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CHARACTERIZATIONS OF NEARLY PSEUDOCOMPACT SPACES AND RELATED SPACES

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ABSTRACT. Nearly pseudocompact spaces were initiated by Henriksen and Rayburn. Several topological characterizations of this space are found in literature but any algebraic description is hardly noticed. In this paper we have furnished a number of algebraic descriptions of nearly pseudocompact spaces. In the process we have also introduced Hard pseudocompact spaces, a generalization of pseudocompact spaces and obtained a few topological characterizations of this space.

1. INTRODUCTION

Through out our paper, X will be assumed to be a Tychonoff space. It is well known that X is pseudocompact if and only if the Stone-Čech compactification βX is identical with the Hewitt realcompactification υX. In 1980, Henriksen and Rayburn [8] generalized the notion of pseudocompactness by defining a space to be nearly pseudocompact when and only when $\upsilon X \setminus X$ is dense in $\beta X \setminus X$. These two authors have furnished several characterizations of nearly pseudocompact spaces, most of which by using the notion of hard sets, an entity introduced by the second author in [15]. The main result of their paper is that - X is nearly pseudocompact if

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and only if X can be expressed as a union of two subspaces X_1 and X_2, where X_1 is a regular closed almost locally compact subspace of X that is pseudocompact, X_2 is a regular closed nowhere locally realcompact subspace of X and $int_X(X_1 \bigcap X_2) = \phi$ [[8],theorem 3.10]. These spaces were subsequently investigated in some details by John Schommer in [17], who treated nearly pseudocompact spaces without using hard sets and rather using co-zero sets, zero sets and special type of open sets. Schommer had in reality adapted the techniques of Blair and van Douwen's paper [3] on nearly realcompact spaces to render a new development of nearly pseudocompact spaces. In the above mentioned papers, the authors have given several characterizations of nearly pseudocompact spaces, most of which are topological in nature. In the present article, our principal intention is to furnish a number of algebraic descriptions of these spaces via some classes of rings of real valued continuous functions on these spaces and then to give a few topological as well as algebraic descriptions of hard pseudocompact spaces, a notion which we have introduced as a second generalization of pseudocompact spaces that agree with nearly pseudocompactness precisely on the class of pseudocompact spaces.

2. Algebraic description of nearly pseudocompact spaces

The main tool to develop the algebraic description of nearly pseudocompact spaces is the notion of hard set, introduced by M. Rayburn in [15].

Definition 2.1. A subset H of X is called hard if it is closed in $K \cup X$, where $K = cl_{\beta X}(\upsilon X \backslash X)$.

The following properties of Hard set are established in [15].

Theorem 2.2. *For any space X*

(1) Every hard subset of X is necessarily closed, and the converse is true if and only if X is realcompact.
(2) Every compact set is a hard set in X and converse is true if and only if X is nearly pseudocompact.
(3) Finite unions, arbitrary intersections, and closed subsets of a hard set are hard.

(4) Let $\delta X = \beta X \setminus (K \setminus X)$. Then a closed subset F of X is hard in X if and only if there exist a compact set T in δX with $T \cap X = F$ and X is nearly pseudocompact if and only if $X = \delta X$

(5) A closed subset F of X is hard in X if and only if there exists a compact set K such that for any open neighborhood V of K there exists a realcompact subset P of X so that $F \setminus V$ and $X \setminus P$ can be completely separated.

The following characterizations of nearly pseudocompact spaces, which are somewhat topological in nature, are due to Henriksen and Rayburn [Theorem 3.2 and theorems 3.8 [8]].

Theorem 2.3. *For a space X, the following statements are equivalent.*

(1) X is nearly pseudocompact
(2) Each hard subset of X is compact
(3) Every regular hard subset of X is compact
(4) Each $f \in C(X)$ is bounded on every hard subset of X
(5) Each $f \in C(X)$ is bounded on every regular hard subset of X
(6) Every decreasing sequence of nonempty regular hard sets has non void intersection

Proof. (1) $\Leftrightarrow$ (2): It follows from the fact that X is nearly pseudocompact if and only if $X = \delta X$ and a closed subset H of X is hard if and only if there exists a compact set L in δX such that $L \cap X = H$.

(2) $\Rightarrow$ (3): It follows trivially.

(3) $\Rightarrow$ (2): Suppose a hard subset of X is not compact. Then there exists a point p in δX such that $p \notin K$. Then we can find an open set V in βX such that $p \in V \subseteq cl_{\beta X} V \subseteq \beta X \setminus K$. Let $W = cl_{\beta X} V \cap X$. $H = cl_X(int_X W)$ is a regular hard set whose closure in βX contains the point p and hence not compact, a contradiction.

(2) $\Leftrightarrow$ (4): It follows from the fact that if a hard set is not compact, then $\varnothing \neq (cl_{\beta X} H \setminus H) \subseteq \delta X \subseteq \beta X \setminus \upsilon X$.

(3) $\Leftrightarrow$ (5): Similar proof as in (2) $\Leftrightarrow$ (4).

(3) $\Rightarrow$ (6): It follows trivially.

(6) $\Rightarrow$ (3): It follows from the fact intersection of two regular closed sets contains another nonempty regular closed set.

