Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No.2, 2005
Pages 641-671

BANACH DOMAINS: COMPUTATIONAL MODELS OF BANACH SPACES

PAWEŁ WASZKIEWICZ

ABSTRACT. This paper initiates research on Banach domains which are domain-theoretic models of Banach spaces.

1. Introduction

This paper is thought as an introduction to the study of domain models of Banach spaces. Domains are certain complete partial orders with an intrinsic notion of approximation, and domain theory is a vast subject arising from considerations in Theoretical Computer Science, which links tightly with Topology: it studies structures of "classical" mathematics, like topological, metric, vector spaces, via properties of suitably constructed partial orders.

We say that a space (X, τ) is modelled by a continuous poset (P, σ), where σ is the Scott topology on P, when X is homeomorphic to the subset of maximal elements of P in the subspace Scott topology: $(X, \tau) \cong (\max(P), \sigma \mid_{\max(P)})$. One then can reason about properties of the modelled topology in terms of the properties of the Scott topology of the underlying domain. Pioneering works of Lacombe [19], Martin-Löf [28], Scott [30], Weihrauch and Schreiber [33], and Kamimura and Tang [16] on modelling spaces have led the way to a variety of applications including: real number computation [11], integration [13], [3], [7] and differential calculus

2000 *Mathematics Subject Classification.* Primary O6B35, 46B99, 54H99, 54D10; Secondary 22A30, 06F99, 52A27, 52A38.

Key words and phrases. Banach domain, continuous domain, Scott topology, partial metric, measurement, convex bodies.

[10], geometry [9], dynamical systems, fractals and measure theory [4], [5], and basic quantum mechanics [2]. Most of the applications are surveyed in [6].

What kind of topologies admit models? Lawson [21], [20] proved, among other things, that Polish spaces do. An explicit construction of such a domain was given by Edalat and Heckmann in [8]. The formal ball model, as it is called, has been later adapted to the special case of Banach spaces in [12] and used to introduce the notion of partial metric to domain theory [15].

In a series of papers [22], [23], [26] following [24] Martin proposed various techniques for studying models of topologies, including the notion of measurement. He generalized Lawson's result by showing that spaces modelled by ω-continuous dcpos are regular iff they are Polish [27]. Moreover, outside the metrizable case, Reed and Martin characterized developable spaces as the ones modelled by continuous dcpos with measurement. Recently, Martin also observed a fundamental connection between order completeness of continuous domains and the Choquet-completeness of topological spaces [25] (the main result is that the maximal elements of a domain are Choquet complete). The role of completeness due to Choquet in domain theory is still not well-understood though.

Motivated by considerations from the area of injective spaces, people became concerned with finding *bounded complete models*. Bounded complete domains have an especially pleasing property that every continuous mapping between modelled spaces extends to a Scott continuous function between the models; moreover, such an extension can be defined in a canonical way. Recently, Kopperman, Künzi and Waszkiewicz characterized all topologies that have bounded complete models [17]. Combining their results with work of Künzi [18] resulted in a construction of such a model for any complete metric space.

In this paper we focus on models of Banach spaces. In fact, Sünderhauf in a conference paper [31] isolated certain properties of the formal ball model of a Banach space and presented them in the axiomatic way. In effect, he ended up with a poset equipped with addition and scalar multiplication and called it a Banach domain. In this paper we explain the rudiments of Banach domains, discuss their axiomatics, examine generalized distances associated with them, study in great detail the Banach domain of convex compacta of the n-dimensional Euclidean space and make first steps towards the domain-theoretic account of duality theory for Banach spaces.

We are assuming familiarity with basics of domain theory to the extent present in the first four chapters of [1]. In particular, the notion of a continuous dcpo and the Scott topology play a major role in this paper. We will also speak about Matthews' partial metrics: For the definitions, examples and basic properties of (weak) partial metrics we direct the reader to [29] and [15]. We occasionally mention measurements in the sense of Keye Martin ([24], [32]) since their theory offers elegant and efficient tools for studying partial metrics and other distances on domains.

2. Basics of Banach domains

The poset of nonempty, closed intervals of the real line under the reverse inclusion is denoted by $\mathbf{I}\mathbb{R}$. The following definition is given by Philipp Sünderhauf in [31].

Definition 2.1. A Banach domain $(D, \sqsubseteq, +, \cdot, \mathbf{0})$ is a continuous dcpo $(D, \sqsubseteq)$, called also a (continuous) domain, together with addition $+\colon D \times D \to D$ and scalar multiplication $\cdot\colon \mathbf{I}\mathbb{R} \times D \to D$, which are both Scott-continuous, and a constant $\mathbf{0} \in D$. The operations are governed by the axioms (B1)-(B8) below:

(B1) $x + (y + z) = (x + y) + z$
(B2) $x + y = y + x$
(B3) $x + \mathbf{0} = x$
(B4) $a(bx) = (ab)x$
(B5) $0x = \mathbf{0}$
(B6) $nx = \underbrace{x + ... + x}_{n \text{ times}}$
(B7) $ax + bx \sqsubseteq (a + b)x$
(B8) $ax + ay \sqsubseteq a(x + y)$

for all $n \in \omega \setminus \{0\}$, for all $a, b \in \mathbf{I}\mathbb{R}$ and $x, y, z \in D$.

(We write $-y$ for $(-1)y$ and $x - y$ for $x + (-1)y$. Intervals of zero length are identified with real numbers. Furthermore, we assume that $\perp + x = x + \perp = \perp$ and $a\perp = \perp a = \perp$ whenever the least element $\perp$ exists in the domain.)

In this paper we have sometimes found it useful to require some additional properties of the algebraic operations, namely:

(B9) $\bigsqcap_{i\in I} ax_i = a \bigsqcap_{i\in I} x_i$
(B10) $\bigsqcap_{i\in I}(x_i + y) \sqsubseteq \bigsqcap_{i\in I} x_i + y$

for all $y \in D, \{x_i\} \subseteq D, a \in \mathbf{I}\mathbb{R}$, whenever the infima exist.

Axioms (B9) and (B10) prove especially useful when the Banach domain is in addition bounded complete (see Theorem 4.8 below).

The axioms presented above capture only the basic intended properties of models of Banach spaces. The example of the formal ball model suggests that the inequalities in (B7) and (B8) cannot, in general, be improved to equality. On the other hand, the examples suggest that we could require the equality in (B7) and (B8) to hold in the case of multiplication by degenerated intervals (that is, real numbers):

(B11) $ax + bx = (a + b)x$ and $ax + ay = a(x + y)$ for all $a, b \in \mathbb{R}$, $x, y \in D$.

Occasionally, we will require that addition and scalar multiplication are strictly monotone:

(B12) If $x \sqsubseteq y$ and $x + z = y + z$ for some $z \in D$, then $x = y$, for all $x, y \in D$.
(B13) if $x \sqsubseteq y$ and $cx = cy$ for some $c \in \mathbf{I}\mathbb{R}$, then $x = y$, for all $x, y \in D$.

Note that $\mathbf{I}\mathbb{R}$ with pointwise operations is a Banach domain.

Definition 2.2. An element x is *invertible* iff $x - x = \mathbf{0}$.

For $c \in \mathbf{I}\mathbb{R}$, the set of complements of c is defined as

$$c^{-1} \stackrel{def}{=} \{z \in \mathbf{I}\mathbb{R} \mid zc \sqsubseteq 1\}.$$

Note that $c^{-1} = \emptyset$ iff $c = 0$.

Most of the properties of Banach domains stated below appear in [31], where they are listed without proof.

Proposition 2.3. *The following hold in any Banach domain D for any $x, y \in D$ and $a, c \in \mathbf{I}\mathbb{R}$:*

(1) $x - x \sqsubseteq \mathbf{0}$,
(2) $a\mathbf{0} = \mathbf{0}$,
(3) $\mathbf{0}$ *is invertible,*
(4) *Every invertible element is maximal,*

(5) *If* $cx = \mathbf{0}$ *and* $c \neq 0$*, then* $x = \mathbf{0}$*,*
(6) *If* cx *is invertible and* $c \neq 0$*, then* x *is invertible,*
(7) *If* c *and* x *are maximal, then* cx *is maximal,*
(8) *If* c *and* x *are invertible, then* cx *is invertible,*
(9) *Addition preserves invertibility,*
(10) *Addition preserves maximality iff every maximal element is invertible.*

Proof. (1) $x - x = x + (-1)x \sqsubseteq (1-1)x = 0x = \mathbf{0}$. (2) $a\mathbf{0} = a(0x) = (a \cdot 0)x = 0x = \mathbf{0}$. (3) $\mathbf{0} - \mathbf{0} = \mathbf{0} + (-1)\mathbf{0} =$ (by (2) above) $\mathbf{0} + \mathbf{0} = \mathbf{0}$.

(4) Assume that x is invertible and $x \sqsubseteq y$. Then $y - y \sqsubseteq \mathbf{0} = x - x$ by the assumption and (1). On the other hand, by monotonicity of $-$, $x - x \sqsubseteq y - y$. Thus $\mathbf{0} = x - x = y - y$, which proves that y is invertible. Now, by monotonicity of $-$ again, $y - x \sqsubseteq y - y$, which yields $y - x \sqsubseteq \mathbf{0}$. By monotonicity of $+$ applied to this last inequality, we have $y = y + 0 = y + (x - x) = (y - x) + x \sqsubseteq \mathbf{0} + x = x$. That is, $y \sqsubseteq x$, which together with the assumption $x \sqsubseteq y$ gives $x = y$. We have proved that x is maximal.

(5) $x = 1x \sqsupseteq (zc)x$ for any $z \in c^{-1}$ by monotonicity of the scalar multiplication. But now, $(zc)x = z(cx) = z\mathbf{0} = \mathbf{0}$. We have shown that $x \sqsupseteq \mathbf{0}$. However, $\mathbf{0}$ is invertible, and hence maximal. Thus, $x = \mathbf{0}$.

(6) We have $c(x - x) \sqsupseteq cx - cx = \mathbf{0}$. But $\mathbf{0}$ is invertible, hence maximal, which gives $c(x - x) = \mathbf{0}$. By (5) above, $x - x = \mathbf{0}$. Therefore x is invertible.

