Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 197-222

EQUICONTINUITY PROPERTIES OF D-DIMENSIONAL CELLULAR AUTOMATA

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Abstract. We introduce a topological classification of cellular automata in dimensions two and higher based on a one-dimensional classification given by Petr Kůrka [*Topological dynamics of cellular automata* in Codes, Systems, and Graphical Models, Ed. Brian Marcus and Joachim Rosenthal. 2001]. We give results, examples, and a diagram showing the dichotomy of sensitive dependence on initial conditions versus the existence of equicontinuity points.

1. Introduction

A cellular automaton is a tool used to model complex systems, making discrete simulations of an intricate process. Originally introduced by John von Neumann, following a suggestion of Stanislaw Ulam in the early 1950's, the purpose of this new tool was to construct a simple mathematical model capable of both universal computation and self-reproduction [1]. High performance computer systems and parallel processing have contributed to the popularity of cellular automata; computer implementation is quite easy due to the local and parallel nature of these objects. Various types of processes are simulated with cellular automata, cutting across many academic disciplines. Spin glass systems, reaction/diffusion

2000 *Mathematics Subject Classification.* 37B15, 37B10, 54H20.

Key words and phrases. cellular automata, equicontinuity, sensitive dependence on inititial conditions, topological dynamics.

This research was facilitated, in part, by a National Physical Science Consortium Fellowship and by stipend support from the National Security Agency.

processes in physics, tumor growth and excitement of muscle tissue in biology, and simulation of Turing machines in computer science are just a few of the existing applications.

Cellular automata were first investigated from a purely mathematical point of view in 1969 with G. A. Hedlund's formative paper [7]. This work was motivated by then-current problems in symbolic dynamics, possibly those of a cryptographic nature. When Stephen Wolfram turned his attention to cellular automata via computer simulation in the early 1980's, the subject gained momentum. Wolfram categorized one-dimensional cellular automata based on features of their asymptotic behavior which could be seen on a computer screen [16], [17]. Robert H. Gilman's work published in 1987 [4] and 1988 [5] was the first attempt to mathematically formalize these characterizations of Wolfram. Gilman utilized the notions of equicontinuity and expansiveness, as well as measure theoretic analogs of each. There are other classifications of one-dimensional cellular automata based on different types of properties; see, for example, [11] and the references therein. While measure is intrinsic to Gilman's partition, Petr Kůrka has a purely topological classification centered on equicontinuity, expansiveness, and sensitivity [10], and Mike Hurley has categorized cellular automata by their attractors [8].

Although Shin'ichirou Ishii has developed a measure theoretic version of Wolfram's classification in dimension two [9], much of the literature devoted to higher dimensional cellular automata pertains to the computational complexity and decidability of various properties. Giovanni Manzini, Luciano Margara, and others have examined a variety of properties of linear cellular automata (those whose local rule is a linear combination of the neighbors) in higher dimensions [3], [12].

Here, we extend the one-dimensional topological classification of Kůrka for cellular automata on the full shift space, to higher dimensional subshift spaces. We are primarily giving the first dichotomy in this classification: sensitive dependence on initial conditions versus equicontinuity points. For the results regarding equicontinuity, we require that the subshift have a dense set of shift periodic points. For the results involving sensitivity, this property is not assumed.

While some properties of one-dimensional cellular automata extend naturally, others require attention to subtleties. Additionally, we provide examples to highlight the differences between one- and two-dimensional cellular automata, as some results do not extend directly from one dimension.

This work is part of the Ph.D. thesis work of the author, which is being done at the University of North Carolina at Chapel Hill under the supervision of Jane Hawkins. The classification is developed further in the thesis; a full classification should also include expansivity, though that is not treated in this paper. A result of Mark A. Shereshevsky is that there are no expansive cellular automata on full shift spaces in dimensions two and higher [14]; in the author's thesis, expansive cellular automata on subshift spaces are investigated.

2. Preliminaries

Let A be a finite set and $|A|$ its cardinality. For $|A| \geq 2$, A is an *alphabet.* A *word* in A is any finite sequence from A, $u = u_0 \cdots u_{n-1}$. The length of u, $|u|$, is n. A D-dimensional generalization of a word is a *pattern* in A, a set of values from A on a finite path-connected (in $\mathbb{Z}^D$) subset of coordinates $E \subseteq \mathbb{Z}^D$. For instance, the following is a two-dimensional pattern of size $(r+1) \times (s+1)$.

$$u = \begin{matrix} u_{0,s} & u_{1,s} & u_{2,s} & \cdots & u_{r,s} \\ & & \vdots & & \\ u_{0,1} & u_{1,1} & u_{2,1} & \cdots & u_{r,1} \\ u_{0,0} & u_{1,0} & u_{2,0} & \cdots & u_{r,0} \end{matrix} . \tag{2.1}$$

Now we form the D-dimensional *full shift spaces*, $A^{\mathbb{Z}^D}$. A point $x \in A^{\mathbb{Z}}$ is a doubly infinite sequence of letters from A,

$$x = \cdots x_{-2}x_{-1}.x_0x_1x_2 \cdots , \tag{2.2}$$

where we use a decimal point to denote the 0^{th} position of x. Points in $A^{\mathbb{Z}^2}$ are doubly infinite sequences of points in $A^{\mathbb{Z}}$, arranged vertically:

(2.3)

$$x = \begin{array}{ccccccc} & & & \vdots & & & \\ \cdots & x_{(-2,2)} & x_{(-1,2)} & x_{(0,2)} & x_{(1,2)} & x_{(2,2)} & \cdots \\ \cdots & x_{(-2,1)} & x_{(-1,1)} & x_{(0,1)} & x_{(1,1)} & x_{(2,1)} & \cdots \\ \cdots & x_{(-2,0)} & x_{(-1,0)} & .x_{(0,0)} & x_{(1,0)} & x_{(2,0)} & \cdots \\ \cdots & x_{(-2,-1)} & x_{(-1,-1)} & x_{(0,-1)} & x_{(1,-1)} & x_{(2,-1)} & \cdots \\ \cdots & x_{(-2,-2)} & x_{(-1,-2)} & x_{(0,-2)} & x_{(1,-2)} & x_{(2,-2)} & \cdots \\ & & & \vdots & & & \end{array},$$

where the decimal point denotes the $(0,0)^{th}$ position of x. Shift spaces in higher dimensions are defined similarly; points in $A^{\mathbb{Z}^D}$ are indexed by D-vectors of integers and have values from A at each coordinate. For a point $x \in A^{\mathbb{Z}^D}$ and a subset $E \subseteq \mathbb{Z}^D$, $x|_E$ is the pattern which results from restricting x to the coordinates given by E. If E is infinite, we call $x|_E$ an *infinite pattern.* For $n < m$, let $\langle n, m\rangle = \{i \in \mathbb{Z} : n \leq i \leq m\}$ be a closed interval of integers.

