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DECOMPOSITION OF UC SPACES II THE UNIFORM CASE

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Abstract. There is a considerable literature on UC metric or UC-uniform spaces, i.e. spaces in which continuity equals uniform continuity. In this paper we study this topic in depth using proximity and the authors' recent work concerning decomposition of UC metric spaces. We study this problem in Tychonoff spaces endowed with a uniformity and a proximity, where proximal continuity and uniform continuity are not equivalent as in metric spaces. So, in this setting the solution to the UC problem is more intricate than in metric spaces. We introduce a new proximity, called Cauchy or C-proximity, and new classes of spaces, namely C-CAU spaces and CAU-PC spaces. These new classes are useful to offer a decomposition of the characteristic property PC spaces, in which real valued functions are proximally continuous.

1. Introduction

It is well known that a continuous function on a compact uniform space to an arbitrary uniform space is uniformly continuous and that the converse is not true. A metric space on which, continuity is equivalent to uniform continuity, is called a UC space. There is a considerable literature on characterizations of UC spaces in

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the setting of a metric space beginning with Nagata [12] followed by Atsuji [1], Rainwater [15], Toader [19] and others. There exists somewhat less literature in the setting of a uniform space, see Atsuji [2]. However, there is very little literature on analogous PC spaces (continuity equals p-continuity) involving proximity (see [7], [4] and [14]). In this paper we study the subject in the setting of a Tychonoff space equipped with a uniformity and a proximity compatible with the underlying topology. Mostly, we deal with *Efremovic* or *EF-proximities*. The word "proximity" refers to EF-proximity and other proximities, which occur sometimes, will be indicated. Our results generalize the existing results and throw considerable light on them.

We also study decomposition of PC spaces similar to our recent work on decomposition of UC metric spaces ([3]). Incidentally, we get a new class of spaces weaker than the class of complete uniform spaces, but coinciding in the setting of metric spaces with the class of complete metric spaces. This generalizes the recent results of Janos in the setting of a uniform space ([8]).

We use the following notation:

$(X,\mathcal{U})$, $(Y,\mathcal{V})$ denote separated uniform spaces, with compatible EF-proximities δ and η, respectively.

$(X^\star,\mathcal{U}^\star)$, $(Y^\star,\mathcal{V}^\star)$ denote the completions of $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, respectively.

For any subset E of $(X,\mathcal{U})$ $cl_X E$, $intE$ and E^c stand for the closure, the interior and the complement of E in $(X,\mathcal{U})$, respectively.

X' = the set of all limit points of X, known as the *derived set* of X.

$C(X,Y)$ = the family of all continuous functions on X to Y.

$P(X,Y)$ = the family of all p-continuous functions on X to Y.

$U(X,Y)$ = the family of all u-continuous functions on X to Y.

$C(X)$ = the family of all real valued continuous functions.

$P(X)$ = the family of all real valued p-continuous functions.

$U(X)$ = the family of all real valued u-continuous functions.

$CAU(X)$ = the family of all real valued Cauchy maps, i.e. maps that preserve Cauchy nets.

$C^\star(X)$ (respectively, $U^\star(X)$, $P^\star(X)$ and $CAU^\star(X)$) is the subfamily of $C(X)$ (respectively, $U(X)$, $P(X)$ and $CAU(X)$) consisting of all bounded functions.

The following conditions are considered:

> **UC**: $C(X) = U(X)$ and $\mathcal{U}$ is the coarsest uniformity for which this holds ([2]).
> **PC**: $C(X) = P(X)$.
> **CAU-PC**: $CAU(X) = P(X)$.

All spaces in this paper are Tychonoff, i.e. completely regular and Hausdorff. Further, for the sake of simplicity, we will assume that all entourages are symmetric.

We give below a short bibliography on UC spaces and the interested reader will find further references in the cited articles.

Whereas, standard references on proximity include [5], [13], [11], [18] and [20].

Let α be a binary relation on the power set of X, where $A\alpha B$ means "A is near B" and $A\not\alpha B$ means its negation "A is far from B".

Definition 1.1. We consider several relations on the power set of a Tychonoff space X equipped with a compatible uniformity $\mathcal{U}$ which are compatible Efremovic or compatible Lodato proximities.

(a) We use the symbol δ to denote the *EF-proximity* induced by $\mathcal{U}$, viz.
$A\delta B$ if, and only if, for each $U \in \mathcal{U}$, $U[A] \cap B \neq \emptyset$.

(b) The symbol δ_F denotes the *fine EF-proximity* on X, viz.
$A\not\delta_F B$ if, and only if, there is an $f \in C^\star(X)$ with $f(A) = 0$ and $f(B) = 1$.

(c) The symbol δ_0 denotes the *Wallman* or *fine LO-proximity*:
$A\delta_0 B$ if, and only if, $clA \cap clB \neq \emptyset$.

We note that X is normal if, and only if, δ_0 is EF (Urysohn's Lemma).

Definition 1.2. (a) If α is a compatible proximity, then $A \ll_\alpha B$ means $A\not\alpha X \setminus B$ and B is called an *α-nbhd* of A.

The family of all α-nbhds of A is denoted by $\mathcal{N}(A, \alpha)$. Note that the δ_0-nbhds of A coincide with the ordinary nbhds of A.

(b) If α and β are two proximities on X, then α is *coarser* than β (or equivalently β is *finer* than α), written $\alpha \leq \beta$, if, and only if,

$A\beta B \Rightarrow A\alpha B$ (or equivalently $A\not\alpha B \Rightarrow A\not\beta B$). Thus, each α-nbhd of A is also a β-nbhd of A but, in general, the converse is not true.

Remark 1.3. (a) We recall that a net (x_λ) in the uniform space $(X,\mathcal{U})$ is *Cauchy* if for each $U \in \mathcal{U}$, eventually, for all λ, λ', $(x_\lambda, x_{\lambda'}) \in U$.