In fact suppose (3) is false. Then there exists a point p in the hyper-real zone such that p lie in the closure of the hard set H which is not compact. Then there exists an $'f'$ in $C(X)$ such that p lie in the βX-closure $Z_n(f)$, where $Z_n(f) = \{x \in X : |f(x)| \geq n\}$ and $\cap_{n\in\mathbb{N}} Z_n(f) = \varnothing$. Note that for each $n \in \mathbb{N}$, $Z_n(f)$ is a regular closed set. Then for each $n \in \mathbb{N}$, there exists a nonempty regular closed set F_n such that $F_n \subseteq Z_n(f) \cap H$.We can choose F_n in such a way that they are decreasing. Thus F_n's are non empty decreasing sequence of regular hard sets with void intersection, a contradiction. □

In the above theorem, $C(X)$ and $C^*(X)$ stand as usual for the rings of real valued continuous functions on X and that of all real valued bounded continuous functions on X respectively. We have introduced here two subrings of $C(X)$ to obtain algebraic descriptions of nearly pseudocompact spaces which are in form similar to $C_K(X)$ and $C_\infty(X)$ - two well known subrings of $C(X)$, in fact subrings of $C^*(X)$ also, where $C_K(X)$ is the family of continuous function with compact support and $C_\infty(X)$ the family of continuous functions which vanish at infinity.

Notation 2.4. Let $C_H(X)$ be the collection of all those f in $C(X)$ which have hard support in the sense that $cl_X(X\backslash Z(f))$ is hard in X, where $Z(f) = \{x \in X : f(x) = 0\}$ and $H_\infty(X)$ is the collection of those f in $C(X)$ for which $\{x\epsilon X : |f(x)| \geq 1/n\}$ is hard for each n in $\mathbb{N}$.

Since every closed subset of a hard set is hard and the finite union of hard subsets of X are hard, it is clear that $C_H(X)$ and $H_\infty(X)$ are subrings of $C(X)$ with $C_H(X) \subseteq H_\infty(X)$ and $C_H(X)$ an ideal of $C(X)$. In a nearly pseudocompact space X, there is no distinction between a hard subset and compact subset and therefore in particular we can write $C_H(X) = C_K(X)$ and $H_\infty(X) = C_\infty(X)$ and consequently then we can also write $C_H(X) \subseteq H_\infty(X) \subseteq C^*(X)$. We have shown as the following proposition manifests that, each of these relations by itself is characteristic of nearly pseudocompact spaces.

Theorem 2.5. *The following statements are equivalent for a space X.*

(1) X is nearly pseudocompact.
(2) $C_k(X) = C_H(X)$.
(3) $C_\infty(X) = H_\infty(X)$
(4) $C_H(X) \subseteq C^*(X)$
(5) $H_\infty(X) \subseteq C^*(X)$
(6) $H_\infty(X) \bigcap C^*(X) = C_\infty(X)$
(7) $C_H(X) \bigcap C^*(X) = C_k(X)$

Proof. $(1) \Rightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (4)$ and $(1) \Rightarrow (5)$ are immediate consequences of theorem 2.3. Assume that the condition (1) of theorem 2.5 is false, then from theorem 2.3, there exists an $f \geq 0$ in $C(X)$ and a regular hard subset H of X such that f is unbounded on H. Since $H = cl_X(int_X H)$, it follows that f is unbounded on $int_X H$ and hence there exists a copy of $\mathbb{N}$={1,2,3...} in $int_X H$ with $f(n) > n2^n$ for each $n \in \mathbb{N}$. Now using the complete regularity of X, we can find out for each $n \in \mathbb{N}$, a $g_n \in C(X)$ such that $g_n(n) = 1$ and $g_n(X \backslash int_X H) = \{0\}$. Set $g = \sum_{n\in\mathbb{N}} |g_n| \bigwedge \frac{1}{2^n}$. Then $g \in C(X)$ and $g(X \backslash int_X H) = \{0\}$. Let $h = fg$, then $h(n) > n$ for each $n \in \mathbb{N}$ so that $h \in C(X) \backslash C^*(X)$. Furthermore the relation $h(X \backslash int_X H) = \{0\}$ implies that $\mathbb{N} \subseteq X \backslash Z(h) \subseteq int_X H$ and consequently $\mathbb{N} \subseteq cl_X(X \backslash Z(h) \subseteq cl_X(int_X H) = H$. Since a closed subset of a hard set is hard, the last relation indicates that $cl_X(X \backslash Z(f))$ is a hard set and consequently $h \in C_H(X)$. Since $h \notin C^*(X)$, it is trivial that $h \notin C_K(X)$ and $h \notin C_\infty(X)$. Thus if (1) is false then each of (2),(3),(4),(5) becomes false.

$(3) \Rightarrow (6), (2) \Rightarrow (7)$: Trivial.

$(6) \Rightarrow (3)$: Suppose (3) is false and $h \in H_\infty(X) \backslash C_\infty(X)$. Let $h' = (-1 \bigvee h) \bigwedge 1$. Since $\{x \in X : |h(x)| \geq \frac{1}{n}\} = \{x \in X : |h'(x)| \geq \frac{1}{n}\}$, $\forall n \in \mathbb{N}$, we can say immediately that $h' \in H_\infty(X) \bigcap C^*(X) \backslash C_\infty(X)$,

$(7) \Rightarrow (2)$: Analogous to $(6) \Rightarrow (3)$. □

Note 2.6. It may be noted in this connection that Schommer has given a characterization of nearly pseudocompact spaces in a manner analogous to the equivalence of the statements 1 and 2 above - indeed he has proved that X is nearly pseudocompact if and only if every f in $C(X)$ with realcompact support has compact support (Vide [17], corollary 3.2).

We shall now approach to few more characterizations of nearly pseudocompact spaces in terms of a subring $'\chi(X)'$ of $C(X)$ containing $C^*(X)$. The following theorems will be necessary tools to describe the ring $'\chi(X)'$.