For a vector of integers, $\vec{\imath} = (i_1, i_2, \cdots, i_D) \in \mathbb{Z}^D$, denote by $||\vec{\imath}||$ the maximum of the components, $\max\{i_1, i_2, \cdots, i_D\}$. We define a metric d on $A^{\mathbb{Z}^D}$ by setting $d(x,y) = 0$ if $x = y$ and for $x \neq y \in A^{\mathbb{Z}^D}$,

$$d(x,y) = 2^{-k}, \text{ where } k = \inf\{||\vec{\imath}|| : x_{\vec{\imath}} \neq y_{\vec{\imath}}\}. \tag{2.4}$$

Under this metric, points in $A^{\mathbb{Z}}$ are close if they agree on a large central word, $x|_{\langle -k,\, k\rangle} = y|_{\langle -k,\, k\rangle}$, points in $A^{\mathbb{Z}^2}$ are close if they agree on a large central square,

$$\begin{array}{ccc} x_{(-k,\,k)} & \cdots & x_{(k,\,k)} \\ \vdots & & \vdots \\ x_{(-k,\,0)} & \cdots & x_{(k,\,0)} \\ \vdots & & \vdots \\ x_{(-k,\,-k)} & \cdots & x_{(k,\,-k)} \end{array}, \tag{2.5}$$

and points in $A^{\mathbb{Z}^D}$ are close if they agree on a large central hypercube.

A basis for the topology determined by this metric is given by the cylinder sets, $[u]_{\vec{\imath}} = \{x \in A^{\mathbb{Z}^D}$ containing the pattern u beginning at the coordinates given by $\vec{\imath}\}$. These sets are both open and closed.

As $A^{\mathbb{Z}^D}$ is a countable product of finite discrete spaces for each $D > 0$, the full shift spaces are compact.

To define a map on $A^{\mathbb{Z}^D}$, we simply describe the element in position $\vec{\imath}$ of the image for arbitrary $\vec{\imath} \in \mathbb{Z}^D$. On each full shift space, we have a $\mathbb{Z}^D$ action given by the *shift transformation.* For each $n \in \mathbb{Z}^D$,

$$(\sigma_{\vec{n}} x)_{\vec{\imath}} = x_{\vec{\imath}+\vec{n}}. \tag{2.6}$$

In one dimension, $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$; the action of σ shifts terms one position to the left,

$$\sigma(\cdots x_{-1}.x_0 x_1 \cdots) \;=\; \cdots x_{-1} x_0 . x_1 x_2 \cdots . \tag{2.7}$$

Let $\{\vec{e_j} : j = 1, \cdots, D\}$ be the standard orthonormal basis vectors of $\mathbb{R}^D$ with a 1 in position j and 0's elsewhere. Generators for the $\mathbb{Z}^D$ action are $\{\sigma_{\vec{e_j}} : j = 1, \cdots, D\}$, the shifts in the directions of each $\vec{e_j}$. It is straightforward to check that the shift is a continuous action on each of the full shift spaces. A closed, shift-invariant subset $X \subseteq A^{\mathbb{Z}^D}$ is a *subshift space*; every subshift space is a compact metric space using the metric d defined in (2.4).

A *cellular automaton, CA*, on a shift space $X \subseteq A^{\mathbb{Z}^D}$ is a continuous function $F : X \to X$ which commutes with the action of the shift. Trivially, $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ itself is a CA. A theorem of M. L. Curtis, Hedlund, and R. C. Lyndon (Theorem 3.4, [7]) states that $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a CA if and only if there is a positive integer r, called the *radius*, and a function $f : A^{2r+1} \to A$, called the *local rule*, such that $F(x)_i = f(x_{i-r}, \cdots, x_{i+r})$. The radius of the shift, σ, is 1 and the local rule is $f(a, b, c) = c$. It is also the case in higher dimensions that a self-map of a shift space is a CA if and only if there is a radius and local rule as above; the proof in [7] extends.

In dynamics, there is large interest in the asymptotic behavior of a system, and this is certainly the case in studying CA's. A useful way to track orbits under a one-dimensional CA consists of writing the iterates of a point underneath one another as follows:

$$\begin{array}{rcccccccc}
x = & \cdots & x_{-2} & x_{-1} & . \, x_0 & x_1 & x_2 & \cdots \\
Fx = & \cdots & (Fx)_{-2} & (Fx)_{-1} & . \, (Fx)_0 & (Fx)_1 & (Fx)_2 & \cdots \\
F^2x = & \cdots & (F^2x)_{-2} & (F^2x)_{-1} & . \, (F^2x)_0 & (F^2x)_1 & (F^2x)_2 & \cdots \\
 & & & & \vdots & & &
\end{array}$$

Example 2.1. Let $A = \{0, 1\}$ and consider the CA $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $(Sx)_i = x_{i-1} + x_{i+1} \pmod 2$. The orbit of the point $x = \cdots 0\ 0\ .1\ 0\ 0 \cdots$ is shown in Figure 1.

$$\begin{array}{rcccccccccccccccc}
x = & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Fx = & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & .0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
F^2x = & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & .0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
F^3x = & \cdots & 0 & 0 & 0 & 1 & 0 & 1 & .0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\
F^4x = & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & .0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
F^5x = & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & .0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
 & & & & & & & & \vdots & & & & & & &
\end{array}$$

FIGURE 1. Orbit of $\cdots 0.10 \cdots$ under the additive CA

A more illustrative way to view points in $\{0, 1\}^{\mathbb{Z}}$ is to let 0's be represented by white space and let 1's be represented by black space. Figure 2 shows the same orbit after more iterations in this fashion.

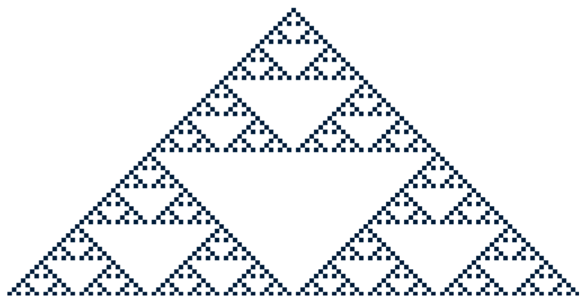

FIGURE 2. Orbit of $\cdots 00100 \cdots$ under S

Example 2.2. Let $A = \{ \blacktriangle, \square, \circ, \quad \}$ and define $P : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ to describe the movement of three different shaped particles in white space as follows. A $\circ$ particle moves both northeast and southwest, a $\blacktriangle$ particle moves both northwest and southeast, and a $\square$ particle is a wall that annihilates any other particle which runs into it. When a $\circ$ and a $\blacktriangle$ particle try to occupy the same location, they annihilate each other. The dynamics are illustrated in figures 3 through 6.