(7) Let $cx \sqsubseteq z$ for some $z \in D$ and c, x maximal. If $c = 0$, we are done. Otherwise, by monotonicity of scalar multiplication, $x = \frac{1}{c}c \sqsubseteq \frac{1}{c}z$. Hence $x = \frac{1}{c}z$ and so $cx = z$, as required.

(8) Let c, x be invertible. We have $\mathbf{0} = x - x = c\frac{1}{c}x - c\frac{1}{c}y \sqsubseteq \frac{1}{c}(cx - cx)$. Hence $\mathbf{0} = \frac{1}{c}(cx - cx)$ and consequently, $\mathbf{0} = c\mathbf{0} = cx - cx$. This shows that cx is invertible.

(9) Suppose x, y invertible. Then $(x+y) - (x+y) \sqsupseteq x + y - x - y = (x - x) + (y - y) = \mathbf{0}$. Hence $(x + y) - (x + y) = \mathbf{0}$ by maximality of $\mathbf{0}$.

(10) Suppose that addition preserves maximality. Let x be maximal. Then $-x$ is maximal by (7). By assumption, $x - x$ is maximal. However, $x - x \sqsubseteq \mathbf{0}$ and so $x - x = \mathbf{0}$. Conversely, suppose that

every maximal element is invertible. Let x, y be maximal and suppose that $x + y \sqsubseteq z$. Then $y \sqsubseteq z - x$ and hence $y = z - x$ by maximality of y. But then $y + x = z - x + x = z$, as required. $\square$

We will now describe a canonical example of a Banach domain from [31].

Example 2.4. Let $(X, ||\cdot||)$ be a Banach space. Then the collection of closed balls

$$C(x, r) = \{y \in X \mid ||x - y|| \leq r\}$$

of radius $r \geq 0$ around the point $x \in X$ forms a domain and, in fact, is order isomorphic to the formal ball model of X (the latter fact is proved in [8]). That is, the set

$$\mathbf{B}X = \{\perp\} \cup \{C(x, r) \mid x \in X, r \geq 0\}$$

is a poset when ordered by $\perp \sqsubseteq A$ for all $A \in \mathbf{B}X$ and

$$C(x, y) \sqsubseteq C(y, s) \iff ||x - y|| \leq r - s$$

(the ordering is readily seen to be the inverse inclusion of the balls). The way-below relation is given by $\perp \ll A$ for all $A \in \mathbf{B}X$ and

$$C(x, y) \ll C(y, s) \iff ||x - y|| < r - s.$$

Now, addition $+\colon \mathbf{B}X \times \mathbf{B}X \to \mathbf{B}X$ is defined as expected:

$$C(x, r) + c(y, s) = C(x + y, r + s)$$

and $\perp + A = A + \perp = \perp$ for all $A \in \mathbf{B}X$. Scalar multiplication is of type $\mathbf{I}\mathbb{R} \times \mathbf{B}X \to \mathbf{B}X$ and is defined by

$$[a - \varepsilon, a + \varepsilon] \cdot C(x, r) = C(ax, \varepsilon||x|| + |a|r + \varepsilon r),$$

where one should note that every interval can be given in terms of its midpoint a and a radius ε as above. We also require $\perp \cdot A = A \cdot \perp = \perp$ to make the definition complete. One can readily check that the formal ball model with the operations presented above is a Banach domain. Moreover, axioms (B9) and (B10) hold. However, this Banach domain is not bounded complete in general. On the other hand, the algebraic operations are the greatest Scott-continuous extensions of addition and scalar multiplication on points.

3. Existence of models for Banach spaces

In this section we show that every Banach space can be modelled by a Banach domain, namely, the domain of formal balls. We demonstrate the converse as well: The subspace of invertible elements of any Banach domain is a Banach space with the norm and metric induced by the intrinsic structure of the model. Both results are due to Philipp Sünderhauf and appear without proof in [31]. In order to understand the situation better, we propose a longer route towards the main results. On the way we define symmetric elements and thoroughly analyze their structure. The symmetrics prove to be a major tool in defining partial norms, which, in turn, induce generalized distance on Banach domains.

Definition 3.1. We say that an element b of a Banach domain is *symmetric*[1] if $b = -b$.

Every symmetric element is below zero (as $b = -b$ implies $2b = b + b = b - b \sqsubseteq \mathbf{0}$ and so $b \sqsubseteq \mathbf{0}$). Every element of the form $[-1, 1]x$ for $x \in D$ is symmetric: $-[-1, 1]x = [-1, 1]x$. Finally, for $y \in D$, the difference $y - y$ is symmetric as $-(y - y) \sqsupseteq -y + y = y - y$ and $y - y = (-1)(-1)(y - y) \sqsupseteq (-1)(-y + y) = -(y - y)$.

Proposition 3.2. *Let D be a Banach domain.*

(1) *x is symmetric iff cx is symmetric for any $c \in \mathbb{R} \setminus \{0\}$.*
(2) *If x is symmetric, then $x + x$ is symmetric.*
(3) *x is symmetric iff $x - x = x + x$.*
(4) *x is symmetric iff $x \sqsubseteq -x$.*
(5) *Partially linear maps*[2] *preserve symmetric elements.*

Proof. (1) If $x = -x$, then $cx = c(-x) = -(cx)$. Conversely, if $cx = -cx$ and $c \neq 0$, then $x = \frac{1}{c}cx = \frac{1}{c}(-cx) = -x$. (2) Immediate by (1), since $x + x = 2x$. (3) If x is symmetric, we have $x + x = x + (-x) = x - x$. Conversely, suppose that x is not symmetric. Then using (1) this is equivalent to say that $2x = x + x$ is not symmetric. But $x + x = x - x$ and the latter element is known to be symmetric, a contradiction. For the nontrivial direction in (4), suppose $x \sqsubseteq -x$. Multiplying the inequality

[1]We could also use the term: "difference element of b" which would agree with terminology of convex analysis in the case when b is a convex body.

[2]See Definition 4.1.

by (-1) gives $-x \sqsubseteq -(-x) = x$, as required. For (5), let u be partially linear and x symmetric. Then $u(x) = u(-x) \sqsubseteq -u(x)$ and so $u(x)$ is symmetric by (4). □

Let x be any element of a Banach domain D. A *symmetrization* of x is the biggest symmetric element below x denoted by $\mathcal{S}(x)$. When does the symmetrization exist for every element of the domain? Surely, if the domain is bounded complete, then the set of symmetric elements of x is nonempty (the bottom belongs to it, and the element $[-1,1]x$ as well), bounded by x, and thus have a supremum $\mathcal{S}(x)$.

Proposition 3.3. *If the scalar multiplication preserves binary infima formed in* $\mathbf{I}\mathbb{R}$, *then every element has a symmetrization given by* $\mathcal{S}(x) = [-1,1]x$.

Proof. Under the assumption about infima, for every symmetric element y we have $y = (-1)y \sqcap 1y = [-1,1]y$. Hence if $z \sqsubseteq x$ and z is symmetric, $z = [-1,1]z \sqsubseteq [-1,1]x$, which proves that $[-1,1]x = \mathcal{S}(x)$. □

In the rest of the section we will assume that we deal with Banach domains where every element has a symmetrization.

What is the structure of the set of symmetric elements $\mathcal{S}(D)$ of D? In the case of the interval domain, they form a chain with supremum $\mathbf{0}$ and infimum $\perp$. (In any case $\mathbf{0}$ and $\perp$ are the biggest and the smallest in $\mathcal{S}(D)$.) Presently, we do not know much about the relative order in $\mathcal{S}(D)$.

Observation 3.4. In any Banach domain D, the subset $\mathcal{S}(D)\setminus\{\mathbf{0}\}$ has a supremum $\mathbf{0}$. In addition, if $\mathbf{0}$ is not compact and $x \ll \mathbf{0}$ for every symmetric element x, then $\mathcal{S}(D)\setminus\{\mathbf{0}\}$ is directed.

Proof. It is clear that $\mathbf{0}$ is an upper bound for $\mathcal{S}(D)\setminus\{\mathbf{0}\}$. Let $z \ll \mathbf{0}$ for some $z \in D$. Since $\mathbf{0} = 0z$, then by Scott-continuity of the scalar multiplication, there exists $\varepsilon > 0$ such that $z \ll [-\varepsilon,\varepsilon]z \sqsubseteq \mathbf{0}z = \mathbf{0}$. But $[-\varepsilon,\varepsilon]z = \varepsilon[-1,1]z$ is symmetric. In effect we have shown that the ideal $\mathop{\downarrow\!\!\!\downarrow}\mathbf{0}$ is contained in the ideal $\downarrow(\mathcal{S}(D)\setminus\{\mathbf{0}\})$ and since $\mathbf{0} = \bigsqcup^{\uparrow}\mathop{\downarrow\!\!\!\downarrow}\mathbf{0}$, the first claim follows.

Now, assume that $x \ll \mathbf{0}$ for every symmetric element $x \in D$. If x_1, x_2 are symmetric and different than $\mathbf{0}$, then by interpolation, there exists x_3 with $x_1, x_2 \ll x_3 \ll \mathbf{0}$. We can assume that

x_3 is different than $\mathbf{0}$, since the latter is not compact. Therefore, $x_1, x_2 \sqsubseteq \mathcal{S}(x_3) \sqsubseteq x_3$, and so the second claim in now proved. □

Observation 3.5. The mapping $\mathcal{S}\colon D \to D$, whenever well-defined, is monotone and idempotent.

Problem 3.6. Find a sufficient and necessary condition for $\mathcal{S}(x) = [-1,1]x$.

Observation 3.7. If the symmetrization is given as a multiplication by $[-1,1]$, then sum of symmetric elements is symmetric.

Proof. For, if w, z are symmetric, then $w+z = [-1,1]w+[-1,1]z \sqsubseteq \mathcal{S}(w+z)$. Since $\mathcal{S}(w+z) \sqsubseteq w+z$ always holds, we have $w+z = \mathcal{S}(w+z)$, which was needed. □

Here we have a chance to "see symmetric elements in action": they are used to define an important part of the intrinsic structure of any Banach domain – a partial norm.

