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NAGATA-SMIRNOV REVISITED: SPACES WITH σ-WHCP BASES

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Abstract. A collection $\mathcal{H}$ of subsets of a space X is *weakly HCP* (wHCP) if, whenever a point $x(H) \in H$ is chosen for each $H \in \mathcal{H}$, the resulting collection $\mathcal{K} = \{x(H) : H \in \mathcal{H}\}$ is a closed discrete subspace of X. We show which spaces with a σ-wHCP bases are metrizable. Specifically, a space X with a σ-wHCP base, is metrizable if X: has $\chi(X) < \aleph_\omega$, is separable, has countable tightness, is Lindelöf, is countably compact, or is a q-space. A list of open questions is included in the closing.

1. Introduction

A metrization theorem gives (necessary and) sufficient conditions for a topological space to be metrizable. One of the classic metrization theorems was provided independently by Nagata and Smirnov in the early 1950's: A topological space is metrizable if and only if it is regular and has a σ*-locally finite* base. A collection $\mathcal{H}$ of subsets of a space X is *hereditarily closure-preserving* (HCP) if, whenever a subset $K(H) \subset H$ is chosen for each $H \in \mathcal{H}$, the resulting collection $\mathcal{K} = \{K(H) : H \in \mathcal{H}\}$ is closure preserving. A σ*-HCP collection* is a collection that can be written as a countable union of HCP collections. In 1975, Burke, Engelking, and Lutzer [2] showed that a topological space is metrizable if and only if it is regular and has a σ*-HCP* base. In that paper, the referee asked if the result could be strengthened by considering the following weaker hypothesis:

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A collection $\mathcal{H}$ of subsets of a space X is *weakly HCP* (wHCP) if, whenever a point $x(H) \in H$ is chosen for each $H \in \mathcal{H}$, the resulting collection $\mathcal{K} = \{x(H) : H \in \mathcal{H}\}$ is a *closed discrete* subspace of X. Burke, Engelking, and Lutzer showed that a *k-space* with a σ-wHCP base is metrizable, but also provided an example of a non-metrizable, hereditarily paracompact space with a σ-weakly HCP base.

In Section 2, we consider when a space with a σ-wHCP base is metrizable. Specifically we show that any space X with a σ-wHCP base and one of the following properties is metrizable: $\chi(X) < \aleph_\omega$, separable, countable tightness, or Lindelöf. We also show in Theorem 8 that closed maps on a space with a σ-wHCP base are compact covering. In Section 3, we turn our attention to D-spaces. Theorem 10 shows that a space with a σ-wHCP base is actually a D-space. This result is used to show that a space with a σ-wHCP base is metrizable if it is countably compact or a q-space. In Section 4, a list of open problems is posed. All spaces are regular and T_1, maps are continuous and onto. Readers may refer to [3] for unstated definitions.

2. When a space with σ-wHCP is metrizable

In the sequel, the k-space result mentioned in the introduction will prove useful, so for reference we restate it as a fact. Recall that a first countable space is clearly a k-space.

Fact 1. [Burke, Engelking, Lutzer: 1975] A *k-space* (*first countable*) with a σ-wHCP base is metrizable.

Theorem 2. *A topological space X with a σ-wHCP base is metrizable if and only if $\chi(X) < \aleph_\omega$.*[1]

Proof. Since a metrizable space is first countable, necessity is clear. For sufficiency, let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega\}$ be a σ-wHCP base for X. Fix non-isolated $x \in X$. We claim that $\chi(x, X) \leq \aleph_0$. Suppose not, then there exists $m \in \mathbb{N}$ such that $\chi(x, X) = \aleph_m$. Let $\mathcal{V} = \{V_\alpha : \alpha \in \omega_m\}$ be a local base at X. We show that for any $n \in \mathbb{N}$, $|\{B \in \mathcal{B}_n : x \in B\}| < \aleph_m$. Otherwise, there exists a $k \in \mathbb{N}$ such that $|\mathcal{K} = \{B \in \mathcal{B}_k : x \in B\}| \geq \aleph_m$. Without loss of generality, we assume $|\mathcal{K}| = \aleph_m$ and for ease of notation we enumerate

[1] $\chi(X)$ denotes the character of X.

$\mathcal{K} = \{B_\beta : \beta < \omega_m\}$. Since x is not isolated, $(V_\alpha \cap B_\beta) \setminus \{x\} \neq \emptyset$ for any $\alpha < \omega_m$ and $\beta < \omega_m$. So, we may pick $x_1 \in (V_1 \cap B_1) \setminus \{x\}$. By induction, suppose $x_1, \ldots, x_\gamma$ have been chosen. Since $\mathcal{B}_k$ is wHCP, $\cup\{x_1, \ldots, x_\gamma\}$ is closed, so $U = X \setminus \{x_1, \ldots, x_\gamma\}$ is open with $x \in U$. Now, $(U \cap V_\alpha \cap B_\alpha) \setminus \{x\} \neq \emptyset$. Again, by induction, pick $x_\alpha \in (U \cap V_\alpha \cap B_\alpha) \setminus \{x\}$ and note $\{x_\alpha : \alpha < \omega_m\}$ is closed discrete, since $x_\alpha \in B_\alpha$. However, $x \in \overline{\{x_\alpha : \alpha < \omega_m\}}$ since $x_\alpha \in V_\alpha$, a contradiction. Hence, for any $n \in \mathbb{N}$, $|\{B \in \mathcal{B}_n : x \in B\}| < \aleph_m$. But this implies $\chi(x, X) < \aleph_m$, contradicting the assumption to the claim. So, $\chi(x, X) \leq \aleph_0$, proving the claim. So, X is first countable and, by Fact 1, metrizable. □