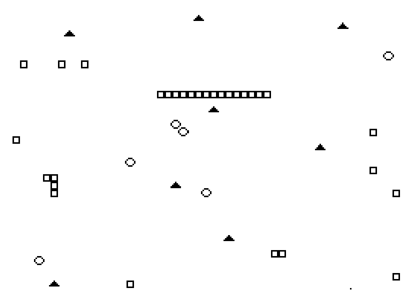

FIGURE 3.
Initial point

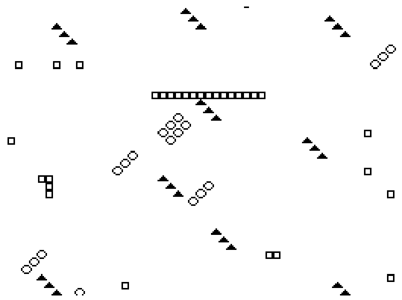

FIGURE 4.
One iteration

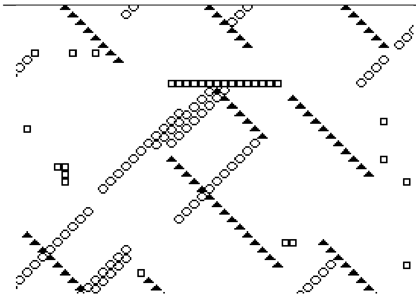

FIGURE 5.
Five iterations

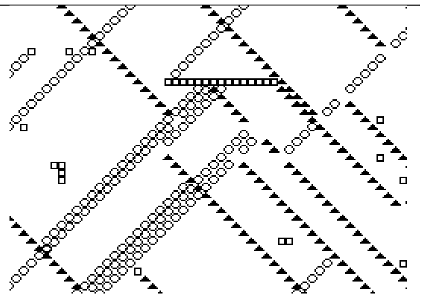

FIGURE 6.
100 iterations

By a *dynamical system*, we will mean a pair (Y, T) consisting of a compact metric space, Y, and a continuous map, $T : Y \to Y$. A point x is an *equicontinuity point* of a dynamical system (Y, T) if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(T^n x, T^n y) < \varepsilon \ \forall \ n \geq 0.$$

A dynamical system is *equicontinuous* if each of its points is an equicontinuity point. Essentially, an equicontinuous system is one for which points initially close have orbits which stay close for all time. We say that a dynamical system is *almost equicontinuous* if the set of equicontinuity points contains an intersection of dense open sets.

A dynamical system (Y, T) has *sensitive dependence on initial conditions* if there exists $\varepsilon > 0$ such that for all $x \in Y$, and $\delta > 0$, there exists y with $d(x, y) < \delta$ and $d(T^n x, T^n y) \geq \varepsilon$ for some $n \geq 0$.

We will refer to this property simply as *sensitive*. Sensitive systems are ones for which given any x, we can find points arbitrarily close to x which eventually get pushed away under iteration of T.

3. Equicontinuity of D-dimensional cellular automata

Theorem 3.1. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and let $F : X \to X$ be a cellular automaton. The following statements are equivalent:*

(1) *F is equicontinuous;*
(2) *there exists $M \geq 0$ such that for $x, y \in X$ with $d(x,y) < 2^{-M}$, $d(F^n x, F^n y) < 1 \ \forall \ n \geq 0$.*

The proof is straightforward, though it does not appear to be in the literature.

Proof: (1) $\Rightarrow$ (2) The equicontinuity of F implies that for $\varepsilon = 1$, there exists a $\delta = 2^{-M}$ satisfying the property given in (2).

(2) $\Rightarrow$ (1) Let $\varepsilon = 2^{-k} > 0$, and take $\delta = 2^{-(k+M)}$. Then for a pair $x, y \in X$ with $d(x,y) < \delta \leq 2^{-M}$, the distance between their iterates being smaller than 1 means that

$$(F^n x)_{\vec{0}} = (F^n y)_{\vec{0}} \ \forall \ n \geq 0.$$

By our choice of δ, we also have

$$d(\sigma_{\vec{\imath}} x, \sigma_{\vec{\imath}} y) < 2^{-M}$$

for $\vec{\imath} = (i_1, \cdots, i_D)$ with $|i_1|, |i_2|, \cdots, |i_D| \leq k$, and so

$$(F^n(\sigma_{\vec{\imath}} x))_{\vec{0}} = (F^n(\sigma_{\vec{\imath}} y))_{\vec{0}}\,.$$

Then for $|i_1|, |i_2|, \cdots, |i_D| \leq k$ and for all $n \geq 0$,

$$(3.1)\quad (F^n x)_{\vec{\imath}} = (\sigma_{\vec{\imath}}(F^n x))_{\vec{0}} = (F^n(\sigma_{\vec{\imath}} x))_{\vec{0}} = (F^n(\sigma_{\vec{\imath}} y))_{\vec{0}} = (\sigma_{\vec{\imath}}(F^n y))_{\vec{0}} = (F^n y)_{\vec{\imath}}.$$

Thus, $d(x,y) < \delta$ implies $d(F^n x, F^n y) < \varepsilon \ \forall \ n \geq 0$, and hence F is equicontinuous. □

The next theorem characterizes equicontinuous CA's as those which are eventually periodic, extending Theorem 5.2 in [11] which is in the setting of one-dimensional CA's on the full shift space. We give the result in the more general setting of a CA on any subshift which has a dense set of shift periodic points. When σ

is a $\mathbb{Z}^D$ action, $x \in X$ is σ-periodic if the set $\{\sigma_{\vec{\imath}}(x) : \vec{\imath} \in \mathbb{Z}^D\}$ is finite. A point is σ-periodic if and only if it has a period for each $\sigma_{\vec{e_j}}$ in the traditional sense. Each of the full shift spaces in all dimensions and the transitive one-dimensional subshifts of finite type (SFT's) has a dense set of shift periodic points. A transitive SFT is one for which, given any pair of allowable words u and v, there is an allowable word w so that uwv is an allowable word. Further, two-dimensional SFT's with strong specification (see [15]) and two-dimensional SFT's with the uniform filling property (see [13]) are also shown to have a dense set of shift periodic points. However, a general characterization of higher dimensional subshifts with this property is unknown.

