lambda,s\delta)}$. Hence there exists a $(\Lambda, s\delta)$-open set F such that $y \in F \subset \{x\}^c$. Since every $(\Lambda, s\delta)$-open set is a Λ_δ^s-D set, we have that (X, τ) is a Λ_δ^s-D_1 space. □

4. $(\Lambda, s\delta)$-continuous functions

Definition 12. *Let (X,τ) and (Y,σ) two spaces. A function $f : (X,\tau) \to (Y,\sigma)$ is called $(\Lambda, s\delta)$-continuous if for each point $x \in X$ and every $(\Lambda, s\delta)$-open set V of Y containing $f(x)$, there exists a $(\Lambda, s\delta)$-open set U of X containing x such that $f(U) \subseteq V$.*

We recall that a function $f : (X,\tau) \to (Y,\sigma)$ is said to be δ-semicontinuous [3] if $f^{-1}(V)$ is δ-semiopen in (X,τ) for each δ-semiopen set V of (Y,σ).

Example 4.1. Let (X,τ) be as in Example 3.1. The function $f : (X,\tau) \to (X,\tau)$ for which $f(a) = c$, $f(b) = a$ and $f(c) = b$ is $(\Lambda, s\delta)$-continuous, but f is not continuous and also is not δ-semicontinuous.

Definition 13. Let X be a space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X. We say that the net $\{x_s, s \in S\}$ $(\Lambda, s\delta)$-converges to x if for every $(\Lambda, s\delta)$-open set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.

Proposition 4.2. *Let X be a space and $A \subseteq X$. A point $x \in A^{(\Lambda,s\delta)}$ if and only if there exists a net $\{x_s, s \in S\}$ of A which $(\Lambda, s\delta)$-converges to x.*

Proof. The existence of such a net clearly implies that $x \in A^{(\Lambda,s\delta)}$. Let us suppose that $x \in A^{(\Lambda,s\delta)}$ and let us denote by $\mathcal{U}$ the set of all $(\Lambda, s\delta)$-open subsets U of X such that $x \in U$ directed by the relation $\leq$ i.e., let us define that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. Easily, the net $\{x_U, U \in \mathcal{U}\}$, where x_U is an arbitrary point of $A \cap U$, $(\Lambda, s\delta)$-converges to x. □

Proposition 4.3. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following are equivalent:*

(1) *f is $(\Lambda, s\delta)$-continuous.*
(2) *$f^{-1}(V)$ is $(\Lambda, s\delta)$-open in (X,τ) for every $(\Lambda, s\delta)$-open set V of (Y,σ).*
(3) *$f^{-1}(F)$ is $(\Lambda, s\delta)$-closed in (X,τ) for every $(\Lambda, s\delta)$-closed set F of (Y,σ).*
(4) *$f(A^{(\Lambda,s\delta)}) \subset [f(A)]^{(\Lambda,s\delta)}$ for each subset A of X.*
(5) *$[f^{-1}(B)]^{(\Lambda,s\delta)} \subset f^{-1}(B^{(\Lambda,s\delta)})$ for each subset B of Y.*

(6) *For every $x \in X$ and every net $\{x_s, s \in S\}$ of X which $(\Lambda, s\delta)$-converges to x in X, the net $\{f(x_s), s \in S\}$ $(\Lambda, s\delta)$-converges to $f(x)$ in Y.*

Proof. (1) ⇒ (2): Let V be any $(\Lambda, s\delta)$-open set of (Y, σ) and $x \in f^{-1}(V)$. Since f is $(\Lambda, s\delta)$-continuous, there exists a $(\Lambda, s\delta)$-open set U_x containing x such that $f(U_x) \subset V$. Therefore, we have $x \in U_x \subset f^{-1}(V)$ and hence $f^{-1}(V) = \cup\{U_x \mid x \in f^{-1}(V)\}$. By Proposition 3.5, $f^{-1}(V)$ is $(\Lambda, s\delta)$-open in (X, τ).

(2) ⇒ (1): This is obvious.

(2) ⇔ (3): This is obvious from Definition 5.

(3) ⇒ (4): Let A be any subset of X. Since $A \subset f^{-1}([f(A)]^{(\Lambda,s\delta)})$, by Proposition 3.7 we have $A^{(\Lambda,s\delta)} \subset f^{-1}([f(A)]^{(\Lambda,s\delta)})$ and hence $f(A^{(\Lambda,s\delta)}) \subset [f(A)]^{(\Lambda,s\delta)}$.