We recall the known result that $C_K(X) = \bigcap_{p \epsilon \beta X \setminus X} O^p =$ intersection of all free ideals of $C^*(X)$, where O^p is the set of those functions $'f'$ in $C(X)$ for which $cl_{\beta X} Z(f)$ is a neighborhood of the point p in βX [see[7], 7E, page 109]. Writing $K = cl_{\beta X}(\upsilon X \setminus X)$, we have for a nearly pseudocompact space X, $K \setminus X = \beta X \setminus X$ and consequently $C_H(X) = C_k(X) = \bigcap_{p \in K \setminus X} O^p$. This is rather an awkward way of writing a natural formula, nevertheless it prompts as the following proposition reveals, the right formula for $C_H(X)$, even when X is not nearly pseudocompact.

Theorem 2.7. *For any space X, $C_H(X) = \bigcap_{p \in (K \setminus X)} O^p$.*

Proof. Let $f \in \bigcap_{p \in K \setminus X} O^p$, then for each p in $K \setminus X$, $cl_{\beta X} Z(f)$ is a neighborhood of p in βX. Since $cl_{\beta X} Z(f)$ is disjoint with $X \setminus Z(f)$, we have therefore $p \notin cl_{\beta X}(X \setminus Z(f))$ and hence $p \notin cl_{\beta X}(cl_X(X \setminus Z(f)))$. Consequently $cl_X(X \setminus Z(f))$ is closed in $X \bigcup K$ and is accordingly a hard subset of X, thus $f \in C_H(X)$. So $\bigcap_{p \in K \setminus X} O^p \subseteq C_H(X)$. For the reversed inclusion relation, let us choose $f \in C_H(X)$, therefore $\forall p \in K \setminus X, p \notin cl_{\beta X}(X \setminus Z(f))$ and so there exists an open neighborhood W of p in βX such that $p \in W \subseteq \beta X \setminus cl_{\beta X}(X \setminus Z(f)) \subseteq cl_{\beta X} Z(f)$. This indicates that $f \in O^p$. Hence $C_H(X) \subseteq \bigcap_{p \in K \setminus X} O^p$. The theorem is completed □

For a nearly pseudocompact X, the set $\beta X \setminus (K \bigcup X)$ is void, nevertheless we have used theorem 2.7 to furnish the following description of points of the set $\beta X \setminus (K \bigcup X)$ in general in terms of the members of the ring $C_H(X)$.

Theorem 2.8. *A point $p \in \beta X$ is outside the set $K \bigcup X$ if and only if there is an f in $C_H(X)$ with $f^*(p) = \infty$, where $f^* : \beta X \to \mathbb{R}^* \equiv \mathbb{R} \bigcup \{\infty\}$ stands for the continuous extension of f to βX.*

The proof of theorem 2.8 requires the following auxiliary result.

Lemma 2.9. *Given any pair of distinct points p,q in βX with $p \in \beta X \setminus \upsilon X$, there is an f in O^q such that $f^*(p) = \infty$.*

Proof. There exists $g \in C^*(X)$ such that $g^\beta(p) = 0$ and $Z(g) = \emptyset$. Since $q \neq p$, there exists an open set V in βX such that $q \in V \subseteq cl_{\beta X} V \subseteq \beta X \backslash \{p\}$ and so there is an $h \in C^*(X)$ with $h^\beta(cl_{\beta X}) = \{0\}$ and $h^\beta(p) = \{1\}$. Since $V \bigcap X \subseteq Z(h)$, it then follows[[7], theorem 7.12(a), page:106], that $h \in O^q$.Now the relation $z(g) = \emptyset$ tells us that $cl_{\beta X} Z(g) = \emptyset$, which in view of Gelfand-Kolmogorov theorem [see[7], theorem:7.3, page-102] implies that $g \notin M^p \equiv M^p_C$, the maximal ideal in $C(X)$ corresponding to the point p. Hence $M^p(g) \neq 0$, where $M^p(g)$ stands for the residue class containing $'g'$ in the residue class field $C(X)/M^p$. Therefore there exists $\xi \in C(X)$ with $M^p(g)M^p(\xi) = 1$. Let $f = h\xi$, then $f \in O^q$ as O^q is an ideal of $C(X)$. Consequently by theorem 7.6 of [7], $|M^p(fg) - 1| = |M^p(h) - 1|$, becomes either 0 or an infinitely small element of the field $C(X)/M^p$, because $h^*(p) = h^\beta(p) = 1$ and this shows by the same theorem 7.6 of [7] that $(fg)^*(p) = 1$. Since $g^*(p) = 0$, it is clear that $f^*(p) = \infty$ □

Proof of theorem 2.8. Let $f \in C_H(X)$ be such that $f^*(p) = \infty$. Then $f \notin O^p$ and from theorem 2.7, $p \notin K \backslash X$, also $p \notin X$. Hence $p \notin K \bigcup X$. To prove the converse assume that $p \notin K \bigcup X$, in particular $p \in \beta X \backslash \upsilon X$. By lemma 2.9, for each $q \in K$, there is an $f_q \in C(X)$ with $f_q^*(p) = \infty$ and $f_q \in O^q$ and we have then $K \subseteq \bigcup_{q \in K}(int_{\beta X}(cl_{\beta X} Z(f_q)))$. The last relation yields in view of the compactness of K in βX that $K \subseteq \bigcup_{i=1}^n (int_{\beta X}(cl_{\beta X} Z(f_{q_i})))$, where $\{f_{q_i}\}_{i=1}^n$ is a suitable finite subfamily of $\{f_q\}_{q \in K}$. Let $f = f_{q_1} f_{q_2} ... f_{q_n}$, then $f^*(p) = \infty$ as each $f^*_{q_i}(p) = \infty$, also $Z(f) = \bigcup_{i=1}^n Z(f_{q_i})$. Now given $q \in K \backslash X$, there is an i with $q \in int_{\beta X}(cl_{\beta X} Z(f_{q_i}))$, which implies that $q \in int_{\beta X}(cl_{\beta X} Z(f))$ and this means $f \in O^q$. Hence by theorem 2.7, it follows that $f \in C_H(X)$. □

It is plain to note that for any point $p \in \beta X$, $f^*(p)$ is real for each f in $C_\infty(X)$ and in particular for each f in $C_K(X)$. The following result indeed takes care of the similar fact for $C_H(X)$ and $H_\infty(X)$.