Proposition 3. *Let X be a space with a σ-wHCP base. If for every non-isolated point, there is a countable subset $A \subset X$ such that $x \in \overline{A \setminus \{x\}}$, then X is metrizable.*

Proof. It suffices to prove that X is first countable at every non-isolated point. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega\}$ be a σ-wHCP base for X. Fix non-isolated $x \in X$. We claim that for any $n \in \mathbb{N}$, $|\{B \in \mathcal{B}_n : x \in B\}| \leq \omega$. Suppose not, then there exists $m \in \mathbb{N}$ such that $|\{B \in \mathcal{B}_m : x \in B\}| > \omega$. Without loss of generality, we can re-index such that $\{B \in \mathcal{B}_m : x \in B\} = \{B_\alpha : \alpha < \omega_1\}$. Since $x \in \overline{A \setminus \{x\}}$, we can pick $x_1 \in B_1 \cap (A \setminus \{x\})$. Suppose for $\beta < \alpha$, x_β has been chosen. Pick $x_\alpha \in B_\alpha \cap (X \setminus \{x_\gamma : \gamma < \beta\}) \cap (A \setminus \{x\})$. But the sequence $\{x_\alpha\}$ is uncountable *and* contained in the countable set A, a contradiction. So, $|\{B \in \mathcal{B} : x \in B\}| \leq \omega$, that is X is first countable and hence metrizable by Fact 1. □

It is easy to see that for every non-isolated point $x \in X$ in a separable space or a space having countable tightness, there is a countable subset $A \subset X$ such that $x \in \overline{A \setminus \{x\}}$. So we have following.

Corollary 4. *A separable space with σ-wHCP base is metrizable.*

Corollary 5. *A topological space X is metrizable if and only if X has a σ-wHCP base and countable tightness*[2].

[2]A space X has countable tightness if whenever $x \in \overline{A}$, $A \subset X$, there exists a countable subset $D \subset A$ such that $x \in \overline{D}$.

A space X is called $\aleph_1$-compact if every subset of X with a cardinality $\aleph_1$ has a cluster point.

Proposition 6. *An $\aleph_1$-compact space X with a σ-wHCP network has a countable network, hence it is separable.*

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega\}$ be a σ-wHCP network of X and let $D_n = \{x \in X : \mathcal{B}_n$ is not point-countable at $x\}$. For each $n \in \omega$, $\{P \backslash D_n : P \in \mathcal{B}_n\}$ is point-countable and wHCP. Since X is $\aleph_1$-compact, then $\{P \backslash D_n : P \in \mathcal{B}_n\}$ is countable. We claim for $n \in \mathbb{N}$, that D_n is countable. Suppose not, by induction, we may obtain an uncountable subset $\{x_\alpha : \alpha < \omega_1\}$ of D_n and an uncountable subfamily $\{P_\alpha : \alpha < \omega_1\}$ of $\mathcal{B}_n$ such that $x_\alpha \in P_\alpha$. Since $\mathcal{B}_n$ is wHCP, $\{x_\alpha : \alpha < \omega_1\}$ is closed discrete subset of X. But X is $\aleph_1$-compact, so $\{x_\alpha : \alpha < \omega_1\}$ has a cluster point, this is a contradiction. Let $\mathcal{P}_n = \{P \backslash D_n : P \in \mathcal{B}_n\} \cup \{\{x\} : x \in D_n\}$ and $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\}$. It is easy to see that $\mathcal{P}$ is a countable network of X. □

By Proposition 6 and Corollary 4, we have the following.

Corollary 7. *A Lindelöf space with a σ-wHCP base is metrizable.*

It is natural to ask if X has a σ-closure preserving and wHCP base, is X metrizable? It turns out the answer is no. In fact, the nonmetrizable space X in Example 9 in [2] has a σ-closure preserving, wHCP base. It is not difficult to check the space is a hereditary M_1-space, thus X has a σ-locally finite network. We can also see that every compact subset of X is finite, so X is an $\aleph$-space (i.e. a space with a σ-locally finite k-network).[3]

A map $f : X \to Y$ is called a *compact covering* map if every compact subset of Y is the image of some compact subset of X under f.