Definition 3.8. Let D be a Banach domain and let $b \ll \mathbf{0}$ be any fixed[3] symmetric element of $D \setminus \{\perp\}$. Define a *partial norm* $||\cdot||\colon D \to \mathbf{I}\mathbb{R}$ to be given by:

$$||x|| = \bigsqcup^{\uparrow}\{[-s,s] \mid s>0, [-s,s]b \sqsubseteq x\}, \qquad x \in D \setminus \{\perp\}$$

and

$$||\perp|| = \perp.$$

The partial norm is well-defined: Let x be any element of $D\setminus\{\perp\}$. Since $b \ll \mathbf{0} = 0x$, Scott-continuity of the scalar multiplication gives a real number, say $1/s$, such that $b \ll [-(1/s), 1/s]x \sqsubseteq (1/s)x$. Therefore, $[-s,s]b \sqsubseteq sb \sqsubseteq x$ and thus we have shown that the set $\{[-s,s] \mid [-s,s]b \sqsubseteq x\}$ is always nonempty.

The mapping $r\colon \mathbf{I}\mathbb{R} \to [0,\infty]^{op}$ defined as $r[a,b] = b$ and $r\perp = \infty$ satisfies $r(x+y) = rx + ry$. It is not monotone and does not satisfy $r(cx) = c(rx)$ for $c \in \mathbb{R}$. However, its restriction to the set of all symmetric elements $r\colon \mathcal{S}(\mathbf{I}\mathbb{R}) \to [0,\infty]^{op}$ is monotone, commutes with the multiplication by reals and in fact, is an order isomorphism between $\mathcal{S}(\mathbf{I}\mathbb{R})$ and $[0,\infty]^{op}$. By the abuse of language

[3] Any other choice of $b' \ll \mathbf{0}$, $b' \neq b$, gives a different partial norm. However, no property of partial norms in the sequel depends on the choice of generator b, and we have found it convenient to speak about *the* partial norm on D.

the triple composition $scale \circ r \circ || \cdot ||$ of type $D \to [0,1]^{op}$, where $scale(t) \stackrel{def}{=} t/(1+t)$ for $t \in [0,\infty)$ and $scale(\infty) = 1$, will be again called a partial norm and denoted by $|| \cdot ||$. It is clearly given by $||x|| = \inf\{s \in (0,1] \mid sb \sqsubseteq x\}$ for any $x \in D$.

Clearly, by definition, the partial norm is monotone. The following elementary observation proves to be very useful in proofs:

Proposition 3.9. *For every $x \in D$, we have $||x|| = ||[-1,1](x)||$.*

Proof. We have $[-s,s]b \sqsubseteq [-1,1][-s,s]b \sqsubseteq [-1,1]x$ for every element $[-s,s]b$ below x. Thus $||x|| \sqsubseteq ||[-1,1]x||$. On the other hand, monotonicity of the partial norm gives $||[-1,1]x|| \sqsubseteq ||x||$, since $[-1,1]x \sqsubseteq x$. □

We will now justify the name "partial norm" by showing that our map satisfies some properties analogous to those of a norm. However, there are some (unexpected?) troubles with the triangle inequality which, as we know, always holds for norms in vector spaces but not necessarily in our generalized setup. Therefore, we can state with confidence only the following:

Proposition 3.10. *The partial norm satisfies:*

1. $||x|| = 0$ *iff* $x = \mathbf{0}$.
2. $||cx|| = |c| \cdot ||x||$, *where* $c \in \mathbb{R}$.
3. $||x|| = \bot$ *iff* $x = \bot$.

Proof. (1) For the nontrivial direction, suppose that $||x|| = 0$. This means that $[-s,s]b \sqsubseteq x$ for every $s > 0$. Hence $\mathbf{0} = 0b = (\bigsqcup^{\uparrow}_{s>0}[-s,s])b = \bigsqcup^{\uparrow}_{s>0}([-s,s]b) \sqsubseteq x$. Therefore, $x = \mathbf{0}$ by maximality of zero. For (2), note that $||-x|| = ||x||$ since $[-s,s]b$ is symmetric for any $s > 0$. Suppose that $c > 0$. We have

$$\begin{aligned} \{[-s,s] \mid sb \sqsubseteq cx\} &= \{[-s,s] \mid (s/c)b \sqsubseteq x\} \\ &= \{[-cz,cz] \mid zb \sqsubseteq x\} \\ &= \{c[-z,z] \mid zb \sqsubseteq x\}. \end{aligned}$$

Therefore,

$$\bigsqcup{}^{\uparrow}\{[-s,s] \mid sb \sqsubseteq cx\} = \bigsqcup{}^{\uparrow}\{c[-z,z] \mid zb \sqsubseteq x\} = c(\bigsqcup{}^{\uparrow}\{[-z,z] \mid zb \sqsubseteq x\}),$$

which proves that $||cx|| = c||x||$. Now, note that $0||x|| = \mathbf{0} = ||\mathbf{0}|| = ||0x||$ and for $c < 0$ we have $||cx|| = ||-(-c)x|| = ||(-c)x|| = (-c)||x||$. We can thus conclude that $||cx|| = |c| \cdot ||x||$ for any real number c. For (3), if $x = \bot$, then by definition, $||x|| = \bot$. Conversely, if $x \neq \bot$, then there exists $\infty > s > 0$ such that $[-s, s]b \sqsubseteq x$. Therefore, $||x|| \sqsupseteq [-s, s] \sqsupset \bot$, as required. □

Nevertheless, the triangle inequality holds when we restrict to a subspace of invertible elements:

Proposition 3.11. *A partial norm on a Banach domain D is a norm when restricted to* $\mathrm{inv}(D)$*. Moreover, any two partial norms give equivalent norms when restricted to* $\mathrm{inv}(D)$*.*

Proof. Since invertible elements are closed under addition and multiplication, the partial norm restricted to $\mathrm{inv}(D)$ satisfies conditions (1) and (2) in Proposition 3.10. On the other hand, the equality $(a + b)x = ax + bx$ holds for any $a, b \in \mathbb{R}$ and $x \in \mathrm{inv}(D)$ and it is easy to see that it implies that the partial norm satisfies the triangle inequality, when restricted to $\mathrm{inv}(D)$. Finally, the restriction of the partial norm never takes the value ∞, since $||x|| = \infty$ iff $x = \bot$. The second part is easy. □

Theorem 3.12 (Sünderhauf)**.** *If D is a Banach domain, then* $(\mathrm{inv}(D), ||\cdot||)$ *is a Banach space. Conversely, every Banach space arises as* $(\mathrm{inv}(D), ||\cdot||)$ *for some Banach domain.*

Proof. (outline) Let D be a Banach domain. By Proposition 3.11, $\mathrm{inv}(D)$ is a normed space. The norm topology and the Scott topology coincide on $\mathrm{inv}(D)$. Completeness of this space is proved in [31]. Conversely, if $(X, ||\cdot||)$ is a Banach space, then the formal ball model, Example 2.4, is a Banach domain. The partial norm $||x|| \stackrel{def}{=} \inf\{s > 0 \mid sC(\mathbf{0}, 1) \sqsubseteq x\}$ coincides with the original norm when restricted to $\mathrm{inv}(D) = \max(D)$. □

The example below demonstrates that sometimes we can expect that the triangle inequality holds on the whole domain:

Example 3.13. Any partial norm $||\cdot||\colon \mathbf{I}\mathbb{R} \to [0, \infty]^{op}$ satisfies

$$||x + y|| \leq ||x|| + ||y||$$

for all $x, y \in \mathbf{I}\mathbb{R}$.

For example, if $b = [-1, 1]$, then $||[\underline{a}, \overline{a}]|| = \max\{|\underline{a}|, |\overline{a}|\}$.

Proof. Fix any $b \in \mathbf{I}\mathbb{R}$, which is symmetric around zero. Let $s_1, s_2 > 0$ be such that $s_1 b \sqsubseteq x$ and $s_2 a \sqsubseteq y$. However, $s_1 b + s_2 b = [s_1 \underline{b}, s_1 \overline{b}] + [s_2 \underline{b}, s_2 \overline{b}] = [(s_1 + s_2)\underline{b}, (s_1 + s_2)\overline{b}] = (s_1 + s_2)b$. Therefore, $(s_1 + s_2)b = s_1 b + s_2 b \sqsubseteq x + y$ by monotonicity of addition. This yields $||x + y|| \leq s_1 + s_2$ and, consequently, $||x + y|| \leq ||x|| + ||y||$. □

Before stating the necessary and sufficient condition for the satisfaction of the triangle inequality for a partial norm, we would need an important concept of a distance induced by a partial norm. In the case of norms, this distance is, of course, a metric. However, in the world of domains and T_0 topologies, the induced distance fails to satisfy the reflexivity axiom (that is, $p(x, x) = 0$) and so we will have to deal with some generalization of the metric.

Definition 3.14. A distance induced by a partial norm on a Banach domain D is the mapping $p \colon D \times D \to \mathbf{I}\mathbb{R}$ given by $p(x, y) = ||x - y||$ for all $x, y \in D$.

Lemma 3.15. *In the interval domain, the following cancellation law holds: whenever $x, y, z \in \mathbf{I}\mathbb{R}$, then*

$$x + y \sqsubseteq z + y \;\Rightarrow\; x \sqsubseteq z.$$

Proof. Assume that $x + y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \sqsubseteq [\underline{z} + \underline{y}, \overline{z} + \overline{y}] = z + y$. This is equivalent to $\underline{x} + \underline{y} \leq \underline{z} + \underline{y}$ and $\overline{z} + \overline{y} \leq \overline{x} + \overline{y}$. This is in turn the same as $\underline{x} \leq \underline{z}$ and $\overline{z} \leq \overline{x}$, which means that $x \sqsubseteq z$. □

The following theorem justifies the claim that weak partial metrics are of major importance in the theory of modelling metric spaces.