(b) Two nets (y_μ), (z_ν) in the uniform space $(X,\mathcal{U})$ form a *Cauchy-pair* if for each $U \in \mathcal{U}$, eventually, for all μ, ν, $(y_\mu, z_\nu) \in U$ (cf. [16] where the term *equivalent* is used).

It is obvious that the nets (y_μ), (z_ν) form a Cauchy-pair if, and only if, they are both Cauchy and they converge to the same point in the completion $X^\star$.

It is easy to see that a net (x_λ) is Cauchy if, and only if, any two of its subnets form a Cauchy-pair.

(c) A net (x_λ) of distinct points in the uniform space $(X,\mathcal{U})$ is *pseudo-Cauchy* if for each $U \in \mathcal{U}$, frequently, there exist λ, λ' with $(x_\lambda, x_{\lambda'}) \in U$.

Pseudo-Cauchy-pair can be defined similarly to Cauchy-pair.

(d) Two nets (y_λ), (z_λ) in the uniform space $(X,\mathcal{U})$ are said to be a *parallel-pair* if they have the same directed set Λ and for each $U \in \mathcal{U}$, eventually, for all λ, $(y_\lambda, z_\lambda) \in U$ (see [3]).

Obviously, a parallel pair is a pseudo-Cauchy pair. The converse, in general fails (see [3]).

(e) If (y_μ), (z_ν) are both Cauchy nets, then (y_μ), (z_ν) form a pseudo-Cauchy pair if, and only if, they form a Cauchy-pair.

(f) If the nets (y_μ), (z_ν) form a pseudo-Cauchy pair and their ranges are both totally bounded, then there are subnets (y_{μ_k}), (z_{ν_l}) of (y_μ), (z_ν) respectively, which form a Cauchy-pair.

Definition 1.4. The symbol δ_c denotes the *Cauchy* or *C-proximity* on X:

> $A\delta_c B$ if, and only if, there is a Cauchy-pair of nets (y_μ), (z_ν) with $y_\mu \in A$ and $z_\nu \in B$.

Remark 1.5. (a) Obviously, $A\delta_c B \Leftrightarrow cl_{X^\star} A \cap cl_{X^\star} B \neq \emptyset$, where $X^\star$ is the completion of X.

(b) The C-proximity δ_c is LO since it is the subspace proximity induced by the Wallman proximity on $X^\star$. Hence, if the completion $X^\star$ is normal, then δ_c is EF.

(c) We note that $\mathbb{R}^{\mathbb{R}}$, with the usual product uniformity, is a complete non-normal space and so $\delta_c = \delta_0$ is a LO-proximity that is not EF.

(d) It is easy to check that $A\delta B \Leftrightarrow$ there is a parallel-pair of nets (y_λ), (z_λ) with $y_\lambda \in A$ and $z_\lambda \in B$. Hence, the various proximities defined so far on X, satisfy the relations:

(i) $\delta \leq \delta_c \leq \delta_0$; $\quad$ *(ii)* $\delta \leq \delta_F \leq \delta_0$.

Moreover, in general, δ_c and δ_F are not comparable. But if δ_c is EF, then

(iii) $\delta \leq \delta_c \leq \delta_F \leq \delta_0$.

Question 1.6. *Is it possible for δ_c to be EF even when $X^\star$ is not normal?*

Definition 1.7. (a) A function $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a (*pseudo-*) *Cauchy* map if f preserves (respectively, pseudo-) Cauchy nets (cf. [16] and [17] where the term *Cauchy-regular* is used).

(b) A function $f\colon (X,\alpha) \to (Y,\beta)$ is *p-continuous* if $A\alpha B$ implies $f(A)\beta f(B)$ (see [5] or [13]).

(c) A function $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is *u-continuous* if for each $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $(x,y) \in U \Rightarrow (f(x), f(y)) \in V$.

Remark 1.8. (a) A function $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a Cauchy map if, and only if, f preserves Cauchy pair of nets.

(b) A function $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is u-continuous $\Leftrightarrow$ f preserves parallel-pair of nets. Thus, if $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is u-continuous and δ and η are the proximities induced by $\mathcal{U}$ and $\mathcal{V}$ respectively, then $f\colon (X,\delta) \to (Y,\eta)$ is p-continuous. The converse in general fails.

(c) However, if $f\colon (X,\delta) \to (Y,\eta)$ is p-continuous and either the uniformity of the domain is pseudo-metrizable or that of the range space is totally bounded, then $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is u-continuous (see [13, Corollary 12.20 and Corollary 12.12 respectively]). As a byproduct, we have $P^\star(X) = U^\star(X)$ since the class of totally bounded sets and the class of bounded sets on $\mathbb{R}$ coincide.

(d) The following always occur:

(1) $U(X,Y) \subset P(X,Y)$;
(2) $P(X,Y) \subset CAU(X,Y)$ (cf. [10, Corollary 5]);
(3) $CAU(X,Y) \subset C(X,Y)$ (cf. [17, Proposition 1]).

2. PC spaces:

We wish to emphasize that all proximities in this section are Efremovic unless otherwise indicated. The situation is quite different if other generalized proximities are considered.

We call a proximity space (X,δ) a *PC-space* if $C(X,Y)=P(X,Y)$ for all proximity spaces (Y,η). In the class of metric spaces PC = UC, but this is no longer true in uniform spaces. Since proximity structures lie between topological and uniform structures, the study of PC spaces is useful in dealing with uniform spaces. Material in this section and the next is related to that in [2], which studies the uniform case but not the proximal one, which is discussed in [7] and [4].