(4) ⇒ (5): Let B be any subset of Y. By (4) and Proposition 3.7, we have $f([f^{-1}(B)]^{(\Lambda,s\delta)}) \subset [f(f^{-1}(B))]^{(\Lambda,s\delta)} \subset B^{(\Lambda,s\delta)}$ and hence $[f^{-1}(B)]^{(\Lambda,s\delta)} \subset f^{-1}(B^{(\Lambda,s\delta)})$.

(5) ⇒ (3): Let F be any $(\Lambda, s\delta)$-closed set in (Y, σ). By Proposition 3.7,

$$[f^{-1}(F)]^{(\Lambda,s\delta)} \subset f^{-1}(F^{(\Lambda,s\delta)}) = f^{-1}(F)$$

and

$$[f^{-1}(F)]^{(\Lambda,s\delta)} \subset f^{-1}(F).$$

Therefore, we obtain $[f^{-1}(F)]^{(\Lambda,s\delta)} = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $(\Lambda, s\delta)$-closed in (X, τ).

(1) ⇒ (6): Let $x \in X$ and $\{x_s \mid s \in S\}$ be a net $(\Lambda, s\delta)$-converging to x. For any $(\Lambda, s\delta)$-open set of (Y, σ) containing $f(x)$, by (1) there exists a $(\Lambda, s\delta)$-open set U of X containing x such that $f(U) \subset V$. Since $\{x_s \mid s \in S\}$ converges to x, there exists $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$. Therefore, $f(x_s) \in V$ for any $s \geq s_0$ and the net $\{f(x_s) \mid s \in S\}$ $(\Lambda, s\delta)$-converges to $f(x)$.

(6) ⇒ (1): Let us suppose that there exists a point $x \in X$ and a $(\Lambda, s\delta)$-open neighbourhood V of $f(x)$ such that for every $(\Lambda, s\delta)$-open set U of X containing x such that $f(U) \nsubseteq V$. Then for every $(\Lambda, s\delta)$-open set U of X such that $x \in U$, we choose an element $x_U \in U$ such that $f(x_U) \notin V$. Let $\mathcal{U}$ be the set of all $(\Lambda, s\delta)$-open sets U of X containing x, directed by the relation $\subseteq$ i.e.,

let us define that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. Observe that the net $\{x_U, U \in \mathcal{U}\}$ $(\Lambda, s\delta)$-converges to x but the net $\{f(x_U), U \in \mathcal{U}\}$ does not $(\Lambda, s\delta)$-converge to $f(x)$ which is a contradiction. Thus there exists a $(\Lambda, s\delta)$-open set U of X such that $x \in U$ and $f(U) \subseteq V$. □

Clearly, if a function $f : (X, \tau) \to (Y, \sigma)$ is δ-semicontinuous, then $f : (X, \tau^{\Lambda_\delta^s {}^*}) \to (Y, \sigma^{\Lambda_\delta^s {}^*})$ is continuous.

Indeed let V be any $\Lambda_\delta^s {}^*$-set of (Y, σ). Then

$$V = \Lambda_\delta^s {}^*(V) = \cup\{W \mid V \supset W \in \delta SC(Y, \sigma)\}.$$

Since f is δ-semicontinuous, we have

$$\begin{aligned} f^{-1}(V) &= \cup\{f^{-1}(W) \mid f^{-1}(V) \supset f^{-1}(W) \in \delta SC(X, \tau)\} \\ &\subset \cup\{U \mid f^{-1}(V) \supset U \in \delta SC(X, \tau)\} \\ &= \Lambda_\delta^s {}^*(f^{-1}(V)). \end{aligned}$$

On the other hand, by Lemma 2.3, we have $f^{-1}(V) \supset \Lambda_\delta^s {}^*(f^{-1}(V))$ and hence $f^{-1}(V)$ is a $\Lambda_\delta^s {}^*$-set of (X, τ). □

By the above we have that if a function $f : (X, \tau) \to (Y, \sigma)$ is δ-semicontinuous, then $f : (X, \tau^{\Lambda_\delta^s}) \to (Y, \sigma^{\Lambda_\delta^s})$ is continuous.