Theorem 2.10. *For any point p in βX, $f^*(p)$ is real for each f in $C_H(X)$ when and only when $f^*(p)$ is real for each f in $H_\infty(X)$.*

Proof. For any subset of A of $C(X)$, we write $\upsilon_A X = \{p \in \beta X : f^*(p) \in \mathbb{R}, \forall f \in A\}$. With this terminology, the statement of theorem 2.10 tells us to show that $\upsilon_{C_H} X = \upsilon_{H_\infty} X$. Since $C_H(X) \subseteq H_\infty(X)$, it is plain to note that $\upsilon_{H_\infty} X \subseteq \upsilon_{C_H} X$. Let $p \in \beta X \backslash \upsilon_{H_\infty} X$ be chosen arbitrarily, then there is an $f \in H_\infty(X)$, such that $f^*(p) = \infty$. Let for each $n \in \mathbb{N}, Z_n = \{x \in X : |f(x)| \geq n\}$, then for each n, $p \in cl_{\beta X} Z_n$ and hence $p \in cl_{\beta X}\{x \in X : |f(x)| \geq \frac{1}{n}\}$. But the sets $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ are all hard as $f \in H_\infty(X)$, hence $p \in \beta X \backslash K \bigcup X$ and therefore by theorem 2.8, p can not belong to $\upsilon_{C_H} X$. This proves that $\upsilon_{C_H} X = \upsilon_{H_\infty} X$. □

3. The ring $\chi(X)$

Let T be the set of all those points p in βX for which $f^*(p)$ is real for each f in $C_H(X)$and for any subset S of βX, $C_S = \{f \in C(X) : f^*(p) \in \mathbb{R}, \forall p \in S\}$. Let $\chi(X) \equiv \chi = C_T(X)$. Then it is easy to verify that $\chi(X)$ is a ring lying between $C^*(X)$ and $C(X)$, which is further of C-type in the sense of Dominguez, Gomez and Mulero [6], as we have established in this section. Before proving that the ring $\chi(X)$ is the smallest subring of $C(X)$ containing $C^*(X)$ and $C_H(X)$ and is of C-type, we need introduce some further notations. Let $\Sigma(X)$ stand for the family of all the rings that lie between $C^*(X)$ and $C(X)$. For any two members $A(X) \equiv A$ and $B(X) \equiv B$ in $\Sigma(X)$, we write $A \sim B$ when and only when $\upsilon_A X = \upsilon_B X$. Then '$\sim$' defines an equivalence relation on $\Sigma(X)$. Suppose for $A \in \Sigma(X)$, $[A(X)]$=$[A]$ stands for the equivalence class which contains the member A. Then there exists a largest member in the equivalence class$[A]$ which is precisely the ring $A_1 = \{f \in C(X) : f$ has a continuous extension over $\upsilon_A X\}$ [see [[1], theorem 2.2]. That these largest members are really special members in the family $\Sigma(X)$, is expressed by the following proposition, which have been proved recently by Acharyya and Dey.

Since the paper [2] by Acharyya and Dey has not yet been published, we are giving the proof of the following theorem in detail for a ready reference to the reader. But prior to that we need some terminology. By a result of Plank [[13], theorem 2.9], it follows that the structure space of any ring $A(X)$ between $C(X)$ and $C^*(X)$, which is nothing but the set of all maximal ideals of $A(X)$ equipped with so-called hull-kernel topology is βX.

The complete list of maximal ideals of A is given by $\{M_A^p : p \in \beta X\}$, where $M_A^p = \{f \in A(X) : (f.g)^*(p) = 0$ for each g in A $\}$ [see Plank [13]].

Theorem 3.1 (Acharyya and Dey)**.** *The following statements are equivalent for any ring $A(X) \equiv A$ in $\Sigma(X)$.*

(1) A is a ring of C-type in the sense that A is isomorphic to the ring $C(Y)$ for some space Y.
(2) A is the largest member of its equivalence class.
(3) there exist a subset S of βX such that $A = C(\upsilon_S X)$.

To prove the above theorem, we shall use the following two lemmas.

Lemma A: The largest member of the equivalence class $[A(X)]$ containing $A(X)$ is given by $\{g|_X : g \in C(\upsilon_A X)\}$. [see [1]]

Lemma B: (Acharyya and Dey). Let $A(X) \in \sum(X)$ be a function ring. Then the map $f \mapsto f^{\upsilon_A}$ defines an isomorphism on $A(X)$ onto the ring $C(\upsilon_A X)$.