Theorem 2.1. *Let $\{A_n\}$, $\{B_n\}$, where $n \in \mathbb{N}$, be countable families of subsets of X. In a proximity space (X,δ) the following are equivalent:*

(a) *$C(X,Y) = P(X,Y)$ for each proximity space (Y,η);*
(b) *$C(X) = P(X)$;*
(c) *if $A_n \ll_{\delta_F} B_n$ and $\{B_n\}$ is a discrete family, then there is an $f \in P(X)$ such that $f(A_n) = n$ and $f(X \setminus \bigcup_{n\in\mathbb{N}} B_n) = 0$;*
(d) *if there is an $f \in C(X)$ such that $f(A_n) = n$, then there is a $g \in P(X)$ such that $g(A_n) = n$;*
(e) *for each pair of disjoint zero sets A, B there is an $f \in P(X)$ such that $f(A) = 0$ and $f(B) = 1$;*
(f) *each pair of disjoint zero sets A, B satisfy $A \not\delta B$;*
(g) *$\delta = \delta_F$;*
(h) *$C^\star(X) = P^\star(X)$.*

Proof. $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$. Since $A_n \ll_{\delta_F} B_n$ for each $n \in \mathbb{N}$, there is an $f_n \in C(X)$ such that $f_n(A_n) = n$ and $f_n(X \setminus B_n) = 0$. Since $\{B_n\}$ is discrete, there is an $f \in C(X) = P(X)$, constructed from $\{f_n\}$ satisfying $f(A_n) = n$ and $f(X \setminus \bigcup_{n\in\mathbb{N}} B_n) = 0$ (cf. [6, Problem 1A, Page 20]).

$(c) \Rightarrow (d)$. Let $B_n = f^{-1}([n-\frac{1}{3}, n+\frac{1}{3}])$. The result follows from (c).

$(d) \Rightarrow (e)$. It is known that if A, B are disjoint zero sets, then there is an $f \in C(X)$ such that $f(A) = 0$ and $f(B) = 1$. By (d) the result follows.

$(e) \Rightarrow (f)$ and $(f) \Rightarrow (g)$ are obvious.

$(g) \Rightarrow (h)$. If (h) is not true, there are sets A, B in X and $f \in C^\star(X) \setminus P^\star(X)$ with $A\delta B$ but $f(A) \not\!\delta f(B)$ in $\mathbb{R}$. Then, there is a $g\colon \mathbb{R} \to \mathbb{R}$ satisfying $g(f(A)) = 0$, $g(f(B)) = 1$, a contradiction.

$(h) \Rightarrow (a)$ If there is an $f \in C(X,Y) \setminus P(X,Y)$, then there are sets A, B in X such that $A\delta B$ but $f(A) \not\!\delta f(B)$ in Y. Then, there is a $g\colon Y \to [0,1]$ satisfying $g(f(A)) = 0$, $g(f(B)) = 1$, i.e. $g \circ f \in C^\star(X) \setminus P^\star(X)$, a contradiction. □

Remark 2.2. (a) In (2.1)(f) we cannot replace zero sets by closed sets. However, we can do so if we are dealing with LO-proximities rather than with EF ones.

(b) If the space (X, δ) is normal, then $\delta_F = \delta_0$ (by Urysohn's Lemma) and suitable modifications can be made in the above result. In this case, for example, we can replace zero sets by closed sets in (2.1)(f).

(c) The space $\mathbf{W}(\omega_1)$ of all ordinals less than the first uncountable ordinal ω_1 is normal and has a unique compatible proximity (and uniformity). It is not complete (cf. [20, Page 10] or [6, Page 238]). So $C(X) = P(X) = U(X)$, but the derived set X' is not compact as it is in the case of a metric space (see [1], [15]).

3. UC spaces:

In this section we consider the uniform case. This case is a bit tricky since $C(X) = U(X)$ does not imply $C(X,Y) = U(X,Y)$ for each uniform space Y and $C^\star(X) = U^\star(X)$ does not imply $C(X) = U(X)$.

So, we have three cases:

(a) **strong UC:** $C(X,Y) = U(X,Y)$ for each uniform space Y.
This case has a simple solution viz. $\mathcal{U}$ must be the fine uniformity.

(b) **Čech UC:** $C^\star(X) = U^\star(X)$.
In this case $\delta = \delta_F$ and this being a proximity property is already studied in Theorem 2.1.

(c) **UC:** $C(X) = U(X)$.

This is an interesting case which generalizes the metric case. It requires much deeper analysis ([2]). And this is the case that we will study in this section.

Theorem 3.1. *Let $\{A_n\}$, $\{B_n\}$, where $n \in \mathbb{N}$, be countable families of subsets of X. For a uniform space $(X, \mathcal{U})$ the following are equivalent:*

(*a*) $C(X) = U(X)$;

(*b*) *if $A_n \ll_{\delta_F} B_n$ and $\{B_n\}$ is a discrete family, then there is an $U \in \mathcal{U}$, such that $U[A_n] \subset B_n$ for each $n \in \mathbb{N}$;*

(*c*) *if $\{A_n\}$ is a sequence of sets such that $f(A_n) = n$ for some $f \in C(X)$, then there is a $U \in \mathcal{U}$, such that for $n \neq m$, $U[A_n] \cap A_m = \emptyset$.*

Proof. $(a) \Leftrightarrow (b)$ is due to Atsuji ([2]).

$(b) \Rightarrow (c)$. Let $\{A_n\}$ be a sequence such that there is an $f \in C(X)$ with $f(A_n) = n$. Let $B_n = f^{-1}([n - \frac{1}{3}, n + \frac{1}{3}])$. Then, $U = \{(x, y)$ such that $\mid f(x) - f(y) \mid < \frac{1}{3}\} \in \mathcal{U}$, $U[A_n] \subset B_n$ and for $n \neq m$, we have $U[A_n] \cap A_m = \emptyset$.