Proposition 4.4. *If $f : (X, \tau) \to (Y, \sigma)$ is a δ-semicontinuous function, then it is $(\Lambda, s\delta)$-continuous.*

Proof. Let F be any $(\Lambda, s\delta)$-closed set of (Y, σ). Then there exist a Λ_δ^s-set T and a δ-semiclosed set D such that $F = T \cap D$. Since f is δ-continuous, $f^{-1}(D)$ is δ-semiclosed and $f^{-1}(T)$ is a Λ_δ^s-set of (X, τ). Therefore, $f^{-1}(F) = f^{-1}(T) \cap f^{-1}(D)$ is a $(\Lambda, s\delta)$-closed set of (X, τ). By Proposition 4.3, f is $(\Lambda, s\delta)$-continuous. □

5. $(\Lambda, s\delta)$-compactness and $(\Lambda, s\delta)$-connectedness

Definition 14. A space X is called $(\Lambda, s\delta)$-compact (resp. δ-semicompact) if every cover of X by $(\Lambda, s\delta)$-open (resp. δ-semiopen) sets has a finite subcover.

Proposition 5.1. *A space X is $(\Lambda, s\delta)$-compact (resp. δ-semicompact) if and only if for every family $\{A_i : i \in I\}$ of $(\Lambda, s\delta)$-closed (resp. δ-semiclosed) sets in X satisfying $\cap\{A_i : i \in I\} = \emptyset$, there is a finite subfamily $A_{i_1}, ..., A_{i_n}$ with $\cap\{A_{i_k} : k = 1, ..., n\} = \emptyset$.*

Proof. Straightforward. □

Proposition 5.2. *For a space (X, τ), the following hold:*

(1) *If $(X, \tau^{\Lambda_\delta^s})$ is compact, then (X, τ) is δ-semicompact.*
(2) *If (X, τ) is $(\Lambda, s\delta)$-compact, then (X, τ) is δ-semicompact.*
(3) *If (X, τ) is $(\Lambda, s\delta)$-compact, then $(X, \tau^{\Lambda_\delta^{s\,*}})$ is compact.*

Proof. (1) Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any δ-semiopen cover of X. By Lemma 2.1 V_α is a Λ_δ^s-set for each $\alpha \in \nabla$. Moreover, by the compactness of $(X, \tau^{\Lambda_\delta^s})$ there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that (X, τ) is δ-semicompact.

(2) Let $\{F_\alpha \mid \alpha \in \nabla\}$ be a family of δ-semiopen sets of (X, τ) such that $\cap\{F_\alpha \mid \alpha \in \nabla\} = \emptyset$. Every δ-semiclosed set is $(\Lambda, s\delta)$-closed set for each $\alpha \in \nabla$. By Proposition 5.1, there exists a finite subset ∇_0 of ∇ such that $\cap\{F_\alpha \mid \alpha \in \nabla_0\} = \emptyset$. Therefore (X, τ) is δ-semicompact.

(3) Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of X by $\Lambda_\delta^{s\,*}$-sets of (X, τ). Since $V_\alpha = V_\alpha \cup \emptyset$ and the empty set is δ-semiopen, by Lemma 2.1 each V_α is $(\Lambda, s\delta)$-open in (X, τ). Since (X, τ) is $(\Lambda, s\delta)$-compact, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that $(X, \tau^{\Lambda_\delta^{s\,*}})$ is compact. □

Proposition 5.3. *The following statements are true:*

(1) *If $f : (X, \tau) \to (Y, \sigma)$ is a $(\Lambda, s\delta)$-continuous surjection and (X, τ) is a $(\Lambda, s\delta)$-compact space, then (Y, σ) is $(\Lambda, s\delta)$-compact.*
(2) *The $(\Lambda, s\delta)$-compactness is preserved by δ-continuous surjections.*

Proof. (1) Let $\{V_i \mid i \in I\}$ be any cover of Y by $(\Lambda, s\delta)$-open sets of (Y, σ). Since f is $(\Lambda, s\delta)$-continuous, by Proposition 4.3 $\{f^{-1}(V_i) \mid i \in I\}$ is a cover of X by $(\Lambda, s\delta)$-open sets of (X, τ). By $(\Lambda, s\delta)$-compactness of (X, τ), there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_i) \mid i \in I_0\}$. Since f is surjective, we obtain $Y = f(X) = \cup_{i \in I_0} V_i$. This shows that (Y, σ) is $(\Lambda, s\delta)$-compact.

(2) This is an immediate consequence of (1) and Proposition 4.3. □

Definition 15. A space X is called $(\Lambda, s\delta)$-connected if X cannot be written as a disjoint union of two non-empty $(\Lambda, s\delta)$-open sets.