Proof. (of lemma B:) Since $A(X)$ is a function ring, there exists a real compact space Y with an isomorphism t from $A(X)$ onto $C(Y)$. As the property of being a real maximal ideal is an algebraic invariant, for any $p \in \upsilon_A X$, $t(M_A^p)$ is a real maximal ideal in $C(Y)$ and therefore it is fixed due to the realcompactness of Y. Accordingly $\bigcap_{g \in t(M_A^p)} Z_Y(g)$ is an one-pointic set, here $Z_Y(g)$ stands for the zero set of the function g in the space Y. We define a mapping $\Psi : \upsilon_A X \mapsto Y$ by the rule $\Psi(p) = \bigcap_{g \in t(M_A^p)} Z_Y(g)$. Then Ψ is clearly one-to-one. Again for any point y in Y if $M_y = \{h \in C(Y) : h(y) = 0\}$ is the corresponding fixed maximal ideal in $C(Y)$, then there is a unique point p in $\upsilon_A X$ with $M_y = t(M_A^p)$ and so $\Psi(p) = y$. Now $\{S_A(f) : f \in A(X)\}$ being a base for closed subsets of βX, where $S_A(f) = \{p \in \beta X : f \in M_A^p\}$ [13] it is obvious that $\{S_A(f) \bigcap \upsilon_A X : f \in A(X)\}$ is a base for closed subsets of $\upsilon_A X$ and we observe that for any $f \in A(X)$, $\Psi(S_A(f) \bigcap \upsilon_A X) = Z_Y(t(f))$. Hence Ψ carries the basic closed sets in $\upsilon_A X$ onto Y. Suppose $s : C(Y) \mapsto C(\upsilon_A X)$ is the isomorphism induced by Ψ, that is, for any g in $C(Y)$, $s(g) = g \circ \Psi$. Since $t : A(X) \mapsto C(Y)$ is already an isomorphism, we see that $s \circ t$ becomes an isomorphism on $A(X)$ onto $C(\upsilon_A X)$. Let us choose an $f \in A(X)$. To prove the theorem it is sufficient to prove that $t(f) \circ \Psi = f^{\upsilon_A}$.

Since $t(f)\circ\Psi$ is clearly a realvalued continuous function on $\upsilon_A X$ it is enough to show that it is an extension of f. Now if we choose $x \in X$ then for any h in the fixed maximal ideal M_A^x of $A(X)$, we have $t(h)(\Psi(x)) = 0$. Hence it follows that $t(f - \underline{f(x)})(\Psi(x)) = 0$ (where $\underline{f(x)}$ is the constant function on X which takes the value $f(x)$ at all points of X). Therefore it is immediate that $t(f)(\Psi(x)) = f(x)$. $\square$

Proof. (of the theorem:) (1) $\Leftrightarrow$ (3): Let T be a subset of βX and $B(X)$ be a member of $\sum(X)$ with $\upsilon_B X = \upsilon_{C_T} X$. We choose f in $A(X)$ and p in T arbitrarily. Since $T \subset \upsilon_{C_T} X$ it follows that $p \in \upsilon_{C_T} X$ and therefore $p \in \upsilon_B X$. Accordingly $f^*(p) \in \mathbb{R}$. Thus $f \in C_T$. Hence $A(X) \subset C_T(X)$, consequently $C_T(X)$ is the largest member of its equivalence class. Conversely let $A(X)$ be the largest member of its equivalence class $[A(X)]$. We shall show that $A(X) = C_{\upsilon_A X}(X)$. Since for each f in $A(X)$ and p in $\upsilon_A X$, $f^*(p)$ is real it is trivial that $A(X) \subseteq C_{\upsilon_A X}(X)$. Conversely let $f \in C_{\upsilon_A X}(X)$ and $p \in \upsilon_A X$. Then $f^*(p)$ is real and therefore $g = f^*|_{\upsilon_A X} \in C(\upsilon_A X)$. Lemma A tells us that $g|_X \in A$, but since $g|_X = f$ it follows that $f \in A(X)$. Thus $C_{\upsilon_A X}(X) \subseteq A(X)$. Hence $A(X) = C_{\upsilon_A X}(X)$.

(1) $\Leftrightarrow$ (2): If $A(X)$ is the largest member of its equivalence class, then we have already observed in the introductory section that $A(X)$ is a function ring.

Conversely let $A(X)$ be not the largest member of its equivalence class. Then from lemma A there exists a $g \in C(\upsilon_A X)$ for which $g|_X$ is not in $A(X)$. Accordingly there can not exists any $f \in A(X)$ with $f^{\upsilon_A} = g$ and therefore the canonical map $T : A(X) \longrightarrow C(\upsilon_A X)$ defined by $T(f) = f^{\upsilon_A}$ is not an isomorphism on $A(X)$ onto $C(\upsilon_A X)$. Hence from lemma B, $A(X)$ is never a function ring. $\square$

We write down also the following example from the paper of Dominguez, Gomez and Mulero [[6], 2.5].

Theorem 3.2 (Dominguez, Gomez and Mulero)**.** *If I is any proper ideal in $C(X)$, then the smallest subring of $C(X)$ containing $C^*(X)$ and I which is the set $C^*(X) + I = \{g + h : g \in C^*(X), h \in I\}$ is a ring of C-type.*

Theorem 3.3. *$\chi(X)$ is the smallest subring of $C(X)$ containing $C^*(X)$ and $C_H(X)$ and there exists a space Y such that $\chi(X)$ is isomorphic to $C(Y)$ as rings.*