Theorem 3.16. *The following are equivalent for a Banach domain D equipped with a partial norm $|| \cdot ||$.*

(1) *The equality $||r + a|| = ||r|| + ||a||$ holds for any element $r \in D$ and any symmetric element $a \in D$;*
(2) *The partial norm satisfies the triangle inequality and the distance induced by the norm satisfies the sharp triangle inequality (in effect, it is a weak partial pseudometric when considered as a map of type $D \times D \to [0, \infty)$).*

Proof. (1)⇒(2): Let x, y, z be arbitrary. Since $[-1,1](x+y) \sqsupseteq [-1,1]x + [-1,1]y$, by monotonicity of the partial norm and the assumption we have:

$$||[-1,1](x+y)|| \sqsupseteq ||[-1,1]x + [-1,1]y|| = ||[-1,1]x|| + ||[-1,1]y||.$$

This is the desired triangle inequality of the norm. We will immediately apply it:

$$\begin{aligned} p(x,y) + p(z,z) &= ||x-y|| + ||z-z|| \\ &= ||x-y+z-z|| \\ &= ||(x-z)+(z-y)|| \\ &\sqsupseteq ||x-z|| + ||z-y|| \\ &= p(x,z) + p(z,y). \end{aligned}$$

For (2)⇒(1), by the triangle inequality, we have $||r+a|| \sqsupseteq ||r||+||a||$. On the other hand, $p(r,\mathbf{0})+p(a,a) \sqsupseteq p(r,a)+p(a,\mathbf{0})$ translates into $||r||+||a-a|| \sqsupseteq ||r-a||+||a||$. Note that since a is symmetric, $r-a = r+a$ and the inequality above is thus equivalent to $||r|| + 2||a|| \sqsupseteq ||r+a||+||a||$. The cancellation law implies that $||r||+||a|| \sqsupseteq ||r+a||$. Finally, antisymmetry of the order yields $||r+a|| = ||r|| + ||a||$, as required. □

Example 3.17. The interval domain and the formal ball model for any Banach space are examples of Banach domains where the sharp triangle inequality holds. In the case of the interval domain, $p(x,y) = \max\{||x-y||, ||x-x||, ||y-y||\}$ is in fact a partial metric for the Scott topology (given also by $p(x,y) = \max\{\overline{x}, \overline{y}\} - \min\{\underline{x}, \underline{y}\}$) and thus $\mu(x) \stackrel{def}{=} ||x-x|| = \text{length}(x)$ is a weakly modular measurement on $\mathbf{I}\mathbb{R}$.

In the case of the formal ball model, $\mathbf{B}X$ the assumptions of the preceding proposition hold as well. The distance mapping $p(C(x,r), C(y,s)) = ||C(x,r) - C(y,s)||$ is a weak partial metric compatible with the Scott topology, given explicitly by

$$p(C(x,r), C(y,s)) = d(x,y) + r + s,$$

where d is the metric of the Banach space X. The self-distance map, assigning to each ball its radius, is a Lebesgue measurement on the entire domain.

In contrast, the partial norm on the domain $\mathbf{K}\mathbb{R}^n$ (introduced in Section 5) does not induce a partial metric as the condition (1) of Theorem 3.16 is violated in general.

For the rest of this section, we assume that the distance induced by the partial norm is a weak partial metric denoted by p. Recall that the symmetrization of p is a pseudometric $d\colon D \times D \to [0, \infty)$ given by

$$d(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2||x - y|| - ||x - x|| - ||y - y||.$$

Proposition 3.18. *Assume that the specialization order of p is compatible with the order on D. Then for all $x, y \in D$ we have*

$$(d(x, y) = 0 \wedge ||x - x|| = ||y - y||) \;\Rightarrow\; x = y.$$

Proof. This is a property of the pseudometric associated with the weak partial pseudometric. □

Proposition 3.19. *Let D be a Banach domain with (B11). For any $x, y, z, w \in D$ we have:*

(1) $d(x + y, z + w) = d(x - z, w - y)$.

(2) $d(x, y) = 0$, *whenever $x, y \in D$ are symmetric.*

Proof. (1) We have

$$\begin{aligned}
&d(x + y, z + w) = \\
&2||x + y - (z + w)|| - ||x + y - (x + y)|| - ||z + w - (z + w)|| = \\
&2||x + y - z - w|| - ||x + y - x - y|| - ||z + w - z - w|| = \\
&2||(x - z) - (w - y)|| - ||(x - x) - (y - y)|| - ||(z - z) - (w - w)|| = \\
&2||(x - z) - (w - y)|| - ||x - x|| - ||y - y|| - ||z - z|| - ||w - w|| = \\
&2||(x - z) - (w - y)|| - (||x - x|| + ||z - z||) - (||y - y|| + ||w - w||) = \\
&2||(x - z) - (w - y)|| - ||(x - z) - (x - z)|| - ||(y - w) - (y - w)|| = \\
&d(x - z, w - y).
\end{aligned}$$

For (2), we calculate $d(x, y) = 2||x - y|| - ||x - x|| - ||y - y|| = 2||x + y|| - 2||x|| - 2||y|| = 0$. □

We will now focus on the case when the partial norm induces a weak partial metric compatible with the Scott topology on D. At present it is unclear what is a sufficient condition for a weak partial metric induced by the norm to be compatible with the Scott topology. However, the assumption on the agreement of the partial metric topology and the Scott topology puts some interesting restrictions on the structure of symmetric elements of the underlying domain:

Proposition 3.20. *If p is compatible with the Scott topology on D, then the restriction of the partial norm to the set of symmetric elements,*

$$||\cdot||: \mathcal{S}(D) \to [0,1]^{op}$$

is an order embedding.

Proof. The proof depends on the characterization of the order induced by the weak partial metric. Suppose that $x, y \in \mathcal{S}(D)$. We have

$$\begin{aligned} x \sqsubseteq y \quad &\text{iff} \quad p(x,y) \leq p(x,x) \\ &\text{iff} \quad ||x-y|| \leq ||x-x|| \\ &\text{iff} \quad ||x|| + ||y|| \leq 2||x|| \\ &\text{iff} \quad ||y|| \leq ||x||. \end{aligned}$$

Since the partial norm is monotone, the proof is now complete. □

Proposition 3.21. *If p is compatible with the Scott topology on D, then for every $y \in D$ we have $\mathcal{S}(y) = [-1,1]y$.*

Proof. For any symmetric $x \in D$ we have $x = [-1,1]x$, since $||x|| = ||[-1,1]x||$ and $||\cdot||$ is an embedding. Hence if $y \in D$ is arbitrary, then $\mathcal{S}(y) \sqsubseteq y$ implies $\mathcal{S}(y) \sqsubseteq [-1,1]y \sqsubseteq y$ and so $\mathcal{S}(y) = [-1,1]y$ since $[-1,1]y$ is a symmetric below y and $\mathcal{S}(y)$ is defined to be the greatest such. □

It happens that under the agreement of topologies, the way-below relation can be easily obtained from a partial metric in most cases:

Theorem 3.22 (way-below)**.** *Let D be a Banach domain with (B12) and (B13). If p is compatible with the Scott topology on D, then*

$$(x \ll y \wedge x \neq y) \Rightarrow p(x,y) < p(x,x).$$

The converse holds if, in addition, $\mathcal{S}$ is strictly monotone.

Proof. Assume that $p(x,y) < p(x,x)$. Then, there exists $\varepsilon > 0$ such that

$$p(x,y) < p(x,x) - \varepsilon. \tag{3.1}$$

Let A be a directed subset of D with supremum z above y. In particular,

$$p(y,z) \leq p(y,y). \tag{3.2}$$

Then the open ball $B_p(z,\varepsilon)$ around z is Scott-open by assumption and thus must contain some element $a \in A$. This implies that

$$p(z,a) \leq p(z,z) + \varepsilon. \tag{3.3}$$

Now,

$$\begin{aligned} p(x,a) &\leq p(x,y) + p(y,z) + p(z,a) - p(y,y) - p(z,z) \\ &< p(x,y) + p(z,a) - p(z,z) \\ &< p(x,x) - \varepsilon + p(z,a) - p(z,z) \\ &< p(x,x) - \varepsilon + p(z,z) - p(z,z) + \varepsilon \\ &= p(x,x). \end{aligned}$$

Note that the above inequalities follow respectively from the sharp triangle inequality, (3.2), (3.1) and (3.3). Hence $x \sqsubseteq a$ and this proves that $x \ll y$.

Conversely, suppose that $x \ll y$ and $x \neq y$. Hence $x \sqsubset y$. By strictness of operations, $x - x \sqsubset x - y$. By assumption $x - x \sqsubset \mathcal{S}(x-y)$ and since the norm is an order-embedding, $p(x,x) = ||x-x|| > ||\mathcal{S}(x-y)|| = ||x-y|| = p(x,y)$, as required. □

Corollary 3.23. *Let D be as in Theorem 3.22. Then if for $x,y \in D$, $d(x,y) = 0$, then either $x \ll y$ or $y \ll x$ or $x = y$.*

Proof. $d(x,y) = 0$ iff $2p(x,y) \leq p(x,x) + p(y,y)$. Now, if $p(x,x) < p(y,y)$, then $2p(x,y) \leq 2p(y,y)$, which implies $y \ll x$. Analogously, if $p(x,x) > p(y,y)$, then $x \ll y$. Finally, the condition $p(x,x) = p(y,y)$ together with $d(x,y) = 0$ is equivalent to $x = y$. □

4. Partially linear operators

Definition 4.1. A Scott-continuous mapping $f\colon D \to E$ between Banach domains is *partially linear* if the following hold:

$$f(\mathbf{0}) = \mathbf{0}, \tag{4.1}$$

$$f(ax) \sqsubseteq af(x), \tag{4.2}$$

$$f(x + y) \sqsubseteq f(x) + f(y), \tag{4.3}$$

for all $x, y \in D$ and $a \in \mathbf{I}\mathbb{R}$.

Surprisingly, the scalar multiplication by non-degenerate real intervals is not partially linear (though multiplication by real numbers is):

Example 4.2. Multiplication $b(\cdot)\colon \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ for $b \in \mathbf{I}\mathbb{R}$ is not partially linear in general.