$(c) \Rightarrow (b)$. Let $A_n \ll_{\delta_F} B_n$ and $\{B_n\}$ a discrete family. For each $n \in \mathbb{N}$ there is $C_n \subset X$ with $A_n \ll_{\delta_F} C_n \ll_{\delta_F} B_n$. Since $C_n \ll_{\delta_F} B_n$, there exists an $f_n \in C(X)$ with $f(C_n) = n$ and $f(X \setminus B_n) = 0$. Since $\{B_n\}$ is discrete, the function $f \colon X \to \mathbb{R}$ defined by $f(x) = f_n(x)$, if $x \in B_n$ for some n, $f(x) = 0$ otherwise, is continuous. By (c) there is a $U_1 \in \mathcal{U}$ such that for $n \neq m$ we have $U_1[C_n] \cap C_m = \emptyset$. Again, since $A_n \ll_{\delta_F} C_n$, there exists a $g_n \in C(X)$ with $g(A_n) = 1$ and $g(X \setminus C_n) = 0$. Since $\{B_n\}$ is discrete and $C_n \subset B_n$ because $C_n \ll_{\delta_F} B_n$, we have that also the family $\{C_n\}$ is discrete. Thus the function $g \colon X \to \mathbb{R}$ defined by $g(x) = g_n(x)$, if $x \in C_n$ for some n, $g(x) = 0$ otherwise, is continuous. Again, by (c) there is a $U_2 \in \mathcal{U}$ such that $U_2[\bigcup A_n] \cap (X \setminus \bigcup C_n) = \emptyset$. Let $U \in \mathcal{U}$ with $U \subset U_1 \cap U_2$. Then, we have $U[A_n] \subset B_n$ for each $n \in \mathbb{N}$. In fact, let $n \in \mathbb{N}$ be fixed and $y \in U[\bigcup A_n]$. Then, $y \in \bigcup C_n$ (because $U[A_n] \subset U_2[\bigcup A_n]$ and $U_2[\bigcup A_n] \cap (X \setminus \bigcup C_n) = \emptyset$). Thus $y \in C_m$ for some $m \in \mathbb{N}$. We claim $m = n$. Assume not. Let $x \in A_n$ such that $(x, y) \in U$. Then, $x \in C_n$ (because $A_n \subset C_n$ since $A_n \ll_{\delta_F} C_n$). Hence $y \in U_1[C_n] \cap C_m$ for $n \neq m$, which contradicts $U_1[C_n] \cap C_m = \emptyset$ for $n \neq m$. □

Theorem 3.2. *Consider the following statements concerning a uniform space* $(X,\mathcal{U})$.

(a) $\mathcal{U}$ *is fine;*
(b) $C(X,Y) = U(X,Y)$ *for each uniform space* Y*;*
(c) $C(X) = U(X)$*;*
(d) *if* $\{A_n\}$ *is a sequence of sets such that* $f(A_n) = n$ *for some* $f \in C(X)$*, then there is a* $U \in \mathcal{U}$*, such that for* $n \neq m$, $U[A_n] \cap A_m = \emptyset$*;*
(e) *for each pair of disjoint zero sets* A, B *of* X*, there is a u-continuous function* $f\colon X \to [0,1]$ *such that* $f(A) = 0$ *and* $f(B) = 1$*;*
(f) *for each pair of disjoint zero sets* A, B *of* X*, there is a* $U \in \mathcal{U}$*, such that* $U[A] \cap B = \emptyset$*;*
(g) $\delta = \delta_F$*;*
(h) $C^\star(X) = U^\star(X)$.

Then, (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) *and none of the arrows can be reversed.*

Theorem 3.3. *Consider the following statements concerning a* **normal** *uniform space* $(X,\mathcal{U})$.

(a) X' *is compact and for each* $U \in \mathcal{U}$ *there is a* $V \in \mathcal{U}$ *such that* $(U[X'])^c$ *is* V*-discrete;*
(b) $\mathcal{U}$ *is Lebesgue, i.e. every open cover of* X *has a refinement* $\{U[x] : x \in X\}$ *for some* $U \in \mathcal{U}$*;*
(c) $\mathcal{U}$ *is fine;*
(d) $C(X,Y) = U(X,Y)$ *for each uniform space* Y*;*
(e) $C(X) = U(X)$*;*
(f) *if* $\{A_n\}$ *is a sequence of sets such that* $f(A_n) = n$ *for some* $f \in C(X)$*, then there is a* $U \in \mathcal{U}$*, such that for* $n \neq m$, $U[A_n] \cap A_m = \emptyset$*;*
(g) $\delta = \delta_0$*;*
(h) $C^\star(X) = U^\star(X)$*;*
(i) $clA \cap clB = \emptyset$ *iff there is* $U \in \mathcal{U}$ *such that* $U[A] \cap B = \emptyset$*;*
(j) *the Vietoris topology on* $CL(X)$ *is coarser than the Hausdorff-Bourbaki uniform topology;*
(k) *every pseudo-Cauchy sequence in* X *of distinct points has a cluster point.*

Then, $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Rightarrow (e) \Leftrightarrow (f) \Rightarrow (g) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (j) \Rightarrow (k)$. *Moreover,* $(j) \not= (k)$ *is not known, and none of the arrows can be reversed.*

Furthermore, if X *is metrizable and* $\mathcal{U}$ *is the compatible metric uniformity, then all the above statements are equivalent.*

4. C-CAU Spaces:

As we have already observed in Remark 2.2(c), the space $X = \mathbf{W}(\omega_1)$ of all ordinals less than the first uncountable ordinal ω_1 is a UC-space although X is not complete in contrast to the case of a metric space. Thus, in uniform setting UC-spaces must enjoy a property weaker than completeness.