Theorem 3.2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift with dense σ-periodic points, and let $F : X \to X$ be a cellular automaton. F is equicontinuous if and only if F is eventually periodic.*

Proof: ($\Leftarrow$) Let r be the radius of F, and assume there exists $m \geq 0,\ p > 0$ such that $F^{m+p} = F^m$. Take $M = r(m+p)$. For $x, y \in X$ with $d(x,y) < 2^{-M}$, we have

$$d(Fx, Fy) < 2^{-M} \cdot 2^r = 2^{-r(m+p-1)} < 1. \tag{3.2}$$

Further, $d(x,y) < 2^{-M}$ implies that for each $n < m+p$,

$$d(F^n x, F^n y) < 2^{-M} \cdot 2^{nr} = 2^{-r(m+p-n)} < 1. \tag{3.3}$$

Now since $F^{m+p} = F^p$, each of the two sequences of patterns consisting of the central symbols of the iterates, $(F^n x)_{\vec{0}}$ and $(F^n y)_{\vec{0}}$, forms an eventually periodic sequence with pre-period m and period p. But since the first $m + p > m$ elements are equal, we have the equality $(F^n x)_{\vec{0}} = (F^n y)_{\vec{0}}$ for all $n \geq 0$. Therefore,

$$d(x,y) < 2^{-M} \Rightarrow d(F^n x, F^n y) < 1 \ \forall\ n \geq 0, \tag{3.4}$$

and by Theorem 3.1, F is equicontinuous.

($\Rightarrow$) Assume F is equicontinuous, and let M be the constant resulting from Theorem 3.1(2). Let $x \in X$, and consider the central $\prod_{j=1}^{D}(2M+1)$ pattern of x, $u_x = x|_{\prod_{j=1}^{D}\langle -M, M\rangle}$. Since the shift periodic points are dense in X, there exists a $z \in [u_x]_{-M\vec{e}}$ which is periodic under the shift, where $\vec{e} = \sum_{j=1}^{D} \vec{e_j}$ is the sum of all basis vectors. Denote by $\vec{p} = (p_1, \cdots, p_D)$ the period vector of z; that is, $\sigma_{\vec{e_j}}^{p_j} z = z$ for each j.

Now as F commutes with the action of the shift, for each $n \geq 0$ and basis vector e_j, we have

$$\sigma_{\vec{e_j}}^{p_j}(F^n z) = F^n(\sigma_{\vec{e_j}}^{p_j} z) = F^n z. \tag{3.5}$$

That is, each iterate $F^n z$ is also periodic for the shift with the same period vector $\vec{p}$. This puts an upper bound on the cardinality of the set of iterates of z, $|\{F^n z : n \geq 0\}| \leq |A|^{p_1 p_2 \cdots p_D}$, which is finite. Hence, there must be a repetition in the set of iterates; let the first one be $F^{m_{u_x}+p_{u_x}} z = F^{m_{u_x}} z$. Thus, the set of iterates $\{F^n z : n \geq 0\}$ forms an eventually periodic sequence with pre-period $m_{u_x} \geq 0$ and period $p_{u_x} > 0$. We use the subscript u_x on both the pre-period and period of this sequence, as these quantities depend only on the pattern u_x. Now for all y in the cylinder $[u_x]_{-M\vec{e}}$, $d(y, z) < 2^{-M}$ and hence, $(F^n y)_{\vec{0}} = (F^n z)_{\vec{0}}$ for all $n \geq 0$. Therefore, $(F^n y)_{\vec{0}}$ is also an eventually periodic sequence with pre-period m_{u_x} and period p_{u_x}. Let

$$m = \max\{m_u\} \quad \text{and let} \quad p = \prod p_u, \tag{3.6}$$

where the maximum and the product are taken over all patterns $u \in A^{\prod_{j=1}^{D}(2M+1)}$. Since for all $x \in X$, the pattern $u_x = x|_{\prod_{j=1}^{D}\langle -M, M\rangle}$ is one of those that the maximum and product are taken over, we have $(F^{m+p}x)_{\vec{0}} = (F^m x)_{\vec{0}}$. Using the commutativity of F and the shift maps gives the equality

$$\begin{aligned}(F^{m+p}x)_{\vec{\imath}} = \big(\sigma_{\vec{\imath}}(F^{m+p}x)\big)_{\vec{0}} = \big(F^{m+p}(\sigma_{\vec{\imath}}x)\big)_{\vec{0}} = \\ (F^m(\sigma_{\vec{\imath}}x))_{\vec{0}} = (\sigma_{\vec{\imath}}(F^m x))_{\vec{0}} = (F^m x)_{\vec{\imath}}\end{aligned} \tag{3.7}$$

for each $\vec{\imath} \in \mathbb{Z}^D$. Hence, $F^{m+p} = F^m$, and so F is eventually periodic. □

Further, an equicontinuous cellular automaton which is also surjective must be periodic. This seems to be well known, but a proof is not available in the literature.

Theorem 3.3. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift with dense σ-periodic points, and let $F : X \to X$ be a cellular automaton. F is both equicontinuous and surjective if and only if F is periodic.*

Proof: ($\Rightarrow$) Suppose F is both equicontinuous and surjective. By the previous theorem, there are minimal integers $m \geq 0$ and $p > 0$

so that $F^{m+p} = F^m$. Assume to the contrary that $m > 0$, i.e., that F is only eventually periodic and not periodic. For an arbitrary $x \in X$, there must be a point $y \in X$ with $Fy = x$. Then we have both

$$F^m y = F^{m+p} y = F^{m+p-1}(Fy) = F^{m+p-1}x, \text{ and} \tag{3.8}$$

$$F^m y = F^{m-1}(Fy) = F^{m-1}x, \tag{3.9}$$

so that $F^{m+p-1}x = F^{m-1}x$. As x was arbitrary, $F^{m-1+p} = F^{m-1}$, and so m is not the pre-period of F. Therefore, $m = 0$, and $F^p = F$ is periodic.

($\Leftarrow$) This direction is trivial, as F periodic of period p implies that for every $x \in X$, $F(F^{p-1}x) = x$; therefore, F is surjective. Equicontinuity of F is then given by Theorem 3.2. □

The proof of Theorem 3.2 suggests a more general result. If we have any CA, F, on a subshift, $X \in A^{\mathbb{Z}^D}$, then even if F is not equicontinuous, we can obtain F-eventually periodic points. These are the points which are periodic under the shift action.