Proposition 5.4. *For a space X, the following statements are equivalent:*

(1) *The space X is $(\Lambda, s\delta)$-connected .*
(2) *The only subsets of X, which are both $(\Lambda, s\delta)$-open and $(\Lambda, s\delta)$-closed are the empty set $\emptyset$ and X.*

Proof. Straightforward. □

Proposition 5.5. *The following statements are true:*

(1) *A space (X, τ) is $(\Lambda, s\delta)$-connected, then $(X, \tau^{\Lambda^s_\delta})$ is connected.*
(2) *If $f : (X, \tau) \to (Y, \sigma)$ is a $(\Lambda, s\delta)$-continuous surjection and (X, τ) is $(\Lambda, s\delta)$-connected, then (Y, σ) is $(\Lambda, s\delta)$-connected.*
(3) *The $(\Lambda, s\delta)$-connectedness is preserved by δ-continuous surjections.*

Proof. (1) Suppose that $(X, \tau^{\Lambda^s_\delta})$ is not connected. There exist nonempty Λ^s_δ-sets G, H of (X, τ) such that $G \cap H = \emptyset$ and $G \cup H = X$. By Proposition 3.3, G and H are $(\Lambda, s\delta)$-closed sets. This shows that (X, τ) is not $(\Lambda, s\delta)$-connected.

(2) Suppose that (Y, σ) is not $(\Lambda, s\delta)$-connected. There exist nonempty $(\Lambda, s\delta)$-open sets G, H of Y such that $G \cap H = \emptyset$ and $G \cup H = Y$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and $f^{-1}(G) \cup f^{-1}(H) = X$. Moreover, $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty $(\Lambda, s\delta)$-open sets of (X, τ). This shows that (X, τ) is not $(\Lambda, s\delta)$-connected. Therefore, (Y, σ) is $(\Lambda, s\delta)$-connected.

(3) This is an immediate consequence of Proposition 4.3. □

6. Some questions

1. Does there exist a $(\Lambda, s\delta)$-closed set which is not Λ^s_δ-set or δ-semiclosed?

2. Does there exist a non $(\Lambda, s\delta)$-symmetric space which is Λ^s_δ-D_1 and it is not Λ^s_δ-D_0?

3. Does there exist a $(\Lambda, s\delta)$-continuous function which is not δ-continuous?

4. Does there exist a δ-semicompact space (X, τ) such that the space $(X, \tau^{\Lambda^s_\delta})$ is not compact?

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References

1. M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Sci. Assiut Univ. 12(1983), 77-90.
2. A. Anderijević, Semi-preopen sets, Mat. Vesnik 38(1986), 24-32.
3. M. Caldas, D.N. Georgiou, S. Jafari and T. Noiri, *More on δ-semiopen sets*, Note di Matematica, 22(2)(2003), 1-14.
4. H. Corson and E. Michael, Metrizability of certain countable unions, Illinois J. Math. 8(1964), 351-360.
5. R. Devi, K. Bhuvaneswari and M. Maki, Weak form on $g\rho$-closed sets, where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$, and digital planes, Mem. Fac. Sci. Kochi Univ. (Math.) 25(2004), 37-54.
6. J. Dontchev, M. Fujimoto, H. Maki, T. Noiri and S. Takigawa, The topological structure and groups of digital n-spaces (submitting).
7. J. Dontchev and M. Ganster, On δ-generalized closed sets and $T_{\frac{3}{4}}$, Mem. Fac. Sci. Kochi Univ. (Math.) 17(1996), 15-31.
8. N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70**(1963), 36–41.
9. G. Di Maio, *A separation axiom weaker than R_0*, Indian J. Pure Appl. Math., **16**(4)(1985), 373–375.
10. A. Mashhour, M. Abd El-Monsef and S. El-Deeb, On pre-continuous and weak pre-continuous mappings , Proc. Math. Phys. Soc. Egypt 53(1982), 47-53.
11. O. Njåstad, On some classes of nealy open sets, Pacific J. Math.15(1965), 961-970.
12. J. H. Park, B. Y. Lee and M. J. Son, *On δ-semiopen sets in topological space*, J. Indian Acad. Math., **19**(1997), 59–67.
13. N. V. Veličko, *H-closed topological spaces*,Mat. Sb., **70**(1966), 98–112; English transl., in Amer. Math. Soc. Transl. **78**(2)(1968), 103–118.

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