Proof. Since $\chi(X) = C_T(X)$ which is isomorphic with $C(\upsilon_T X)$ via the isomorphism $f \mapsto f^*|_{\upsilon_T X}$ where $\upsilon_{C_T} X = \upsilon_{C_H} X$, it follows from the equivalence of the statements (1) and (3) of theorem 3.1 that $\chi(X)$ is isomorphic to the ring $C(Y)$ for some space Y (in fact here Y is $\upsilon_{C_T} X$). It is quite clear from the definition of $\chi(X)$ and from theorem 2.10 that $\upsilon_\chi X = \upsilon_{C_H} X$ from which it follows that $\upsilon_\chi X = \upsilon_{C^*(X)+C_H} X$. This indicates that the two rings $\chi(X)$ and $C^*(X) + C_H(X)$ belong to the same equivalence class of $\Sigma(X)$ modulo the relation $'\sim'$ on it. But from theorem 3.2, it is clear that the ring $C^*(X) + C_H(X)$ is of C-type and is contained in $\chi(X)$. Hence from theorem 3.1, it follows that each of the rings $\chi(X)$ and $C^*(X) + C_H(X)$ is the largest member of the equivalence class of $\chi(X)$, in particular, $\chi(X) = C^*(X) + C_H(X)$ □

As promised earlier, a characterization of nearly pseudocompactness of X in terms of $\chi(X)$ has also been furnished.

Theorem 3.4. *For a space X, the following statements are equivalent.*

(1) X is nearly pseudocompact
(2) $\chi(X) = C^*(X)$
(3) $|\beta X \setminus \upsilon_\chi X| \lneqq 2^c$

Proof. (1) ⇒ (2): If X is nearly pseudocompact then from theorem 2.5, $C_k(X) = C_H(X)$ so that $\chi_(X) = C^*(X) + C_H(X) = C^*(X)$.

(2) → (3): Trivial.

(3) → (1): Suppose X is not nearly pseudocompact . By theorem 2.5, there exists $f \in C_H(X) \setminus C_K(X)$, so that $cl_{\beta X}(X \setminus Z(f)) \bigcap \upsilon_\chi X = cl_X(X \setminus Z(f))$ and $X \setminus Z(f)$ contains a C-embedded copy of $\mathbb{N}$ along which f tends to infinity. Then it follows [see [7], theorem 6.9] that $cl_{\beta X}\mathbb{N} = \beta\mathbb{N}$ and hence $\beta\mathbb{N} \setminus \mathbb{N} \subseteq \beta X \setminus \upsilon_\chi X$, as because $\mathbb{N}$ is closed in X which immediately implies that $|\beta X \setminus \upsilon_\chi X| \geq 2^c$, a contradiction to (3). □

It is well known in the theory of rings of continuous functions that $C_\infty(X) = \bigcap_{p \in \beta X \setminus X} M^{*p} \equiv$ the intersection of all free maximal ideals of $C^*(X)$, where $M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}$ where f^β denotes the Stone-extension of f. We have generalized this result by finding a suitable description of the ring $H_\infty(X)$.

Theorem 3.5. *For any space X, $H_\infty(X) = \bigcap_{p \in K\setminus X} M_\chi^p$, where we recall once again that M_χ^p is the maximal ideal in the ring $\chi(X) \equiv \chi$ corresponding to the point p in βX.*

Proof. We first note that $H_\infty(X) \subseteq \chi(X)$. Let $f \in H_\infty(X)$ and $p \in K\setminus X$ be chosen arbitrarily. Then since $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is hard for each $n \in \mathbb{N}$, p cannot belong to the βX-closure of these sets, consequently $p \in cl_{\beta X}\{x \in X : |f(x)| < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Hence $|f^*(p)| \leq \frac{1}{n}$ for all n and so $f^*(p) = 0$. Since $\upsilon_{H_\infty} X\setminus X \supseteq K\setminus X$, for any $g \in \chi(X)$, $g^*(q)$ is real for each q in $K\setminus X$. Thus for each q in $K\setminus X$, $(fg)^*(q) = f^*(q)g^*(q) = 0$ and hence by Plank's result [[13], theorem 2.9], $f \in M_\chi^p$ for each p in $K\setminus X$. Therefore $H_\infty(X) \subseteq \bigcap_{p \in K\setminus X} M_\chi^p$.

Conversely suppose $f \in \bigcap_{p \in K\setminus X} M_\chi^p$. Then $f^*(p) = 0$, for each $p \in K\setminus X$. So definitely $p \notin cl_{\beta X}\{x \in X : |f(x)| \geqq \frac{1}{n}\}$, for each $n \in \mathbb{N}$ and $p \in K\setminus X$. Thus $cl_{\beta X}\{x \in X : |f(x)| \geqq \frac{1}{n}\}$ is closed in $X \cup K$, for each n and hence is hard in X. Thus $f \in H_\infty(X)$. $\square$

It may be pointed out in this connection in view of theorem 3.5 that this formulae for $H_\infty(X)$ agrees with the formulae $C_\infty(X) = \bigcap_{p \in \beta X\setminus X} M^{*p}$, for a nearly pseudocompact space X.

There are characterizations of pseudocompact spaces X in terms of extensions of ring homomorphism from $C^*(X)$ onto $\mathbb{R}$. Indeed it was established in [1] that X is pseudocompact if and only if every nonzero ring homomorphism from $C^*(X)$ into $\mathbb{R}$ extends to a ring homomorphism from $C(X)$ to $\mathbb{R}$. In the next theorem we have obtained an analogous characterization of nearly pseudocompact spaces.