Proposition 3.4. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space and let $F : X \to X$ be a cellular automaton. If $x \in X$ has $\sigma_{\vec{\imath}}x = x$ for some $\vec{\imath} \in \mathbb{Z}^D$, then there exist integers $m \geq 0$ and $p > 0$ so that $F^{m+p}x = F^m x$.*

Proof: Suppose that $\sigma_{\vec{\imath}}x = x$ for some point $x \in X$ and vector $\vec{\imath} = (i_1, \cdots, i_D) \in \mathbb{Z}^D$. Then for each $n \geq 0$, we have $\sigma_{\vec{\imath}}(F^n x) = F^n(\sigma_{\vec{\imath}}x) = F^n x$, and so each iterate $F^n x$ is also fixed under $\sigma_{\vec{\imath}}$. As in the proof of Theorem 3.2, we have the bound $|\{F^n x : n \geq 0\}| \leq |A|^{\prod_{j=1}^D i_j}$; the collection of iterates of x is a finite set. Therefore, there must be a repetition, $F^m x = F^{m+p}x$, for some $m \geq 0$ and $p > 0$. Thus, x is F-eventually periodic. □

4. Examples of equicontinuous CA's

Example 4.1. Let A be any finite set and let $O : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be the zero map, i.e., $O(x)_{\vec{\imath}} = 0$ for all $\vec{\imath} \in \mathbb{Z}^D$ and $x \in A^{\mathbb{Z}^D}$. For $\varepsilon > 0$, let $\delta = \varepsilon$. Then for $x, y \in A^{\mathbb{Z}^D}$ with $d(x, y) < \delta$, clearly the 0^{th} iterates of x and y are within epsilon, and since for any $n > 0$, $(O^n x)_{\vec{\imath}} = (O^n y)_{\vec{\imath}} = 0$ for all $\vec{\imath} \in \mathbb{Z}^D$, we have $d(O^n x, O^n y) = 0 < \varepsilon$ for $n > 0$ also. Hence, O is equicontinuous. Since there is only one point in the image of O, it is certainly not surjective, and we see that $m = 1,\ p = 1$.

Example 4.2. Let A be any finite set and let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a CA with radius 0 and local rule $f : A \to A$. For $\varepsilon = 2^{-k} > 0$, again let $\delta = \varepsilon$. Now for $x, y \in A^{\mathbb{Z}^D}$ with $d(x, y) < \delta$, we have $x_{\vec{\imath}} = y_{\vec{\imath}}$ for $|i_1|, |i_2|, \cdots, |i_D| \leq k$. Then for $|i_1|, |i_2|, \cdots, |i_D| \leq k$ and $n \geq 0$, $(F^n x)_{\vec{\imath}} = f^n(x_{\vec{\imath}}) = f^n(y_{\vec{\imath}}) = (F^n y)_{\vec{\imath}}$, and therefore, $d(F^n x, F^n y) < \varepsilon$ for all $n \geq 0$. Thus, any radius 0 CA is equicontinuous.

Example 4.3. A class of two-dimensional examples can be obtained from equicontinuous one-dimensional cellular automata. We define the two-dimensional action by letting the one-dimensional CA act on the rows of points in $A^{\mathbb{Z}^2}$. Precisely, let A be a finite set and let $G : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an equicontinuous one-dimensional CA. For $j \in \mathbb{Z}$, let $H_j : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}}$ be the restriction map onto the j^{th} row, given by $(H_j\, x)_i = x_{(i,j)}$. Now define the map $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ by $(Fx)_{(i,j)} = (G \circ H_j\, x)_i$. We notice that as $(G \circ H_j\, x)$ and $(H_j \circ Fx)$ both represent the j^{th} row of Fx, we have $(F^n x)_{(i,j)} = (G^n \circ H_j\, x)_i$. To see that F is equicontinuous, let $\varepsilon > 0$. As G is equicontinuous, there exists $\delta_G = 2^{-k} > 0$ such that for $x, y \in A^{\mathbb{Z}}$ with $d(x, y) < \delta_G$, $d(G^n x, G^n y) < \varepsilon\ \forall\ n \geq 0$. Let $\delta = \delta_G$. Then for $x, y \in A^{\mathbb{Z}^2}$ with $(x, y) < \delta$, we have $d(H_j\, x, H_j\, y) < \delta = \delta_G$ for $|j| \leq k$. Now $d(F^n x, F^n y) = d(G^n \circ H_j\, x, G^n \circ H_j\, y) < \varepsilon\ \forall\ n \geq 0$.

This construction extends to higher dimensions so that from a D-dimensional equicontinuous CA, we can create a $(D+1)$-dimensional equicontinuous CA on the same alphabet.

5. An example of a CA on a subshift space

Consider the *Golden Mean Shift Space*, the two-dimensional subshift

$$\mathcal{X} = \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : x|_{\{(i,j),(i+1,j)\}} \neq 1\ 1 \text{ and } x|_{\{(i,j),(i,j+1)\}} \neq \begin{matrix} 1 \\ 1 \end{matrix} \right\}.$$

Define $L : \mathcal{X} \to \mathcal{X}$ by the radius one local rule which sends the pattern $\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$ to 1 and all other 3×3 patterns to 0. First, we note that the image of L is, in fact, contained in $\mathcal{X}$, as can be seen from the local rule definition. The dynamics of this CA are illustrated in figures 7 through 10.

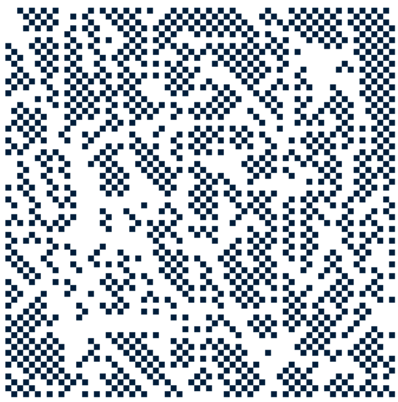

FIGURE 7.
Initial point

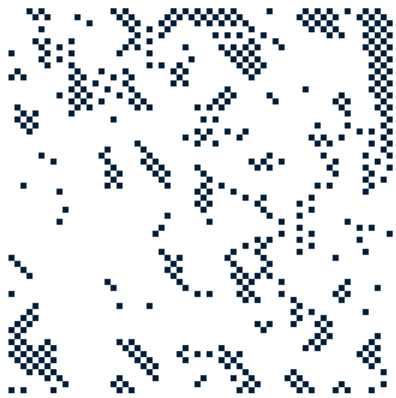

FIGURE 8.
One iteration

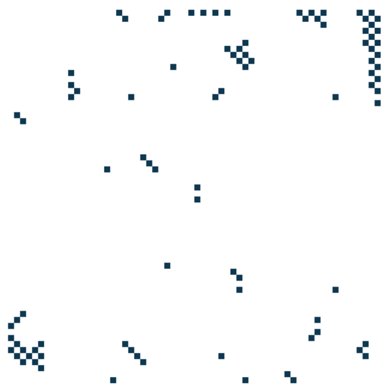

FIGURE 9.
Two iterations

FIGURE 10.
Three iterations

We next show that the point x_0 consisting entirely of 0's is a point of equicontinuity for L. Note that x_0 is a fixed point of L. Fix an $\varepsilon = 2^{-k}$ and suppose that $d(x, x_0) < 2^{-(k+1)}$. Since every 0 in x in this central region has 0's in at least two of its left, right, top, and bottom neighbors, none of them will get mapped to a 1 in Lx. Similarly, all iterates $L^n x$ will retain the central $(2(k+1)+1)\times(2(k+1)+1)$ pattern of 0's, and so $d(L^n x, L^n x_0) < \varepsilon$ for all $n \geq 0$.