Proof. We have $[-2,-1]([1,2] + [-1,1]) = [-6,0] \sqsupset [-6,1] = [-2,-1][1,2] + [-2,-1][-1,1]$, which demonstrates the claim. In fact, any partially linear mapping of type $[\mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}]$ must be an extension of the multiplication by a real number, i.e. of the form $c(\cdot)$, where $c \in \mathbb{R}$. □

Proposition 4.3. *The poset of all partially linear functions between Banach domains D and E is a dcpo with respect to the pointwise order. It is not a semilattice in general.*

Proposition 4.4. *If D, E are Banach domains, then $[D \to E]$ is another, providing it is continuous. The addition and scalar multiplication in $[D \to E]$ are defined as follows:*

$$(f + g)(x) \stackrel{def}{=} f(x) + g(x),$$

$$(af)(x) \stackrel{def}{=} af(x).$$

In addition, the zero function $\mathbf{0}\colon D \to E$ is defined as:

$$\mathbf{0}(x) \stackrel{def}{=} \mathbf{0}.$$

Unfortunately, if D, E are bounded complete models of non-degenerate Banach spaces X, Y, respectively, then the Banach domain of Scott-continuous maps cannot be the model of continuous mappings from X to Y. This fact (demonstrated below) seems not to be a flaw of the existing definition of invertible elements nor partially linear operators; it is rather a problem with the definition of the model itself. It happens very rarely that the domain of Scott-continuous between models is itself a model of continuous maps between spaces.

Proposition 4.5. *Let D, E be bounded complete Banach domains. The only invertible partially linear mapping of $[D \to E]$ is the zero function.*

Proof. Assume that $f - f = \mathbf{0}$. In particular, this yields $f(\bot) - f(\bot) = 0$; that is, $f(\bot)$ is invertible (hence maximal) in E. Since $f(\bot) \sqsubseteq f(x)$ holds for every x by monotonicity, we have $f(\bot) = f(x)$. Hence the mapping f is constant. However, for every x, $f(x) = f(\bot) = f(\mathbf{0}) = \mathbf{0}$, as required. □

On the other hand, for bounded complete Banach domains there exists a close correspondence between linear operators on modelled Banach spaces and partially linear operators between corresponding models. First, we need two auxiliary lemmata:

Lemma 4.6. *Suppose that A, B are Banach spaces modelled by Banach domains D, E respectively. Let $f\colon A \to B$ be a linear map and let $a \in \mathbf{I}\mathbb{R}$. Then $f[aq] = a \cdot f(q)$ holds for all $q \in \max(D)$.*

Proof. We have $u \in f[aq]$ iff there exists $v \in aq$ with $f(v) = u$ iff there is $z \in a$ such that $f(zq) = u$. Equivalently, $zf(q) = u$ by linearity, and hence $u \in af(q)$. □

Lemma 4.7. *Let $f\colon D \to E$ be a map between Banach domains that satisfies properties (4.2) and (4.3) of Definition 4.1. Then it satisfies property (4.1) iff it preserves invertible elements.*

Proof. Let x be an invertible element of D. Then $\mathbf{0} = f(\mathbf{0}) = f(x - x) \sqsubseteq f(x) - f(x)$. By maximality of zero, $f(x) - f(x) = \mathbf{0}$. Therefore, $f(x)$ is invertible.

Conversely, we have $f(\mathbf{0}) = f(0x) \sqsubseteq 0f(x) = \mathbf{0}$. However, $\mathbf{0}$ is invertible in D, and so by assumption $f(\mathbf{0})$ is invertible, thus maximal in E. Therefore, $f(\mathbf{0}) = \mathbf{0}$. □

Theorem 4.8. *Let A, B be Banach spaces modelled by Banach domains D, E, respectively. Assume that D, E are bounded complete, satisfy (B9), (B10), and that* $\mathrm{inv}(D) = \max(D)$, $\mathrm{inv}(E) = \max(E)$. *Then:*

(1) *Every linear map $f\colon A \to B$ has the greatest Scott-continuous extension to a partial linear mapping $\bar{f}\colon D \to E$ between corresponding models.*

(2) *Every partially linear operator $F\colon D \to E$ restricts to a linear operator between modelled Banach spaces.*

Proof. Let $a \in \mathbf{I}\mathbb{R}$ and consider $\bar{f}(ax)$ for some $x \in D$. Recall that $\bar{f}(ax) = \bigsqcup^{\uparrow}\{\bigsqcap f[\uparrow\!\!\uparrow y \cap \max(D)] \mid y \ll ax\}$. Let $w \ll \bar{f}(ax)$. This means that $w \sqsubseteq \bigsqcap f[\uparrow\!\!\uparrow y \cap \max(D)]$ for some $y \ll ax$, by definition of the way-below relation. Consequently, $w \sqsubseteq f(p)$ for all $p \gg y$. By Scott-continuity of the scalar multiplication, there exists $z \ll x$ such that $az \gg y$. Thus, for each $q \in \uparrow\!\!\uparrow z \cap \max(D)$ we have $aq \sqsupseteq az \gg y$. This yields $aq \gg y$ and so $f[aq] \sqsupseteq w$. By Lemma 4.6, $af(q) \sqsupseteq w$. However, this implies that w is a lower bound for all elements $af(q)$, where $q \in \uparrow\!\!\uparrow z \cap \max(D)$. Therefore, $\bigsqcap af[\uparrow\!\!\uparrow z \cap \max(D)] \sqsupseteq w$. By (B9), $a \bigsqcap f[\uparrow\!\!\uparrow z \cap \max(D)] \sqsupseteq w$. Consequently, $a\bar{f}(x) \sqsupseteq w$ (recall that $z \ll x$). We have shown that $w \ll \bar{f}(ax)$ implies $w \sqsubseteq a\bar{f}(x)$. But since $\bar{f}(ax)$ is the directed supremum of all elements way-below it, we must have $\bar{f}(ax) \sqsubseteq a\bar{f}(x)$, as required.

Now, we will show that $\bar{f}(x+y) \sqsubseteq \bar{f}(x)+\bar{f}(y)$. Suppose that $w \ll \bar{f}(x+y)$. Hence $w \sqsubseteq f(p)$ for all $p \gg r$, where $r \in D$ is some element with $r \ll x+y$. By Scott-continuity of addition, there are elements $x' \ll x$ and $y' \ll y$ such that $x' + y' \gg r$. Hence, for all $(q_1, q_2) \in (\uparrow\!\!\uparrow x' \cap \max(D)) \times (\uparrow\!\!\uparrow y' \cap \max(D))$ we have $q_1 + q_2 \sqsupseteq x' + y' \gg r$. Consequently, $q_1+q_2 \gg r$ and thus $w \sqsubseteq f(q_1+q_2)$. Since the map f is linear at maximal elements, $w \sqsubseteq f(q_1) + f(q_2)$ for all $q_1 \in \uparrow\!\!\uparrow x' \cap \max(D)$ and $q_2 \in \uparrow\!\!\uparrow y' \cap \max(D)$. By (B10), $w \sqsubseteq \bigsqcap f[\uparrow\!\!\uparrow x' \cap \max(D)] + f(q_2)$ and $w \sqsubseteq \bigsqcap f[\uparrow\!\!\uparrow x' \cap \max(D)] + \bigsqcap f[\uparrow\!\!\uparrow y' \cap \max(D)]$. Consequently, $w \sqsubseteq \bigsqcup^{\uparrow}_{(x',y')\ll(x,y)}(\bigsqcap f[\uparrow\!\!\uparrow x' \cap \max(D)] + \bigsqcap f[\uparrow\!\!\uparrow y' \cap \max(D)])$. By Scott-continuity of addition, $w \sqsubseteq \bar{f}(x) + \bar{f}(y)$. We conclude that $\bar{f}(x+y) = \bigsqcup^{\uparrow}\downarrow\!\!\downarrow \bar{f}(x+y) \sqsubseteq \bar{f}(x) + \bar{f}(y)$, as required.

Lastly, $\bar{f}(\mathbf{0}) = f(\mathbf{0})$, since $\mathbf{0}$ is maximal. By linearity, $f(\mathbf{0}) = \mathbf{0}$. This yields $\bar{f}(\mathbf{0}) = \mathbf{0}$.

For the converse, we have to prove that any partially linear map between Banach domains restricted to the space of maximal elements is linear. Firstly, note that by Lemma 4.7 partially linear maps preserve invertible elements (which in our case coincide with maximals).

Let $f \colon A \to B$ be a restriction of F to the maximal elements. By the previous paragraph, f is well-defined. Let $a \in \mathbf{I}\mathbb{R}$ and $x \in D$ be maximal. Then ax is maximal in D, because scalar multiplication preserves invertibility and maximals equal invertibles.

Thus, $f(ax) \sqsubseteq aF(x) = af(x)$ by definition of F. But $f(ax)$ is maximal, and thus $f(ax) = af(x)$. Next, if x, y maximal, then so is $x + y$, since $\mathrm{inv}(D) = \max(D)$ and addition preserves invertibility. Therefore, $f(x+y) \sqsubseteq F(x)+F(y) = f(x)+f(y)$. But $f(x+y)$ is maximal and we can conclude that $f(x + y) = f(x) + f(y)$. We have shown that the restriction of a partially linear map between Banach domains is a linear map between Banach spaces. □

Theorem 4.9. *Let $D_1, ..., D_k, E$ be Banach domains and assume that the partial norm on E satisfies the triangle inequality. Let u be a partially multilinear mapping from $D_1 \times ... \times D_k$ into E. Then the following conditions are equivalent:*

(1) *There exists a number $a > 0$ such that for all $(x_1, ..., x_k) \in D_1 \times ... \times D_k$ we have*
$$||u(x_1, ..., x_k)|| \le a \cdot ||x_1|| \cdot ... \cdot ||x_n||;$$
(2) *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if*
$$||x_1 - c_1|| \le \delta, ..., ||x_k - c_k|| \le \delta,$$
then
$$||u(x_1, ..., x_k) - u(c_1, ..., c_k)|| \le \varepsilon.$$

Proof. We will give a proof for $k = 2$. (1)⇒(2): since
$$u(x_1 - c_1, x_2) \sqsubseteq u(x_1, x_2) - u(c_1, x_2)$$
and
$$u(c_1, x_2 - c_2) \sqsubseteq u(c_1, x_2) - u(c_1, c_2),$$
we have
$$u(x_1-c_1, x_2)+u(c_1, x_2-c_2) \sqsubseteq u(x_1, x_2)-u(c_1, c_2)+u(c_1, x_2)-u(c_1, x_2).$$
Therefore,
$$\begin{aligned} ||u(x_1, x_2) - u(c_1, c_2)|| &\le ||u(x_1, x_2) - u(c_1, c_2) + u(c_1, x_2) - u(c_1, x_2)|| \\ &\le ||u(x_1 - c_1, x_2) + u(c_1, x_2 - c_2)|| \\ &\le a(||x_1 - c_1|| \cdot ||x_2|| + ||c_1|| \cdot ||x_2 - c_2||). \end{aligned}$$
Now, for any $\delta \in (0, 1)$, suppose that $||x_i - c_i|| \le \delta$, for $i = 1, 2$. Then,
$$||x_2|| \le ||x_2 + c_2 - c_2|| \le ||x_2 - c_2|| + ||c_2|| \le ||c_2|| + \delta \le 1 + ||c_2||.$$
Hence
$$||u(x_1, x_2)-u(c_1, c_2)|| \le a(\delta(||c_2||+1)+\delta||c_1||) = a\delta(||c_1||+||c_2||+1).$$

To prove the claim, having $\varepsilon > 0$, it is therefore enough to take $\delta = \varepsilon/(a(||c_1|| + ||c_2|| + 1))$.