Theorem 8. *Let $f : X \to Y$ be a closed map. If X has a σ-wHCP base, then f is a compact covering map.*

Proof. Let L be a compact subset of Y. We first show that L is metrizable. Since X has a σ-wHCP base and f is a closed map.

[3]A cover $\mathcal{P}$ of a space X is a k-network if whenever $K \subset U$ with K compact and U open, there is a finite subfamily $\mathcal{P}'$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset U$.

Then Y has a σ-wHCP network. Since L is compact, by Proposition 6, L has a countable network, hence L is metrizable. L is a compact metric space, so it is separable. Let D be a countable dense subset of L. For each $y \in D$, pick $x_y \in f^{-1}(y)$. Let $D_1 = \{x_y : y \in D\}$. Notice that $|D_1| \leq \omega$. By Corollary 4, $\overline{D_1}$ is metrizable. Since f is a closed continuous map, we have

$$f(\overline{D_1}) = \overline{f(D_1)} = \overline{D} = L.$$

It is well known that a closed map on a metric space is a compact covering map, hence there exists a compact subset K with $K \subset \overline{D_1}$ such that $f(K) = L$. $\square$

3. D-spaces and σ-wHCP bases

D-spaces were first introduced by van Dowen in 1979. Recently, there has been considerable interest in this topic. The interested reader can refer to [1] for more on D-spaces.

Definition 9 (van Dowen, 1979)**.** A topological space X is a *D-space* if given any collection $\{G(x) : x \in X\}$ of open sets in X with $x \in G(x)$ for each $x \in X$, there is a closed discrete subset D of X such that $\cup\{G(x) : x \in D\}$ covers X.

Theorem 10. *A space with a σ-wHCP base is a D-space.*

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega\}$ be a σ-wHCP base of X. Let $\{G(x) : x \in X\}$ be a collection of open sets in X such that for all $x \in X$, $x \in G(x)$. Since $\mathcal{B}$ is a base, for each $x \in X$ there exists a $B(x) \in \mathcal{B}$ such that $x \in B_x \subseteq G(x)$. $\{B(x) : x \in X\}$ is a cover of X and a subfamily of $\mathcal{B}$, so we can consider $\{B(x) : x \in X\} = \cup\{\mathcal{B'}_n : n \in \omega\}$ where $\mathcal{B'}_n \subseteq \mathcal{B}_n$ is a wHCP family for all $n \in \omega$.

We now construct the desired closed discrete subset D of X. For each $B \in \mathcal{B'}_1$, pick a $x_B \in B$ such that $B(x_B) = B$ and let $D_1 = \{x_B : B \in \mathcal{B'}_1\}$. Note D_1 is closed discrete. By induction, suppose D_n has been chosen. If $X \setminus \cup\{B(x) : x \in \cup_{i=1}^n D_i\} = \emptyset$, let $D = \cup_{i=1}^n D_i$, otherwise let

$$D_{n+1} = \left\{y \in X : y \in X \setminus \cup\{B(x) : x \in \cup_{i=1}^n D_i\} \text{ and } \mathrm{B(y)} \in \mathcal{B'}_{\mathrm{n+1}}\right\}.$$

Let $D = \cup_{i=1}^\infty D_i$. It is straightforward to show that $X = \cup\{B(x) : x \in D\}$. We now show that D is locally finite.

For $z \in X$, let m be the smallest natural number such that $z \in B(x_z)$ and $x_z \in D_m$, then

$$V = (B(x_z) \setminus \cup_{i=1}^{m} D_i) \cup \{x_z\}$$

is an open neighborhood of z. Since $V \cap (\cup_{i=1}^{m} D_i) = \{x_z\}$ and $V \cap (\cup_{i>m} D_i) = \emptyset$, we have $V \cap D = \{x_z\}$. That is, D is locally finite, hence closed discrete, as desired. □

It is well known that a countably compact D space is compact and hence a k-space. So by Fact 1, we have the following:

Corollary 11. *A countably compact space with a σ-wHCP base is metrizable.*

Recall, a space X is a *q-space* if for every $x \in X$, there exists a collection $\{U_n : n \in \omega\}$ of open neighborhoods of $x \in X$ such that for every choice $x_n \in U_n$, the sequence $\{x_n : n \in \omega\}$ has a cluster point. Given a q-space X with a σ-wHCP base, for each $x \in X$, there exists countably compact $L_x = \cap\{U_n : n \in \omega\} \subset X$. By Corollary 11, it is easy to show each L_x is metrizable, hence X is first-countable, which leads to the next corollary.

Corollary 12. *A q-space with a σ-wHCP base is metrizable.*

4. Future work

Based on the above work, the following questions are natural:

Question 13. Is every space with a σ-wHCP base a meta-Lindelöf space?

Question 14. Is every point of a space with a σ-wHCP base a G_δ-set?

Question 15. Is every pseudocompact space with a σ-wHCP base metrizable?

Question 16. Is a σ-wHCP base preserved under a perfect map? (It is easy to see that closed and open maps preserve σ-wHCP bases.)

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