However, L is not an equicontinuous CA, as we next show. Consider the "checkerboard" period 2 point,

$$y_0 = \cdots \begin{array}{ccccc} & & \vdots & & \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ & & \vdots & & \end{array} \cdots \mapsto y_1 = \cdots \begin{array}{ccccc} & & \vdots & & \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ & & \vdots & & \end{array} \cdots .$$

Fix $\varepsilon = 1$, and suppose there is a $\delta = 2^{-M}$ so that

$$d(x, y_0) < \delta \Rightarrow d(L^n x, L^n y_0) < 1 \,\text{for all}\, n \geq 0.$$

Take x^* to agree with y_0 in the central $(2M+1)\times(2M+1)$ pattern and to be filled with 0's in all other coordinates. Now the 0's in the $\pm M^{th}$ rows and columns have only two 1's in their left, right, top, and bottom neighbors, and so they will not change to 1's in Lx^*. Under the next iteration of L, we see that the 0's in the $\pm(M-1)^{st}$ rows and columns do not change to 1's, and as we continue to apply L, the 1's eventually disappear completely. In particular, $L^n x^* = x_0$ for $n \geq M+1$. But as $(L^n y_0)_{(0,0)}$ alternates between 0 and 1 as n increases, $d(L^n x^*, L^n y_0) \not< 1$ for all n.

6. Almost equicontinuous results

The property of being equicontinuous at every point is too restrictive for classifying CA's, for a number of cellular automata are not equicontinuous but still have points of equicontinuity. Recall that *almost equicontinuous* refers to a topological property; it means that the set of equicontinuity points is residual. In one dimension, Kůrka has shown that every CA is either almost equicontinuous or has sensitive dependence on initial conditions [11]. Therefore, we study the property of almost equicontinuity in higher dimensions. We begin by noting the well-known fact that any dynamical system without isolated points which is almost equicontinuous cannot be sensitive.

Theorem 6.1. *Assume that X has no isolated points, and let (X,T) be an almost equicontinuous dynamical system. Then T is not sensitive.*

Proof: Since T is almost equicontinuous, T has a point of equicontinuity, x. Suppose that T is also sensitive. Then there is an $\varepsilon > 0$ such that for all $z \in X$ and $\delta > 0$, there exists $y \in X$ with $d(y,z) < \delta$ and $d(T^n y, T^n z) \geq \varepsilon$ for some $n \geq 0$. But by the definition of an equicontinuity point, the above property does not apply to x, and hence T is not sensitive. □

In one dimension, almost equicontinuous CA's are characterized by the existence of blocking words, first introduced by F. Blanchard and P. Tisseur [2]. We extend the notion of blocking to patterns in dimensions two and higher and examine the existence of such patterns and their relation to almost equicontinuity.

We say that a word (in one-dimension) or a pattern (in higher dimensions) u *occurs* in a point $x \in X \subseteq A^{\mathbb{Z}^D}$ if there exists a finite subset, $E \subseteq \mathbb{Z}^D$, so that $x|_E = u$.

A word u is called *s-blocking* for a CA $F : X \to X$ on a one-dimensional subshift space if $s \leq |u|$ and there exists a non-negative integer $p \leq |u| - s$ so that for all $x, y \in [u]_0$, $F^n x_{\langle p,p+s)} = F^n y_{\langle p,p+s)}$ $\forall\ n \geq 0$. Illustrated in Figure 11, this means that each occurrence of the pattern u in x determines a length s word in all iterates of x.

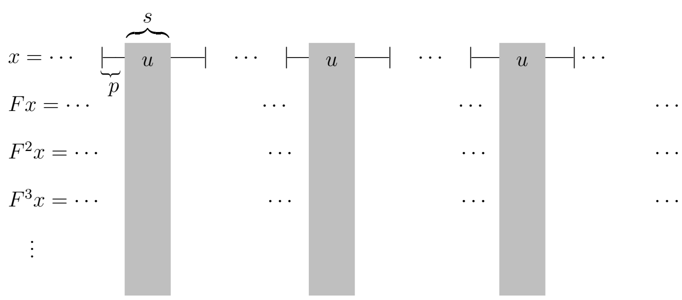

FIGURE 11. A blocking word for a 1D CA

We extend blocking first to two dimensions, where the definition is easier to state. A rectangular pattern u of size $k \times l$ is said to be (r, s)-*blocking* if there exist non-negative integers $p \leq k - r$ and $q \leq l - s$ such that for all $x, y \in [u]_{0,0}$ and $n \geq 0$, we have

$$(F^n x)|_{\langle p,p+r) \times \langle q,q+s)} = (F^n y)|_{\langle p,p+r) \times \langle q,q+s)}. \tag{6.1}$$

That is to say, if the pattern u occurs in a point x at the coordinates of E, the values of $F^n x$ at a subset of the coordinates of E are determined for all time n. If $r = k$, $s = l$, and $p = q = 0$, we say that u is *fully blocking*. A fully blocking pattern is one whose occurrence in x at E determines the values of $F^n x$ in all coordinates of E for all time n. In higher dimensions, a pattern u of size $k_1 \times k_2 \times \cdots \times k_D$ is $(r_1, r_2, \cdots, r_D)$-*blocking* if the occurrence of u in a point x determines the values of $F^n x$ in a $r_1 \times r_2 \times \cdots \times r_D$ hypercube of coordinates for all n. If each $k_i = r_i$, then u is said to be *fully blocking*; that is, the values of $F^n x$ are determined in all coordinates where u occurs in x for all time n. For the example in section 5 on the Golden Mean Shift Space, any square pattern of 0's with side length at least 2 is fully blocking; once such a pattern occurs, there will always be 0's in those positions under iteration of L.

Theorem 6.2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and $F : X \to X$ be a CA with radius r that is not sensitive. Then there exists an $(r, r, \cdots, r)$-blocking pattern for F.*

Proof: Let $m \in \mathbb{Z}$ be such that $2m + 1 \geq r$. Since F is not sensitive, for $\varepsilon = 2^{-m}$, there exists $x \in X$ and $\delta = 2^{-m-p}, p \geq 0$, such that for all $y \in X$, $d(x, y) < \delta \Rightarrow d(F^n x, F^n y) < \varepsilon \ \forall \ n \geq 0$. Let the pattern u be a central hypercube of x,

$$u = x|_{\prod_{j=1}^{D} \langle -(m+p), m+p \rangle} \in A^{\prod_{j=1}^{D}(2m+2p+1)}.$$

Now for $y, z \in [u]_{-(m+p)\vec{e}}$, we have $d(F^n y, F^n z) < \varepsilon \ \forall \ n \geq 0$, since $d(x, y) < \delta$ and $d(x, z) < \delta$ imply that both $d(F^n x, F^n y) < \varepsilon$ and $d(F^n x, F^n z) < \varepsilon$ for all $n \geq 0$. Thus, u blocks a hypercube of size $(2m+1) \times \cdots \times (2m+1)$, and as m was chosen so that $2m+1 \geq r$, u is an $(r, r, \cdots, r)$-blocking pattern. □

Theorem 6.2 shows that if a CA is not sensitive, then a blocking pattern exists. If we assume the pattern is fully blocking on a full shift space, we obtain almost equicontinuity of the CA. The proof we give relies on the fact that in a full shift space, patterns can always be fit together.