Theorem 3.6. *A space X is nearly pseudocompact if and only if every non-zero ring homomorphism from $C^*(X)$ into $\mathbb{R}$ extends to a homomorphism from the ring $\chi(X)$ into $\mathbb{R}$.*

Proof. If X is nearly pseudocompact then it is trivial in view of theorem 3.4 that every non zero ring homomorphism from $C^*(X)$ to $\mathbb{R}$ extends to a homomorphism from $\chi(X)$ to $\mathbb{R}$. Conversely let X be not nearly pseudocompact, then by theorem 3.4, there exists a point $p \in \beta X\setminus \upsilon_\chi X$. Accordingly we can select an h from $\chi(X)$ with $h \geq 1$ and $h^*(p) = \infty$. Let $g = \frac{1}{h}$. Then $g \in C^*(X)$ and $g^\beta(p) = 0$.

Let $\psi : C^*(X) \to \mathbb{R}$ be the map defined by $\psi(f) = f^\beta(p)$, which is clearly non zero ring homomorphism. We assert that ψ has no extension to a ring homomorphism from $\chi(X)$ to $\mathbb{R}$. Suppose not and $\phi : \chi(X) \to \mathbb{R}$ is a ring homomorphism extending ψ. Then $\phi(1) = \phi(g)\phi(h) = g^\beta(p)\phi(h) = 0$, a contradiction to the obvious relation that $\phi(1) = 1$. The theorem is completed □

It is well known and proved long back in 1948 by Hewitt [10], that a space X is pseudocompact if and only if the relative topology on $C^*(X)$, due to the m-topology on $C(X)$ coincides with the uniform norm topology on $C^*(X)$ [see also [7], exercise 2N, page 35]. We have proved that the following theorem offers an analogous characterization of nearly pseudocompact spaces. But A little technicality is needed to describe these. On any ring, $A(X)$ lying between $C^*(X)$ and $C(X)$, we can define the topology, which we call m_A topology in which the neighborhood system at each point $g \in A(X)$ is given as follows: $B_u^A(g) = \{f \in A(X) : |f - g| \leq u\}$, u is a positive unit of $A(X)$. Then $A(X)$ becomes a topological ring under m_A topology. The only difficulty that may appear to show it a topological ring is the continuity of product. In this case for any positive unit u of $A(X)$, choose $u_1 = \frac{u}{u+|f|+|fg|}$ and $u_2 = \frac{u}{1+|f|}$. Then it can be shown that for any $f' \in B_{u_1}^A(f)$ and $g' \in B_{u_2}^A(g)$, $|fg - f'g'| \leq u$. Clearly in this terminology m_C topology on $C(X)$ is the same as the m-topology on it.

Theorem 3.7. *X is nearly pseudocompact if and only if the m_χ topology on $\chi(X)$ relativised to $C^*(X)$ coincides with the uniform norm topology on $C^*(X)$.*

Proof. Suppose X is not nearly pseudocompact, then $\upsilon_\chi X \neq \beta X$. Choose $p \in \beta X \backslash \upsilon_\chi X$. There exists an $f \in \chi(X)$, with $Z(f) = \emptyset$ and $f > 1$ such that $f^*(p) = \infty$. Since $\frac{1}{f} \in C^*(X)$, f is a unit of $\chi(X)$ and so is $\frac{1}{f}$. Therefore $B_{\frac{1}{f}}^\chi(0)$ becomes a neighborhood of 0 in $\chi(X)$ which does not contain any neighborhood of 0 with respect to the norm topology on $C^*(X)$ as because units in $C^*(X)$ are bounded away from 0. □

4. Hard pseudocompact spaces

In this section we have defined a space to be hard pseudocompact if $\chi(X) = C(X)$. Then it follows from theorem 3.4 that X is pseudocompact if and only if it is hard pseudocompact and nearly pseudocompact. The notion hard pseudocompactness as defined here is essentially algebraic, nevertheless we have furnished some topological characterizations of the same in the following result.

Theorem 4.1. *For a space X, the following statements are equivalent:*

(1) X is hard pseudocompact
(2) for a point $p \in \beta X, p \in \upsilon X$ if and only if $f^*(p)$ is real for each f in $\chi(X)$
(3) $\upsilon X \backslash X$ is closed in $\beta X \backslash X$
(4) Every non-void zero subset of the space $\upsilon_\chi X \equiv \{p \in \beta X : f^*(p)$ is real for each f in $\chi(X)\}$ intersects X.
(5) $|\upsilon_\chi X \backslash \upsilon X| < 2^c$

Since the relation $K\backslash X = \upsilon_{C_H} X\backslash X = \upsilon_\chi X\backslash X$ holds as we have observed in the preceding section and $K = cl_{\beta X}(\upsilon_X \backslash X)$, we have the following result, which we note down for our possible reference.

Lemma 4.2. $\upsilon_\chi X\backslash X = cl_{\beta X\backslash X}(\upsilon X\backslash X)$

Proof of theorem 4.1. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): Suppose (2) holds, then $\upsilon_\chi X = \upsilon_C X$ which says that $\chi(X)$ and $C(X)$ belong to the same equivalence class of $\Sigma(X)$, modulo the relation $' \sim '$. On the other hand both of $\chi(X)$ and $C(X)$ are rings of C-type, hence from theorem 3.1, it is plain that $\chi(X) = C(X)$, which means that X is hard pseudocompact.

(2) $\Leftrightarrow$ (3) follows trivially from lemma 4.2.

(2) $\Rightarrow$ (4): Suppose the condition (4) is false and Z be a non empty zero subset of the space $\upsilon_\chi X$ which misses X. Now there exists a zero subset Z_1 of the space βX with $Z_1 \bigcap \upsilon_\chi X = Z$. Since a zero subset of βX, which misses X misses also υX as can easily be seen, it follows that $\upsilon_\chi X\backslash \upsilon X$ is non empty and hence the condition (2) becomes false.