Conversely, by assumption (for $(c_1, c_2) = (\mathbf{0}, \mathbf{0})$) we have that there exists $r > 0$ such that if $||x_1||, ||x_2|| \leq r$, then $||u(x_1, x_2)|| \leq 1$. Now, let (x_1, x_2) be arbitrary. Firstly, suppose that $x_1, x_2 \neq \mathbf{0}$. If $z_1 = rx_1/||x_1||$ and $z_2 = rx_2/||x_2||$, then $||z_1|| = ||z_2|| = r$, and hence $||u(z_1, z_2)|| \leq 1$. But $u(z_1, z_2) \sqsubseteq r^2 u(x_1, x_2)/||x_1|| \cdot ||x_2||$ and therefore

$$r^2||u(x_1, x_2)||/||x_1|| \cdot ||x_2|| \leq ||u(z_1, z_2)|| \leq 1.$$

This yields $||u(x_1, x_2)|| \leq a \cdot ||x_1|| \cdot ||x_2||$, where $a = r^{-2}$. If $x_1 = x_2 = \mathbf{0}$, $u(x_1, x_2) = \mathbf{0}$ and the preceding inequality still holds. □

Since all partially linear maps are Scott-continuous, we expect that in analogy with the Banach theorem, they must be bounded.

Corollary 4.10. *Let $D_1, ..., D_k, E$ be Banach domains such that the Scott topology is given by a partial metric induced by the norm. Then every partially multilinear mapping u from $D_1 \times ... \times D_k$ into E is bounded, i.e. there exists a number $a > 0$ such that for all $(x_1, ..., x_k) \in D_1 \times ... \times D_k$ we have*

$$||u(x_1, ..., x_k)|| \leq a \cdot ||x_1|| \cdot ... \cdot ||x_n||.$$

Proof. ($k = 2$) Since the Scott topology is given by a partial metric induced by one of the equivalent norms, the continuity of u at $(\mathbf{0}, \mathbf{0})$ is characterized by the following condition:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_1 \in D_1, x_2 \in D_2.\ (||x_1||, ||x_2|| \leq \delta \Rightarrow ||u(x_1, x_2)|| \leq \varepsilon).$$

(Note that $||(\mathbf{0}, \mathbf{0})|| = ||u(\mathbf{0}, \mathbf{0})|| = 0$). In particular, for $\varepsilon = 1$,

$$\exists r > 0\ \forall x_1 \in D_1, x_2 \in D_2.\ (||x_1||, ||x_2|| \leq r \Rightarrow ||u(x_1, x_2)|| \leq 1).$$

For the rest of the proof, we proceed like in (the proof of) Theorem 4.9. □

5. The Banach domain of convex bodies

In this section we study another (apart from the formal ball model) canonical example of a Banach domain: the domain of convex compacta of the Euclidean space $\mathbb{R}^n$. In contrast to the domain of formal balls, this domain is always bounded complete. On the other hand, the partial norm does not induce a weak partial metric! Let us start with necessary preparations: The collection

$\mathbf{K}\mathbb{R}^n$ of all compact, convex, nonempty subsets of $\mathbb{R}^n$ ordered by the inverse inclusion is a bounded complete ω-continuous dcpo. The n-dimensional Euclidean space is homeomorphic to the subset of maximal elements of $\mathbf{K}\mathbb{R}^n$ in the subspace Scott topology. Note that the interval domain $\mathbf{I}\mathbb{R}$ arises as $\mathbf{K}\mathbb{R}^1$. One can readily see that any finite product $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$ is modelled by the bounded complete domain $\mathbf{K}\mathbb{R}^{n_1} \times ... \times \mathbf{K}\mathbb{R}^{n_k}$. Any metric ρ on $\mathbb{R}^n$ compatible with the natural topology is a restriction of the corresponding Hausdorff metric ρ_H on the compact convex sets.

It is well-known that – categorically speaking – any bounded complete domain D in its Scott topology is an injective space with respect to dense subspace embedding in the ambient category of topological T_0 spaces. In the language of mortals this means that every continuous map $f : X \to D$ extends to a continuous $\bar{f} : Y \to D$ for any space Y containing X as a dense subspace. In fact, we can always find the greatest such extension [14], which is explicitly defined as:

$$\bar{f}(y) \stackrel{def}{=} \bigsqcup^{\uparrow}\{ \bigsqcap f(U \cap X) \mid y \in U\},\ U \in \Omega Y.$$

In what follows, we will prove that the Minkowski operations on the compact convex sets arise as greatest Scott-continuous extensions of the natural operations in $\mathbb{R}^n$ to the domain $\mathbf{K}\mathbb{R}^n$. In the sequel, we will make a frequent use of the following elementary properties of convex sets:

Proposition 5.1. *For any convex subsets $x, y, \{x_i\}_{i\in I}$ of $\mathbb{R}^n$, $p, q \subseteq \mathbb{R}^n$, for any $a, b, c, s, z \in \mathbb{R}$ and $u, u_1, ..., u_k \in \mathbb{R}^n$ the following hold:*

(1) $(s + z)x = sx + zx$,
(2) $s(x + y) = sx + sy$,
(3) $su \in sx$ *iff* $u \in x$, *providing that* $s \neq 0$,
(4) $z \bigcap_{i\in I} x_i = \bigcap_{i\in I}(zx_i)$,
(5) $z \bigcup_{i\in I} x_i = \bigcup_{i\in I}(zx_i)$,
(6) *For every* $k \in \omega$, $s \cdot \mathrm{conv}(u_1, ..., u_k) = \mathrm{conv}(su_1, ..., su_k)$,
(7) $\mathrm{conv}(sq) = s \cdot \mathrm{conv}(q)$,
(8) $s(x \cup y) = sx \cup sy$,
(9) $\mathrm{conv}(q \cup p) = \mathrm{conv}(\mathrm{conv}(q) \cup \mathrm{conv}(p))$,
(10) *If* $a \leq b \leq c$, *then* $bx \subseteq \mathrm{conv}(ax \cup cx)$,
(11) $\bigcup_{i\in I} x_i + y = \bigcup_{i\in I}(x_i + y)$.

Here, and in the sequel, we abbreviate any expression of the form $\mathrm{conv}(\{u_1, ..., u_k\})$ *to* $\mathrm{conv}(u_1, ..., u_k)$.

To start with, note that any continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ is trivially a map $f\colon \mathbb{R}^n \to \mathbf{I}\mathbb{R}$ continuous from the Euclidean topology on $\mathbb{R}^n$ to the Scott topology on $\mathbf{I}\mathbb{R}$. Now, since $\mathbb{R}^n \cong \max(\mathbf{K}\mathbb{R}^n)$, the extension $\bar{f}\colon \mathbf{K}\mathbb{R}^n \to \mathbf{I}\mathbb{R}$ is given by:

$$\bar{f}(K) = \bigcap\{f[L] \mid K \subseteq \mathbf{int}(L)\},\ L \in \mathbf{K}\mathbb{R}^n,$$

or, equivalently,

$$\bar{f}(K) = \bigcap\{f[(K)_{2^{-n}}] \mid n \in \omega\}.$$

Since f is continuous, we have

$$f[K] = [\min_{x\in K} f(x), \max_{x\in K} f(x)] \in \mathbf{I}\mathbb{R}$$

for any $K \in \mathbf{K}\mathbb{R}^n$ (the map f attains extremal values on compacta). This allows us to write the expression for $\bar{f}$ more concisely as:

$$\bar{f}(K) = f[K],\ K \in \mathbf{K}\mathbb{R}^n.$$

In fact, the argument carries without major changes if one wishes to consider any continuous function $g\colon \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k} \to \mathbb{R}^m$.

$$\begin{aligned} \bar{g}(K) &= \bigcap\{\mathrm{conv}(g[(K)_{2^{-n}}]) \mid n \in \omega\} \\ &= \mathrm{conv}(\bigcap\{(g[(K)_{2^{-n}}]) \mid n \in \omega\}) \\ &= \mathrm{conv}(g[K]). \end{aligned}$$

In the sequel, if no ambiguity arises, we will drop the bar $^-$.

Consider addition $+\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. The extension is

$$K + L = \mathrm{conv}(\{x + y \mid x \in K,\ y \in L\}) = \{x + y \mid x \in K,\ y \in L\}.$$

This is the well-known Minkowski addition of compact convex sets of type $\mathbf{K}\mathbb{R}^n \times \mathbf{K}\mathbb{R}^n \to \mathbf{K}\mathbb{R}^n$. In a similar fashion, the scalar product $\cdot\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ extends to the operation of type $\mathbf{I}\mathbb{R} \times \mathbf{K}\mathbb{R}^n \to \mathbf{K}\mathbb{R}^n$ and a substraction $K - L$ can be readily defined as $K + (-1) \cdot L$.

The following theorem characterizes one of the (domain-theoretic) bases of $\mathbf{K}\mathbb{R}^n$ and thus the way-below relation of the domain.

Proposition 5.2. *Polyhedra are dense in the way-below relation on* $\mathbf{K}\mathbb{R}^n$, $n \in \omega$.