Theorem 6.3. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a CA with radius r. If there exists a fully blocking pattern of size $k \times k \times \cdots \times k$ for F, where $k \geq r$, then F is almost equicontinuous.*

Proof: Let $u \in A^{\prod_{j=1}^{D} k}$ be a fully blocking pattern, where $k \geq r$. That is, u is a size $k \times k \times \cdots \times k$ pattern and when u occurs in a point of $A^{\mathbb{Z}^D}$, the values in the $k \times k \times \cdots \times k$ frame of the coordinates where u is are determined for all iterates of that point.

Let the sets G_n be the following:

$$G_n = \{x \in A^{\mathbb{Z}^D} : \exists\, \delta = \delta(n,x) \text{ such that } d(x,y) < \delta \Rightarrow d(F^i x, F^i y) < 2^{-n}\ \forall\, i \geq 0\}. \tag{6.2}$$

Clearly, $\bigcap_{n\geq 0} G_n$ is the set of equicontinuity points for F. We will show that G_n is open, and using u, we will also show that G_n is dense for each n. This will prove that the set of equicontinuity points contains a residual set; that is, that F is almost equicontinuous.

CLAIM 1. G_n is open for each $n \geq 0$.

Fix $n \geq 0$, and let $x \in G_n$. Let δ be the $\delta(n,x)$ guaranteed by (6.2), the definition of G_n. We claim that $B_\delta(x) \subseteq G_n$. To see this, let $y \in B_\delta(x)$. In order for y to be in G_n, we need an $\alpha = \alpha(n,y)$ so that $d(y,z) < \alpha \Rightarrow d(F^i y, F^i z) < 2^{-n}\ \forall\, i \geq 0$. Let

$$\alpha = \min\left\{\frac{d(x,y)}{2}, \frac{\delta - d(x,y)}{2}\right\}. \tag{6.3}$$

Then, for $d(y,z) < \alpha$, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \frac{\delta - d(x,y)}{2} = \frac{\delta}{2} + \frac{d(x,y)}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{6.4}$$

Thus, for each $i \geq 0$, $d(F^i x, F^i z) < 2^{-n}$. That is, $(F^i x)_{\vec{\jmath}} = (F^i z)_{\vec{\jmath}}$ for each $i \geq 0$ and $\vec{\jmath} = (j_1, j_2, \cdots, j_D)$ with $|j_1|, |j_2|, \cdots, |j_D| \leq n$. Now by the choice of δ, we also have

$$y \in B_\delta(x) \Rightarrow d(F^i x, F^i y) < 2^{-n}\ \forall\, i \geq 0, \tag{6.5}$$

and thus, $d(F^i y, F^i z) < 2^{-n}\ \forall\ i \geq 0$. Therefore,

$$d(y,z) < \alpha \Rightarrow d(F^i y, F^i z) < 2^{-n}\ \forall\ i \geq 0, \tag{6.6}$$

and hence, $y \in G_n$. Thus, $B_\delta(x) \subseteq G_n$ and so G_n is open.

CLAIM 2. G_n is dense for each $n \geq 0$.

Let B be a hypercube pattern with side length a multiple of k. We will show that for all $\vec{p} \in \mathbb{Z}^D$, $G_n \cap [B]_{\vec{p}} \neq \emptyset$; thus, G_n is dense. We build a point $x \in [B]_{\vec{p}}$ by placing the pattern B in the proper place, then filling out the rest of the coordinates with the pattern u. We will get the following picture for x in $A^{\mathbb{Z}^2}$.

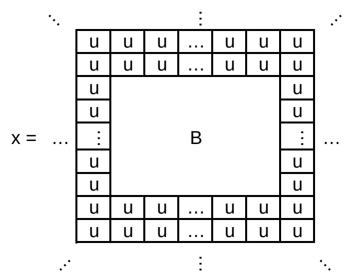

FIGURE 12. Creation of the point $x \in [B]_{\vec{p}} \cap G_n$

We have $x \in [B]_{\vec{p}}$, and we will now show that $x \in G_n$ also. Let $\delta = 2^{-(k+m)}$, where $m = \max\{n,$ side length of B$\}$. Suppose $y \in A^{\mathbb{Z}^D}$ has $d(x,y) < \delta$; we will show that all of the iterates $F^i x$ and $F^i y$, $i \geq 0$, are within 2^{-n}. By construction of δ, since y is within δ of x, y must contain the pattern B and at least one border of u patterns around B in the same location as x. Further, y will contain a border of u patterns inside the central $n \times n \times \cdots \times n$ region, again by construction of δ. Now as u is a $(k,k,\cdots,k)$-blocking pattern, the values at the coordinates of y which contain u (the frame around B) will be determined for all iterates $F^i y$. Since F has radius r and u has side length $k \geq r$, for each iterate $F^i y$, the values at the coordinates where y has the pattern B can depend only on the values at those same coordinates as well as the coordinates where y has the pattern u. That is, for all $i \geq 0$, the values in the central $n \times n \times \cdots \times n$ region of $F^i y$ are determined by the values of the coordinates in the central $(k+m) \times (k+m) \times \cdots \times (k+m)$

region of y, which equal the values of the coordinates in the central $(k+m)\times(k+m)\times\cdots\times(k+m)$ region of x. Therefore, $d(x,y) < \delta \Rightarrow d(F^i x, F^i y) < 2^{-n}\ \forall\ i \geq 0$, and thus $x \in G_n$.

Therefore, we have that G_n is an open, dense set for each $n \geq 0$, and that the intersection of all G_n's is contained in the set of equicontinuity points of F. Hence, F is almost equicontinuous. □

The preceding three theorems in this section attempt to extend Theorem 5.1 of [11], which proves the equivalence of the conditions of being almost equicontinuous, of being not sensitive, and of having a blocking pattern for one-dimensional cellular automata. What we have shown in higher dimensions is that almost equicontinuous implies not sensitive, not sensitive implies the existence of blocking patterns, and the existence of fully blocking patterns implies almost equicontinuity. However, not all almost equicontinuous CA's have fully blocking patterns; one such example is given in the next section. It has not yet been determined if there are any two-dimensional CA's which have a non-fully blocking pattern but are not almost equicontinuous.

The next result guarantees almost equicontinuity when there is a blocking pattern which allows for no cracks when the pattern is used to form a boundary around the coordinates in the epsilon region. This shows that fully blocking is not a necessary condition for almost equicontinuity.