(4) $\Rightarrow$ (2): Assume that the condition (2) is false and choose a point p from $\upsilon_\chi X\backslash \upsilon X$. Then there exists an f in $C(X)$ such that $f \geq 1$ and $f^*(p) = \infty$. Set $g = \frac{1}{f}$, then $g \in C^*(X)$ and

$g^\beta(p) = 0$, g^β does not vanish anywhere on υX. Consequently $\{q \in \upsilon_\chi X : g^\beta(q) = 0\}$ becomes a non void zero subset of $\upsilon_\chi X$ which does not intersect X and the condition (4) becomes false then.

(2) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (2): Suppose (2) is false and choose $p \in \upsilon_\chi X \backslash \upsilon X$.

Then there exists f in $C(\upsilon X)$ such that $f^*(p) = \infty$. Since $p \in cl_{\beta X}(\upsilon X \backslash X) \equiv K$, f is unbounded on $T = \upsilon X \bigcap K$. Thus T contains a copy of $\mathbb{N}$, C-embedded in υX and hence $cl_{\upsilon X}\mathbb{N} = \beta\mathbb{N}$. Now $cl_{\beta X}\mathbb{N} \subseteq cl_{\beta X} T = K$, a compact subset of βX. $\mathbb{N}$ being closed in υX, $\beta\mathbb{N}\backslash\mathbb{N} \subseteq \upsilon_\chi X \backslash \upsilon X$ which implies that $|\upsilon_\chi X \backslash \upsilon X| \geq 2^c$. □

From the above theorem it follows that a realcompact space is hard pseudocompact. It also establishes the known fact that X is pseudocompact if and only if $\beta X = \upsilon X$, if and only if $|\beta X \backslash \upsilon X| < 2^c$, [[7], chapters 8, 9]. It may be mentioned in this context that Rayburn in 1978 asked in [14] if there is any internal characterization in terms of subsets of X of the property that $\upsilon X \backslash X$ is closed in $\beta X \backslash X$, that is in our terminology, of hard pseudocompact spaces. Henriksen and Rayburn in 1987 in [9] furnished some necessary conditions, one of which tells that every C-embedded realcompact subset of X is hard in X, for this to happen. However these conditions are not sufficient as shown by the same authors in the same paper. However we have obtained two more characterizations of hard pseudocompact spaces in terms of the ring $\chi(X)$.

Theorem 4.3. *A space X is hard pseudocompact if and only if every non zero ring homomorphism from $\chi(X)$ into $\mathbb{R}$ extends to a ring homomorphism from $C(X)$ into $\mathbb{R}$.*

Proof. Analogous to the proof of theorem 3.6. □

Theorem 4.4. *X is hard pseudocompact if and only if the m-topology on $C(X)$ relativised to $\chi(X)$ is the same as m_χ topology on $\chi(X)$.*

Proof. Suppose X is not hard pseudocompact. Then $\upsilon_\chi X \neq \upsilon X$. Choose $p \in \upsilon_\chi X \backslash \upsilon X$. There exists f in $C(X)$, such that $Z(f) = \emptyset$, $f > 0$, $f^*(p) = 0$. Consider the neighborhood of 0, $B_f^C(0)$ in $C(X)$.

Then there does not exist any positive unit u of $\chi(X)$ such that $0 \in B_u^X(0) \subseteq B_f^C(0)$, for if there exists a positive unit u of $\chi(X)$ satisfying the above property, then $u \leq f$ and thus $u^*(p) = 0$ and so $(ug)^*(p) = 0, \forall g \in \chi(X)$ which implies that $u \in M_\chi^p$, a contradiction. □

We have concluded this article after giving an example of a hard pseudocompact space which is neither pseudocompact nor real compact.

Example: Consider the space $X = \omega_1 - \{\omega_0\}$, where ω_1 denotes the first uncountable ordinal and ω_0 is the first infinite ordinal. Then X is an example of non realcompact, not pseudocompact, hard pseudocompact space. Since ω_0 is copy of $\mathbb{N}$, C-embedded in X, is closed in υX and $[\omega_0 + 1, \omega_1)$ is a relatively pseudocompact subspace of X therefore $\upsilon X = cl_{\upsilon X} X = cl_{\upsilon X} \omega_0 \bigcup cl_{\upsilon X}[\omega_0+1, \omega_1) = \omega_0 \bigcup [\omega_0 + 1, \omega_1]$. Thus $\upsilon X - X = \{\omega_1\}$ which is closed in $\beta X - X$.

This example further prompts us to write down the following theorem.

Theorem 4.5. *If a space X can be expressed as a union of two subsets A and B, where A is relatively pseudocompact in X, and B is a realcompact subset of X, which either C-embedded in X or co-zero subset of X, then X is hard pseudocompact.*

Proof. (a) If B is $C - embedded$ in X:

Since B is a realcompact $C - embedded$ subset of X, $B = cl_{\upsilon X} B$. A being relatively pseudocompact subset of X, $cl_{\beta X} A \subseteq \upsilon X$. The rest follows immediately from above example.

(b) If B is a co-zero subset of X:

Since B is a co-zero subset of X, by [3], $\upsilon B = \upsilon X - cl_{\upsilon X}(X - B)$. But B is realcompact, so, $B = \upsilon X - cl_{\upsilon X}(X - B)$. Now A is a relatively pseudocompact subset of X containing $X - B$. But $X - B$ is a zero set $Z(f)$ for some $f \in C(X)$. As $Z(f)$ is contained in the relatively pseudocompact subset A of X and $cl_{\beta X} Z(f) \bigcap (\beta X - X) = \upsilon X - X$, $\upsilon X - X$ is closed in $\beta X - X$. □

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