Proof. Suppose $y \ll x$. Then there exists $\varepsilon > 0$ with $x \subseteq (x)_\varepsilon \subseteq \mathbf{int}(y)$. Define $C = \mathbf{cl}((x)_{\frac{2\varepsilon}{3}} \setminus (x)_{\frac{\varepsilon}{3}})$. The set C is closed and bounded, and hence compact. Thus any cover of C by a family of cubes of diameter smaller than $\frac{\varepsilon}{6}$ has a finite subcover, say $\{c_1, ..., c_k\}$. We define the interpolating polyhedron as

$$z \stackrel{def}{=} \mathrm{conv}(\{c_1, ..., c_k\}).$$

Now, if $r \in (x)_{\frac{\varepsilon}{6}}$, then any straight line through r punctures the cover $\{c_1, ..., c_k\}$ in at least two points, say u, v on the opposite sides of r. Hence there are cubes, say c, c', such that $u \in c, v \in c'$. Consequently,

$$r \in \mathrm{conv}(u, v) \subseteq \mathrm{conv}\{c, c'\} \subseteq z$$

and this proves that $x \subseteq (x)_{\varepsilon/6} \subseteq z$, that is, $z \ll x$. On the other hand, by construction,

$$z \subseteq (x)_{\varepsilon/6} \subseteq (x)_\varepsilon \subseteq \mathbf{int}(y),$$

which gives $y \ll z$, as required. □

Proposition 5.3. *The diameter measures* (*in the sense of K. Martin* [24]) *the space of maximal elements of* $\mathbf{K}\mathbb{R}^n$, $n \in \omega$.

Proof. The diameter is clearly Scott-continuous. Let $x \in \mathbb{R}^n$. Suppose that $z \ll x$ for $z \in \mathbf{K}\mathbb{R}^n$. We have to show that there is $\varepsilon > 0$ such that $z \ll \mathrm{diam}(x, \varepsilon) = \{y \mid x \in y \ \& \ \mathrm{diam}(y) < \varepsilon\}$. Since $z \ll x$, there exists $\delta > 0$ such that $(x)_\delta \subseteq z$. Set $\varepsilon = \delta/3$. Now, if $y \in \mathrm{diam}(x, \varepsilon)$, then $y \subseteq (x)_{2\varepsilon} \subset (x)_\delta \subseteq z$. This proves that diameter is a measurement.

It is clear that if $x \in \mathbb{R}^n = \max(\mathbf{K}\mathbb{R}^n)$, then $\mathrm{diam}(x) = 0$, which means that $x \in \ker(\mathrm{diam})$. Conversely, if $x \in \mathbf{K}\mathbb{R}^n \setminus \mathbb{R}^n$, then there are $a, b \in x$ with $a \neq b$. By convexity, $\mathrm{conv}(a, b) \subseteq x$. Therefore, $0 < \mathrm{diam}(\mathrm{conv}(a, b)) \leq \mathrm{diam}(x)$. That is, $x \notin \ker(\mathrm{diam})$. □

Proposition 5.4. *For any* $n \in \omega$ *we have* $\mathrm{inv}(\mathbf{K}\mathbb{R}^n) = \max(\mathbf{K}\mathbb{R}^n)$.

Proof. Let $x \in \mathbf{K}\mathbb{R}^n$ be invertible. Then $0 \leq \mathrm{diam}(x) \leq \mathrm{diam}(x - x) = \mathrm{diam}(\mathbf{0}) = 0$, which gives $x \in \ker(\mathrm{diam})$. The diameter is a measurement and thus $x \in \max(\mathbf{K}\mathbb{R}^n)$ follows. For the converse, if $x \in \max(\mathbf{K}\mathbb{R}^n)$, then $x \in \ker(\mathrm{diam})$ and so $\mathrm{diam}(x) = 0$. This means that $x \in \mathbb{R}^n$ and so $x - x = \mathbf{0}$. That is, $x \in \mathrm{inv}(\mathbf{K}\mathbb{R}^n)$. □

Definition 5.5. The (extended) scalar multiplication $\mathbf{I}\mathbb{R} \times \mathbf{K}\mathbb{R}^n \to \mathbf{K}\mathbb{R}^n$ is defined as:

$$[a,b]x = \mathrm{conv}(\bigcup_{z\in[a,b]} zx),$$

where $[a,b] \in \mathbf{I}\mathbb{R}$ and $x \in \mathbf{K}\mathbb{R}^n$.

It coincides with the Minkowski scalar multiplication when restricted to type $\mathbb{R} \times \mathbf{K}\mathbb{R}^n \to \mathbf{K}\mathbb{R}^n$.

Lemma 5.6.

$$[a,b]x = \mathrm{conv}(ax \cup bx).$$

Proof. Clearly, $ax, bx \subseteq \bigcup_{s\in[a,b]} sx$. Thus, $\mathrm{conv}(ax \cup bx) \subseteq [a,b]x$. Conversely, if $u \in [a,b]x$, then by Carathéodory's Theorem,

$$\exists k\ \exists u_1, ..., u_k \in \bigcup_{s\in[a,b]} sx.\ u \in \mathrm{conv}(u_1, ..., u_k).$$

Equivalently,

$$\exists k\ \exists v_1, ..., v_k \in x\ \exists s_1, ..., s_k \in [a,b].\ u \in \mathrm{conv}(s_1v_1, ..., s_kv_k).$$

However, for each $i = 1, ..., k$, we have $s_iv_i \subseteq \mathrm{conv}(av_i, bv_i)$ by Proposition 5.1(10) and so

$$\mathrm{conv}(s_1v_1, ..., s_kv_k) \subseteq \mathrm{conv}(av_1, ..., av_k, bv_1, ..., bv_k).$$

Now, by Proposition 5.1(9),

$$\begin{aligned} u &\in \mathrm{conv}(\mathrm{conv}(av_1, ..., av_k) \cup \mathrm{conv}(bv_1, ..., bv_k)) \\ &= \mathrm{conv}(a\ \mathrm{conv}(v_1, ..., v_k) \cup b\ \mathrm{conv}(v_1, ..., v_k)) \\ &\subseteq \mathrm{conv}(ax \cup bx). \end{aligned}$$ □

Lemma 5.7. *The extended scalar multiplication of type* $\mathbf{I}\mathbb{R} \times \mathbf{K}\mathbb{R}^n \to \mathbf{K}\mathbb{R}^n$ *satisfies the following properties:*

(1) $0x = \mathbf{0}$,
(2) $nx = \underbrace{x + ... + x}_{n \text{ times}}$,
(3) $[a,b]([c,d]x) = ([a,b][c,d])x$,
(4) $[a,b]x + [c,d]x \supseteq ([a,b] + [c,d])x$,
(5) $[a,b]x + [a,b]y \supseteq [a,b](x + y)$,
(6) *preserves directed suprema in* $\mathbf{K}\mathbb{R}^n$,
(7) *preserves infima in* $\mathbf{K}\mathbb{R}^n$.

Proof. For (1), $0x = \{(\underbrace{0,...,0}_{n \text{ times}})\} = \{\mathbf{0}\}$, which is identified with $\mathbf{0}$. Part (2) follows immediately from Proposition 5.1. For (3), by Proposition 5.1(7),(8),(10):

$$\begin{aligned}
[a,b]([c,d]x) &= \text{conv}(a\ \text{conv}(cx \cup dx) \cup b\ \text{conv}(cx \cup dx)) \\
&= \text{conv}(\text{conv}(acx \cup adx) \cup \text{conv}(bcx \cup bdx)) \\
&= \text{conv}(acx \cup adx \cup bcx \cup bdx) \\
&= \text{conv}(\min\{ac,ad,bc,bd\}x \cup \max\{ac,ad,bc,bd\}x) \\
&= [\min\{ac,ad,bc,bd\}, \max\{ac,ad,bc,bd\}]x \\
&= ([a,b][c,d])x.
\end{aligned}$$

For (4), $([a,b]+[c,d])x = \text{conv}((a+c)x \cup (b+d)x)$, which by Proposition 5.1(1) is $\text{conv}((ax+cx) \cup (bx+dx))$. It is easy to see that $(ax+cx) \cup (bx+dx) \subseteq (ax \cup bx) + (cx \cup dx)$, and therefore, $\text{conv}((ax+cx) \cup (bx+dx)) \subseteq \text{conv}((ax \cup bx) + (cx \cup dx))$. However, the latter set is the same as $\text{conv}(ax \cup bx) + \text{conv}(cx \cup dx)$, which is precisely $[a,b]x + [c,d]x$.

For (5),

$$\begin{aligned}
[a,b](x+y) &= \text{conv}(\bigcup_{z \in [a,b]} z(x+y)) \\
&= \text{conv}(\bigcup_{z \in [a,b]} (zx+zy)) \\
&\subseteq \text{conv}(\bigcup_{z \in [a,b]} zx + \bigcup_{s \in [a,b]} sy) \\
&= \text{conv}(\bigcup_{z \in [a,b]} zx) + \text{conv}(\bigcup_{s \in [a,b]} sy) \\
&= [a,b]x + [a,b]y.
\end{aligned}$$

Note that we have used Proposition 5.1(2) and distributivity of the convex closure over the Minkowski addition.

For (6), extended scalar multiplication is clearly monotone in $\mathbf{K}\mathbb{R}^n$ and so $[a,b](\bigsqcup^{\uparrow} D) \sqsupseteq \bigsqcup^{\uparrow}_{d \in D}([a,b]d)$ for any directed set $D \subseteq \mathbf{K}\mathbb{R}^n$. For the converse, we have to show that

$$\bigcap_{d \in D}([a,b]d) \subseteq [a,b](\bigcap_{d \in D} d).$$

Suppose $u \in \bigcap_{d\in D}([a,b]d)$. Hence

$$\forall d \in D\ \exists k, z_1, ..., z_k, u_1 \in z_1 d, ..., u_k \in z_k d.\ u \in \mathrm{conv}(u_1, ..., u_k).$$

However, $u_1 \in z_1 d$ for all $d \in D$ and so

$$u_1 \in \bigcap_{d\in D}(z_1 d) = z_1 \bigcap_{d\in D} d \subseteq \bigcup_{z\in[a,b]} z \bigcap_{d\in D} d,$$

and similarly for $u_2, ..., u_k$. Therefore,

$$u \in \mathrm{conv}(u_1, ..., u_k) \subseteq \mathrm{conv}(\bigcup_{z\in[a,b]} z \bigcap_{d\in D} d) = [a,b](\bigcap_{d\in D} d),$$

as required.