Definition 4.1. A uniform space $(X,\mathcal{U})$ is called a *C-CAU space* if $C(X) = CAU(X)$, i.e. X is a space in which all real valued continuous functions are Cauchy.

We recall that a function $f\colon X \to Y$ is continuous if, and only if, $f\colon (X,\delta_0) \to (Y,\eta)$ is p-continuous (note that η is any compatible proximity on Y). Compare this statement with the next Lemma.

Lemma 4.2. (cf. [3, (1.19)]) *Let* $(X,\mathcal{U})$, $(Y,\mathcal{V})$ *be uniform spaces with C-proximities* δ_c, η_c *respectively. Then,* $f\colon X \to Y$ *is a Cauchy map if, and only if,* $f\colon (X,\delta_c) \to (Y,\eta_c)$ *is p-continuous.*

Proof. The result follows from the fact that f is p-continuous if, and only if, f preserves the Cauchy-pair nets and by (a) in Remark 1.8 the claim holds. □

Theorem 4.3. *Let* $(X,\mathcal{U})$ *be a uniform space and* $\{A_n\}$, $\{B_n\}$, $n \in \mathbb{N}$, *countable families of subsets of* X. *The following are equivalent:*

- (a) $C(X,Y) = CAU(X,Y)$ *for each normal uniform space* $(Y,\mathcal{V})$;
- (b) $C(X) = CAU(X)$;
- (c) *if* $A_n \ll_{\delta_F} B_n$ *and* $\{B_n\}$ *is a discrete family, then there is an* $f \in CAU(X)$ *such that* $f(A_n) = n$ *and* $f(X \setminus \bigcup_{n\in\mathbb{N}} B_n) = 0$;
- (d) *if there is an* $f \in C(X)$ *such that* $f(A_n) = n$, *then there is a* $g \in CAU(X)$ *such that* $g(A_n) = n$;
- (e) *for each pair of disjoint zero sets* A, B *there is an* $f \in CAU(X)$ *such that* $f(A) = 0$ *and* $f(B) = 1$;
- (f) *each pair of disjoint zero sets* A, B *satisfy* $A \not\delta_c B$;

(g) $\delta_F \leq \delta_c$ *on* X*;*
(h) $C^\star(X) = CAU^\star(X)$.

Further, if the compatible C-proximity δ_c *on* X *is EF, then* (g) *is equivalent to* $(g^\star)$ $\delta_F = \delta_c$ *on* X.

Proof. $(a) \Rightarrow (b)$ is trivial and $(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ is similar to $(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ in Theorem 2.1.

$(f) \Rightarrow (g)$. Note that if a pair A, B of subsets of X, are δ_F-far, then there exists a pair of zero sets A', B' of X with $A \subset A'$, $B \subset B'$ and $A' \cap B' = \emptyset$. By (f) $A' \not\delta_c B'$. Since $A \subset A'$, $B \subset B'$, we have $A \not\delta_c B$ and the claim holds.

$(g) \Rightarrow (h)$. If (h) is not true, there are an $f \in C^\star(X) \setminus CAU^\star(X)$ and a Cauchy net (x_λ) in X such that the corresponding net $(f(x_\lambda))$ is not Cauchy in $[0,1]$. Nevertheless, the net $(f(x_\lambda))$ has at least a cluster point p in $[0,1]$. Let $p \in V \subset clV \subset W$, where V and $W \neq [0,1]$ are open subsets of $[0,1]$. Since $(f(x_\lambda))$ does not converge but has p as a cluster point, there are subnets (z_ν) and (y_μ) of (x_λ) such that $\{f(z_\nu)\} \subset V$ and $\{f(y_\mu)\} \subset W^c$. Then, there is a $g \colon \mathbb{R} \to \mathbb{R}$ satisfying $g(\{f(z_\nu)\}) = 0$, $g(\{f(y_\mu)\}) = 1$. Set $A = (g \circ f)^{-1}(\{0\})$ and $B = (g \circ f)^{-1}(\{1\})$. Clearly, A, B are disjoint zero sets, but $A \delta_c B$ (in fact, the above pair of nets (y_μ), (z_ν) is a Cauchy-pair with $y_\mu \in A$ and $z_\nu \in B$), a contradiction.

$(h) \Rightarrow (a)$. If there is an $f \in C(X,Y) \setminus CAU(X,Y)$, then there are sets A, B in X such that $A \delta_c B$, but $f(A) \not\eta_0 f(B)$ in Y (use Lemma 4.2 and the fact that $\eta_c \leq \eta_0$). Since Y is a normal space, there is a continuous function $g \colon Y \to [0,1]$ satisfying $g(f(A)) = 0$, $g(f(B)) = 1$, i.e. $g \circ f \in C^\star(X) \setminus CAU^\star(X)$; a contradiction.

The last statement of the Theorem follows from (iii) in Remark 1.5 (d). □

Proposition 4.4. *Every complete uniform space* $(X,\mathcal{U})$ *is a C-CAU space. The converse fails even in the class of normal uniform spaces.*

Proof. Since X is complete, every continuous real valued function is Cauchy. That the converse in general does not hold has been observed at the beginning of this section. □

Proposition 4.5. *Let $(X,\mathcal{U})$ be a uniform space. X is sequentially complete if, and only if, every pair of infinite closed disjoint sets have infinite subsets that are δ-far, i.e. they are* **weakly separated** (see [8]).

Proof. Necessity. Suppose X is not sequentially complete. Then, there exists a Cauchy sequence (x_n) which does not converge and we may assume that $x_n \neq x_m$ for $n \neq m$. Let $A = \{x_n : n$ is even$\}$ and $B = \{x_n : n$ is odd$\}$. Then A and B form a pair of infinite closed disjoint sets which are not weakly separated, since every infinite sequence extracted from A or B is again a Cauchy sequence.

To show the converse, suppose $(X,\mathcal{U})$ is sequentially complete and A and B are infinite, closed and disjoint subsets of X. If A is sequentially compact, then there is a sequence $\{a_n : n \in \mathbb{N}\} \subset A$ converging to a point $a \in A$. Thus, we have that $A_1 = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is a compact subset of A disjoint from B. Hence A_1 and B are δ-far. So, it remains to deal with the case when both A and B fail to be sequentially compact. Since they are closed and X is sequentially complete, they cannot be totally bounded. So, there are an entourage U and two infinite sets $A_1 = \{a_n : n \in \mathbb{N}\} \subset A$ and $B_1 = \{b_n : n \in \mathbb{N}\} \subset B$ such that $(a_n, a_m) \notin U$ as well as $(b_n, b_m) \notin U$ for $n \neq m$. If the sets A_1, B_1 are δ-far, then we are done again. So that it remains to consider the case that they are δ-near, which means that $V[A_1] \cap B_1 \neq \emptyset$ for each $V \in \mathcal{U}$. Let $V \in \mathcal{U}$ such that $V \circ V = V^2 \subset U$ and for each $n \in \mathbb{N}$ choose $V_n \in \mathcal{U}$ such that $V_n^n = V_n \circ V_n^{n-1} \subset V$. Since for each $n \in \mathbb{N}$, $V_n[A_1] \cap B_1 \neq \emptyset$ we have that there are $a_n^\star \in A_1$ and $b_n^\star \in B_1$ with $(a_n^\star, b_n^\star) \in V_n$. Let $A^\star = \{a_n^\star : n \text{ odd}\}$ and $B^\star = \{b_n^\star : n \text{ even}\}$. We claim that $V[A^\star] \cap B^\star = \emptyset$. Assume not. So, there are some $a_{2n+1}^\star$ and $b_{2m}^\star$ such that $(a_{2n+1}^\star, b_{2m}^\star) \in V$. But $(a_{2m}^\star, b_{2m}^\star) \in V_{2m} \subset V_{2m}^{2m} \subset V$. As a result we have $(a_{2n+1}^\star, a_{2m}^\star) \in U$, which contradicts $(a_n, a_m) \notin U$ for $n \neq m$. □

Proposition 4.6. *Let $(X,\mathcal{U})$ be a uniform space. If X is normal and C-CAU, then every pair of infinite closed disjoint sets are weakly separated. Furthermore, in general, the converse fails.*

Proof. The result is shown as in metric case [3] and the failure of the converse was shown by Janos [8] in the form of a counter example. □

Remark 4.7. (cf. [8]). If in the above Proposition the uniformity $\mathcal{U}$ is a metric uniformity, then the following are equivalent:

(a) X is complete;
(b) X is a C-CAU space;
(c) every pair of infinite closed disjoint sets of X are weakly separated.

Corollary 4.8. *Consider the following statements concerning a* **normal** *uniform space* $(X,\mathcal{U})$.

(a) X *is complete;*
(b) $C(X,Y)=CAU(X,Y)$ *for each normal uniform space* $(Y,\mathcal{V})$*;*
(c) $C(X) = CAU(X)$*;*
(d) *if* $A_n \ll_{\delta_F} B_n$ *and* $\{B_n\}$ *is a discrete family, then there is an* $f \in CAU(X)$ *such that* $f(A_n) = n$ *and* $f(X \setminus \bigcup_{n\in\mathbb{N}} B_n) = 0$*;*
(e) *if there is an* $f \in C(X)$ *such that* $f(A_n) = n$*, then there is a* $g \in CAU(X)$ *such that* $g(A_n) = n$*;*
(f) *for each pair of disjoint zero sets* A*,* B *there is an* $f \in CAU(X)$ *such that* $f(A) = 0$ *and* $f(B) = 1$*;*
(g) $\delta_F = \delta_0 = \delta_c$ *on* X*;*
(h) $C^\star(X) = CAU^\star(X)$*;*
(i) *every pair of infinite closed disjoint sets are weakly separated;*
(j) X *is sequentially complete.*

The following relations hold:
$(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f) \Leftrightarrow (h) \Rightarrow (i) \Leftrightarrow (j)$. *None of the arrows can be reversed. Further, if* X *is metrizable and* $\mathcal{U}$ *is the associated metric uniformity, then all the above statements are equivalent.*

5. CAU-PC Spaces:

In this section we explore a new class of spaces which we call CAU-PC spaces in view of (b) in Theorem 5.1. In these spaces all real valued Cauchy maps are p-continuous. We point out that in metric spaces, CAU-PC = CAU-UC, but in the uniform setting this equality, in general, does not hold.

Theorem 5.1. *For a uniform space $(X,\mathcal{U})$ the following are equivalent:*

(a) $CAU(X,Y) = P(X,Y)$ *for each proximity space* (Y,η)*;*
(b) $CAU(X) = P(X)$*;*
(c) *the completion* $X^\star$ *of* X *is a PC-space;*
(d) *if for* A, $B \subset X$ *there is an* $f \in CAU(X)$ *with* $f(A) = 1$ *and* $f(B) = 0$*, then there is a* $g \in P(X)$ *with* $g(A) = 1$ *and* $g(B) = 0$*;*
(e) *if the closures in* $X^\star$ *of two subsets* A, B *of* X *are contained in two disjoint zero sets, then* A, B *are* δ*-far;*
(f) $CAU^\star(X) = P^\star(X)$.

Proof. $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$. $f\colon X^\star \to \mathbb{R}$ is continuous $\Rightarrow f\colon X^\star \to \mathbb{R}$ is Cauchy $\Rightarrow f/X\colon X \to \mathbb{R}$ is Cauchy $\Rightarrow f/X\colon (X,\delta) \to (\mathbb{R},\eta)$ is p-continuous $\Rightarrow f\colon (X^\star,\delta) \to (\mathbb{R},\eta)$ is p-continuous.