For (7), let $x_i, i \in I$ be any family of convex compacta in $\mathbb{R}^n$. We have to show that $[a,b]\prod_{i\in I} x_i = \prod_{i\in I}([a,b]x_i)$, or, in another words,

$$[a,b]\mathrm{conv}(\bigcup_{i\in I} x_i) = \mathrm{conv}(\bigcup_{i\in I}[a,b]x_i).$$

We have

$$\begin{aligned}
[a,b]\mathrm{conv}(\bigcup_{i\in I} x_i) &= \mathrm{conv}(a\cdot\mathrm{conv}(\bigcup_{i\in I} x_i) \cup b\cdot\mathrm{conv}(\bigcup_{i\in I} x_i)) \\
&= \mathrm{conv}(\mathrm{conv}(\bigcup_{i\in I} a x_i) \cup \mathrm{conv}(\bigcup_{i\in I} b x_i)) \\
&= \mathrm{conv}(\bigcup_{i\in I} a x_i \cup \bigcup_{i\in I} b x_i) \\
&= \mathrm{conv}(\bigcup_{i\in I}(a x_i \cup b x_i)) \\
&= \mathrm{conv}(\bigcup_{i\in I}[a,b]x_i).
\end{aligned}$$

The equalities above are justified by Lemma 5.6, Proposition 5.1, parts (5),(7),(9), and the definition. □

We have proved:

Theorem 5.8. *For every $n \in \omega$:*

(1) *the poset $\mathbf{K}\mathbb{R}^n$ is a bounded complete Banach domain with (B9) when considered with Minkowski addition, extended scalar multiplication and zero element $\mathbf{0} = (\underbrace{0, ..., 0}_{n})$.*

(2) *The Euclidean n-dimensional space coincides with invertible elements of* $\mathbf{K}\mathbb{R}^n$.
(3) *The diameter* $\mathrm{diam}\colon \mathbf{K}\mathbb{R}^n \to [0,\infty)^{op}$ *measures* $\max(\mathbf{K}\mathbb{R}^n) = \mathrm{inv}(\mathbf{K}\mathbb{R}^n)$.
(4) *Polyhedra form a basis for* $\mathbf{K}\mathbb{R}^n$.

We conclude this section with an observation that the partial norm for $\mathbf{K}\mathbb{R}^n$ does not satisfy the triangle inequality. Hence the associated distance is not a weak partial metric. However, the domain $\mathbf{K}\mathbb{R}^n$ admits a meaningful generalized distance, namely the Hausdorff quasimetric between convex bodies:

$$q_H(K,L) \stackrel{def}{=} \inf\{\varepsilon > 0 \mid K \subseteq (L)_\varepsilon\}.$$

Proposition 5.9. *The dual Hausdorff quasimetric* $q_H^*\colon \mathbf{K}\mathbb{R}^n \times \mathbf{K}\mathbb{R}^n \to [0,\infty)$ *induces the Scott topology on* $\mathbf{K}\mathbb{R}^n$.

Proof. Assume that $K \in \twoheaduparrow L$ for some $K, L \in \mathbf{K}\mathbb{R}^n$. We want to show that $B_{q_H^*}(K,\varepsilon) \subseteq \mathbf{int}(L)$ for some $\varepsilon > 0$. Since $K \subseteq \mathbf{int}(L)$, there is $\varepsilon > 0$ with $(K)_\varepsilon \subseteq \mathbf{int}(L)$. Hence if $Z \in B_{q_H^*}(K,\varepsilon)$, then $\inf\{\delta \mid Z \subseteq (K)_\delta\} < \varepsilon$, that is, $Z \subseteq (K)_\varepsilon \subseteq \mathbf{int}(L)$.

Conversely, if $Z \in B_{q_H^*}(K,\varepsilon)$ for some K and $\varepsilon > 0$, then there are positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ with $q_H^*(K,Z) < \varepsilon_3 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Hence $Z \subseteq (K)_{\varepsilon_3} \subseteq \mathbf{cl}((K)_{\varepsilon_2}) \subseteq (K)_{\varepsilon_1} \subseteq (K)_\varepsilon$. Setting $L = \mathbf{cl}((K)_{\varepsilon_2})$ yields $Z \in \mathbf{int}(L)$. That is, $Z \ll L$. Now, if $R \in \twoheaduparrow L$, then $R \subseteq \mathbf{int}(L) \subseteq (K)_{\varepsilon_1}$, and so, $\inf\{\delta \mid R \subseteq (K)_\delta\} \leq \varepsilon_1 < \varepsilon$. This means that $q_H^*(K,R) < \varepsilon$, that is, $R \in B_{q_H^*}(K,\varepsilon)$, as required. □

6. Conclusions and future work

We proposed Banach domains as computational models for Banach spaces. The rudiments of the theory of Banach domains have been given in [31]; here, we offer a much more thorough (still incomplete, though) study of models of Banach spaces. We discussed possible changes in axiomatics proposed in [31], defined partially linear operators, showed that they are bounded (Corollary 4.10) and proved an important extension theorem (Theorem 4.8) for bounded complete Banach domains. Moreover, we observed that under certain precisely defined conditions (Theorem 3.16) a weak partial metric can be defined from a partial norm in a canonical way. We noted that whenever the partial metric topology agrees with the Scott topology, the approximation relation of the underlying domain can be characterized by the induced distance (Theorem 3.22). Finally, we proposed a canonical Banach domain model of the n-dimensional Euclidean space and thoroughly analyzed its structure.

A future work will focus on Banach domains with the property that partial norm induces the Scott topology via the associated weak partial metric. Another separate task is to reflect duality theory for Banach spaces in the world of Banach domains. Finally, we conjecture that every Banach space has a bounded complete model which is a Banach domain.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, 1994.

[2] B. Coecke and K. Martin. A partial Order on Classical and Quantum States. Technical Report PRG-RR-02-07, Oxford University Computing Laboratory, 2002.

[3] A. Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.

[4] A. Edalat. Dynamical systems, measures and fractals via domain theory. *Information and Computation*, 120(1):32–48, 1995.

[5] A. Edalat. Power domains and iterated function systems. *Information and Computation*, 124:182–197, 1996.

[6] A. Edalat. Domains for Computation in Mathematics, Physics and Exact Real Arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997.

[7] A. Edalat and M.H. Escardó. Integration in Real PCF (full version). *Information and Computation*, 160:128–166, 2000.

[8] A. Edalat and R. Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193(1–2):53–73, 28 February 1998.

[9] A. Edalat and A. Lieutier. Foundation of a Computable Solid Modelling. In *Proceedings of ACM's Symposium on Solid Modeling'99*. ACM, 1999. Extended abstract. To appear in Theoretical Computer Science.

[10] A. Edalat and A. Lieutier. Domain Theory and Differential Calculus. In *Proceedings of IEEE Symposium on Logic in Computer Science*, 2002. Extended abstract.

[11] A. Edalat and P. Sünderhauf. A Domain-theoretic Approach to Real Number Computation. *Theoretical Computer Science*, 210:73–98, 1998.

[12] A. Edalat and P. Sünderhauf. Computable Banach Spaces via Domain Theory. *Theoretical Computer Science*, 219:169–184, 1999.

[13] M.H. Escardó. *PCF extended with real numbers: a domain theoretic approach to higher-order exact real number computation*. PhD thesis, Department of Computing, Imperial College of Science, Technology and Medicine, University of London, 1997.

[14] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics*. Cambridge University Press, first edition, 2003.

[15] R. Heckmann. Approximation of metric spaces by partial metric spaces. *Applied Categorical Structures*, 7:71–83, 1999.

[16] T. Kamimura and A. Tang. Total objects of domains. *Theoretical Computer Science*, 34:275–288, 1984.

[17] R. Kopperman, H-P. A. Künzi, and P. Waszkiewicz. Bounded complete models of topological spaces. *Topology and Appl.*, 139:285–297, 2004.

[18] H-P. A. Künzi. Cocompactness and quasi-uniformizability of completely metrizable spaces. *Topology Appl.*, 133:89–95, 2003.

[19] D. Lacombe. Quelques procédés de définitions en topologie recursif. In *Constructivity in Mathematics*, pages 129–158. North-Holland, 1959.

[20] J. Lawson. Computation on metric spaces via domain theory. *Topology and its Applications*, 85:274–263, 1998.

[21] J. D. Lawson. Spaces of maximal points. *Mathematical Structures in Computer Science*, 7(5):543–555, October 1997.

[22] K. Martin. Domain theoretic models of topological spaces. In *COMPROX III*, volume 13 of *Electronic Notes in Theoretical Computer Science*, 1998.

[23] K. Martin. Nonclassical techniques for models of computation. *Topology Proceedings*, 24, 1999.

[24] K. Martin. *A Foundation for Computation*. PhD thesis, Department of Mathematics, Tulane University, New Orleans, LA 70118, 2000.

[25] K. Martin. Topological games in domain theory. *Topology and Appl.*, 129(2), 2002.

[26] K. Martin. Ideal models of spaces. *Theoretical Computer Science*, 305, 2003.

[27] K. Martin. The Regular Spaces with Countably Based Models. *Theoretical Computer Science*, 305, 2003.

[28] P. Martin-Löf. *Notes on Constructive Mathematics*. Almquist & Wiksell, 1970.

[29] S. G. Matthews. Partial metric topology. In *Proceedings of the 8th Summer Conference on Topology and Its Applications*, volume 728, pages 176–185, 1992.

[30] D. Scott. Outline of the mathematical theory of computation. In *4th Princeton Conference on Informations Science*, 1970.

[31] P. Sünderhauf. Continuous domains and Banach spaces. *Electronic Notes in Theoretical Computer Science*, 24, 1999.

[32] P. Waszkiewicz. *Quantitative Continuous Domains*. PhD thesis, School of Computer Science, The University of Birmingham, Edgbaston, B15 2TT, May 2002.

[33] K. Weihrauch and U. Schreiber. Embedding metric spaces into cpo's. *Theoretical Computer Science*, 16:5–24, 1981.

Boole Centre for Research in Informatics, University College, Cork

Current address: Jagiellonian University, ul. Nawojki 11, 30-072 Kraków, Poland

E-mail address: `pqw@ii.uj.edu.pl`