$(c) \Rightarrow (d)$. Let $f \in CAU(X)$ with $f(A) = 1$ and $f(B) = 0$. There is a continuous function $f^\star\colon X^\star \to \mathbb{R}$ which extends f to the completion $X^\star$ of X (see [17]). Clearly, $f^\star(cl_{X^\star}A) = 1$ and $f^\star(cl_{X^\star}B) = 0$. By (c) the claim holds.

$(d) \Rightarrow (e)$. Let A, $B \subset X$ with $cl_{X^\star}A \subset A^\star$, $cl_{X^\star}B \subset B^\star$ and $A^\star$, $B^\star$ disjoint zero sets in $X^\star$. Then, there is an $f \in C(X^\star)$ with $f(A^\star) = 1$ and $f(B^\star) = 0$. Since $X^\star$ is complete, $C(X^\star) = CAU(X^\star)$ and so the restriction f/X of f to X is Cauchy with $f/X(A) = 1$, $f/X(B) = 0$. By (d) the result follows.

$(e) \Rightarrow (f)$. If (f) is not true, then there are an $f \in CAU^\star(X) \setminus P^\star(X)$ and subsets C, D of X with $C\delta D$ but $f(C) = 1$, $f(D) = 0$. Its extension $f^\star \in CAU^\star(X^\star) \setminus P^\star(X^\star)$ and $f^\star(C) = f(C) = 1$, $f^\star(D) = f(D) = 0$. Let $A^\star = f^{\star-1}(\{1\})$, $B^\star = f^{\star-1}(\{0\})$, $A = A^\star \cap X$ and $B = B^\star \cap X$. By construction $C \subset A \subset A^\star$, $D \subset B \subset B^\star$ with $A^\star$, $B^\star$ disjoint zero sets in $X^\star$. Thus, $cl_{X^\star}A \subset A^\star$, $cl_{X^\star}B \subset B^\star$ but $A\delta B$ (in fact $C \subset A$, $D \subset B$ and $C\delta D$) contradicting (e).

$(e) \Rightarrow (a)$. Assume not. Then, there are an $f \in CAU(X,Y) \setminus P(X,Y)$ and subsets A, B of X such that $A\delta B$ but $f(A) \not\eta f(B)$ in Y. From Theorem 7.12 in [13] there is a $g \in P^\star(Y)$ satisfying $g \circ f(A) = 1$, $g \circ f(B) = 0$. By a result of Leader (cf. [10, Corollary 5]) $g \in CAU^\star(Y)$. It follows $g \circ f \in CAU^\star(X) \setminus P^\star(X)$, a contradiction. ☐

Remark 5.2. The above Theorem 5.1 with Theorem 4.3 offers a decomposition of the property of PC-spaces.

Theorem 5.3. *Consider the following statements concerning a uniform space* $(X,\mathcal{U})$.

(a) $\delta = \delta_c$ *on* X;

(b) *every Cauchy map* $f\colon X \to Y$, *where* $(Y,\mathcal{V})$ *is any uniform space, is p-continuous w.r.t. proximities* δ, η_c *on* X, Y *respectively;*

(c) *every Cauchy map* $f\colon X \to Y$, *where* $(Y,\mathcal{V})$ *is any uniform space, is p-continuous w.r.t. proximities* δ, η *on* X, Y *respectively;*

(d) $CAU(X) = P(X)$;

(e) *the completion* $X^\star$ *of* X *is a PC-space;*

(f) $CAU^\star(X) = P^\star(X)$.

Then $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$. *Further, if the completion* $X^\star$ *is normal, then all the above statements are equivalent. Thus, if* X *is metrizable and* $\mathcal{U}$ *is the associated metric uniformity, then all the above statements are equivalent.*

Proof. $(a) \Rightarrow (b)$. Let $f\colon X \to Y$ be a Cauchy map. By Lemma 4.2 $f\colon (X,\delta_c) \to (Y,\eta_c)$ is p-continuous. Since $\delta = \delta_c$, the claim holds.

$(b) \Rightarrow (a)$. The identity map $i : X \to X$ is a Cauchy map. By (b) $i\colon (X,\delta) \to (X,\delta_c)$ is p-continuous. So, $A\delta B$ implies $A\delta_c B$, i.e. $\delta_c \leq \delta$. On the other hand, it is always true that $\delta \leq \delta_c$ (cf. (i) in Remark 1.5 (d)). Thus, $\delta = \delta_c$.

$(b) \Rightarrow (c)$. Let $f\colon X \to Y$ be a Cauchy map. By (b) $f\colon (X,\delta) \to (Y,\eta_c)$ is p-continuous. Since $\eta \leq \eta_c$, $f\colon (X,\delta) \to (Y,\eta)$ is p-continuous, too.

$(c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ are in Theorem 5.1.

Now, suppose the completion $X^\star$ is normal and (e) holds. Then, $X^\star$ is a normal PC-space. Thus, $\delta = \delta_0$ on $X^\star$. It follows $\delta/X = \delta_0/X$, i.e. $\delta = \delta_c$ on X. □

6. CAU-UC Spaces

In this section we consider the uniform case. This case is a bit tricky since $CAU(X) = U(X)$ does not imply $CAU(X,Y) = U(X,Y)$ for each uniform space Y and $CAU^\star(X) = U^\star(X)$ does not imply $CAU(X) = U(X)$.

So, we have three cases:

(a) **strong CAU-UC:** $CAU(X,Y) = U(X,Y)$ for each uniform space Y.

(b) **Čech CAU-UC:** $CAU^\star(X) = U^\star(X)$.
In this case $U^\star(X) = P^\star(X)$ (cf. (c) and (d) in Remark 1.8) and this being a proximity property is already studied in Theorem 5.1.

(c) **UC:** $CAU(X) = U(X)$.

Theorem 6.1. *For a uniform space* $(X,\mathcal{U})$ *the following are equivalent:*

(a) $CAU(X) = U(X)$ *(*$CAU(X,Y) = U(X,Y)$ *for each uniform space* Y*);*

(b) $(X^\star,\mathcal{U}^\star)$ *is UC (respectively,* $(X^\star,\mathcal{U}^\star)$ *is strong UC), where* $(X^\star,\mathcal{U}^\star)$ *is the completion of* $(X,\mathcal{U})$*.*

Proof. Let $X^\star$ (respectively, $Y^\star$) the completion of X (respectively, the completion of Y). To show the equivalence of (a) and (b) note that the class $CAU(X)$ (respectively, $CAU(X,Y)$) equals the class $C(X^\star)$ (respectively, $C(X^\star,Y^\star)$) as well as $U(X)$ (respectively, $U(X,Y)$) equals $U(X^\star)$ (respectively, $U(X^\star,Y^\star)$). □

Remark 6.2. By the above Theorem 6.1 we have:

(a) $(X,\mathcal{U})$ is a strong CAU-UC space if, and only if, the trace on X of the fine uniformity of $X^\star$ equals $\mathcal{U}$.

(b) $(X,\mathcal{U})$ is a CAU-UC space if, and only if, $\mathcal{U}^\star$ is the coarsest uniformity for which $C(X^\star) = U(X^\star)$.

From the above Remark and the results in Sections 4 and 5 the following are clear and do not need proofs.

Theorem 6.3. *Let* $(X,\mathcal{U})$ *be a uniform space and* $\{A_n\}$*,* $\{B_n\}$*,* $n \in \mathbb{N}$*, countable families of subsets of* X*. The following are equivalent:*

(a) $CAU(X) = U(X)$*;*

(b) *if* $\{cl_{X^\star}B_n\}$ *is a discrete family in* $X^\star$ *and for each* $n \in \mathbb{N}$ *there is a function* $f_n \in CAU(X)$*,* $0 \le f_n \le 1$ *with values* 1 *on* A_n *and* 0 *on the complements of* B_n*, then there is an* $U \in \mathcal{U}$*, such that* $U[A_n] \subset B_n$ *for each* $n \in \mathbb{N}$*;*

(c) *if* $\{A_n\}$ *is a sequence of sets such that* $f(A_n) = n$ *for some* $f \in CAU(X)$*, then there is a* $U \in \mathcal{U}$*, such that for* $n \ne m$*,* $U[A_n] \cap A_m = \emptyset$*.*

Theorem 6.4. *Consider the following statements concerning a uniform space* $(X,\mathcal{U})$.

(a) $\mathcal{U}^\star$ *is fine;*
(b) $CAU(X,Y) = U(X,Y)$ *for each uniform space* Y*;*
(c) $CAU(X) = U(X)$*;*
(d) *if* $\{cl_{X^\star}B_n\}$ *is a discrete family in* $X^\star$ *and for each* $n \in \mathbb{N}$ *there is a function* $f_n \in CAU(X)$, $0 \leq f_n \leq 1$ *with values* 1 *on* A_n *and* 0 *on the complements of* B_n, *then there is an* $U \in \mathcal{U}$, *such that* $U[A_n] \subset B_n$ *for each* $n \in \mathbb{N}$*;*
(e) *if* $\{A_n\}$ *is a sequence of sets such that* $f(A_n) = n$ *for some* $f \in CAU(X)$, *then there is a* $U \in \mathcal{U}$, *such that for* $n \neq m$, $U[A_n] \cap A_m = \emptyset$*;*
(f) *for each pair of sets* A, B *of* X *which have a Cauchy map* $f\colon X \to [0,1]$ *such that* $f(A) = 0$ *and* $f(B) = 1$, *there is a uniformly continuous function* $g\colon X \to [0,1]$ *with* $g(A) = 0$ *and* $g(B) = 1$*;*
(g) *if the closures in* $X^\star$ *of two subsets* A, B *of* X *are contained in two disjoint zero sets, then there is a* $U \in \mathcal{U}$, *such that* $U[A] \cap B = \emptyset$.

Then, $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Rightarrow (f) \Leftrightarrow (g)$ *and none of the arrows can be reversed.*

Theorem 6.5. *Consider the following statements concerning a uniform space* $(X,\mathcal{U})$ *such that the completion* $(X^\star,\mathcal{U}^\star)$ *is normal.*

(a) $(X^\star)'$ *is compact and for each* $U^\star \in \mathcal{U}^\star$ *there is a* $V^\star \in \mathcal{U}^\star$ *such that* $(U^\star[(X^\star)'])^c$ *is* $V^\star$*-discrete;*
(b) $\mathcal{U}^\star$ *is Lebesgue, i.e. every open cover of* $X^\star$ *has a refinement* $\{U^\star[x^\star] : x^\star \in X^\star\}$ *for some* $U^\star \in \mathcal{U}^\star$*;*
(c) $\mathcal{U}^\star$ *is fine;*
(d) $CAU(X,Y) = U(X,Y)$ *for each uniform space* Y*;*
(e) $CAU(X) = U(X)$*;*
(f) $\delta = \delta_c$*;*
(g) $X^\star$ *is a normal PC-space.*

Then, $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Rightarrow (e) \Rightarrow (f) \Leftrightarrow (g)$, *and none of the arrows can be reversed.*

Furthermore, if X *is metrizable and* $\mathcal{U}$ *is the associated metric uniformity, then all the above statements are equivalent.*

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