Theorem 6.4. *Let $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be a cellular automaton with radius r. If there exists a pattern u which blocks a cross containing an $r \times r$ square for F, then F is almost equicontinuous.*

Proof: This proof is essentially identical to the proof of Theorem 6.3. The only difference is that the iterates of the point $x \in [B]_{p,q} \cap G_n$ formed using the pattern u do not have all values determined in coordinates outside of those containing the pattern B. We have the picture in Figure 13 for x, where the shaded coordinates represent the cross determined by the occurrence of u which contains an $r \times r$ square. Since the shaded region of x, the coordinates whose values are determined for all time, has no gaps, x will be in each set G_n, and the proof of Theorem 6.3 holds here as well. □

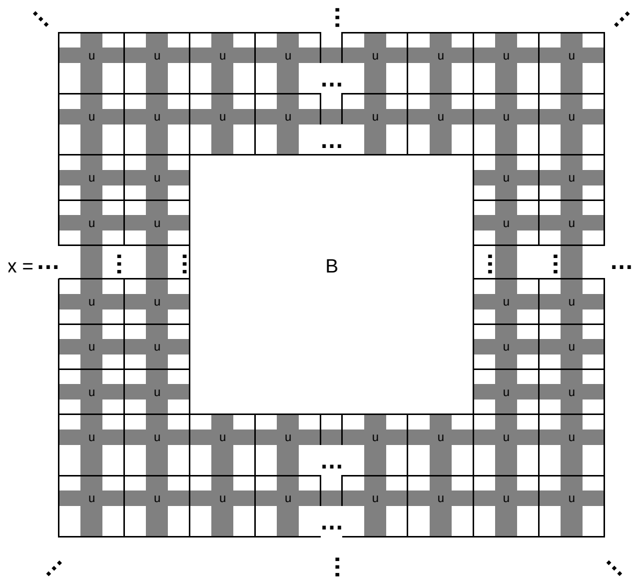

FIGURE 13. Creation of the point $x \in [B]_{p,q} \cap G_n$

7. EXAMPLES OF ALMOST EQUICONTINUOUS CA'S

Example 7.1. Let $A = \{0,1\}$, and let $R : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be given by the radius 1 local rule,

$$r\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = \begin{cases} 0 & \text{if } \sum_{i=1}^{9} x_i \leq 3 \\ 1 & \text{if } \sum_{i=1}^{9} x_i \geq 4 \end{cases}. \tag{7.1}$$

Neither 0 nor 1 is a (1,1)-blocking pattern for R, but consider the pattern $u = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$. Since each 1 neighbors at least three other 1's, $u \mapsto u$, and hence u is a fully blocking pattern. Then by Theorem 6.3, R is almost equicontinuous.

Example 7.2. Let $A = \{0, 1\}$, and let $M : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be given by the radius 1 local rule,

$$m\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = \begin{cases} 0 & \text{if } \sum_{i=1}^{9} x_i \leq 4 \\ 1 & \text{if } \sum_{i=1}^{9} x_i \geq 5 \end{cases} . \tag{7.2}$$

M successively homogenizes the space, illustrated in figures 14 and 15.

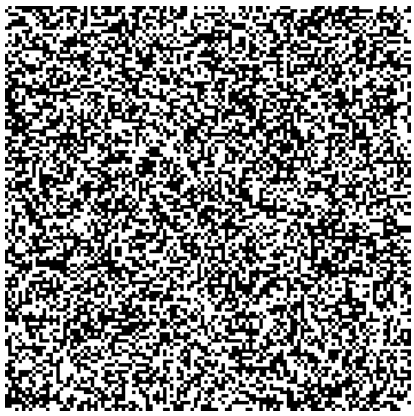

Figure 14.
An initial point

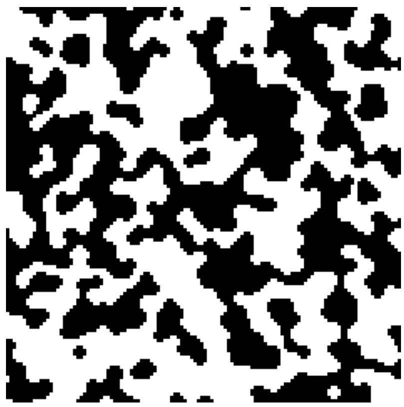

Figure 15.
Twenty-two iterations

There are no fully blocking $k \times k$ patterns for this CA, since the values in the corners of a square neighbor only three other known values, and at least four neighboring values must be known in order to determine the value in the next iteration. However, consider the pattern $u = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$. After one iteration, $u \mapsto \begin{array}{cccc} & 1 & 1 & \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ & 1 & 1 & \end{array}$, which is invariant, since each of the 1's neighbors at least four other 1's. Thus, u is a blocking pattern; it determines a cross of 1's containing a 2×2 square for all time. Now as M has radius 1, M is almost equicontinuous by Theorem 6.4. Note that this CA is not equicontinuous though, for consider the point

$$x = \begin{array}{ccccccccc} & & & & \vdots & & & & \\ & & 1 & 0 & 1 & 0 & 1 & & \\ & & 0 & 1 & 0 & 1 & 0 & & \\ \cdots & & 1 & 0 & 1 & 0 & 1 & & \cdots \\ & & 0 & 1 & 0 & 1 & 0 & & \\ & & 1 & 0 & 1 & 0 & 1 & & \\ & & & & \vdots & & & & \end{array} .$$

Since each value is surrounded by four 0's and four 1's, this is a fixed point under M. It is not an equicontinuity point though: for $\varepsilon = 2^{-k}$ and $\delta = 2^{-m}$, let $y \in B_\delta(x)$ so that $y_{\vec{i}} = 0$ for each $||\vec{i}|| \geq m+1$. That is, y agrees with x in a large central square and has 0's elsewhere. However, under iteration of M, the 0's propagate towards the center until $M^{m+1}y$ consists solely of 0's.

Example 7.3. Let $A = \{ \blacktriangle, \square, \circ, \quad \}$, and $P : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be the CA given in Example 2.2, describing the movement of three different shaped particles through white space.

Clearly, $\square$ is a (1,1)-blocking pattern, and since P has radius 1, Theorem 6.3 implies that $(A^{\mathbb{Z}^2}, P)$ is almost equicontinuous. As the point consisting of a single $\circ$ particle in all white space evolves under iteration to a longer and longer diagonal line of $\circ$'s in white space, P cannot be eventually periodic, and hence is not equicontinuous.

8. Sensitive dependence on initial conditions

In Example 4.3, we used a one-dimensional equicontinuous CA to build a two-dimensional equicontinuous CA. In fact, in building a two-dimensional CA by letting a one-dimensional CA act on rows of two-dimensional points, the properties of sensitive and of not sensitive are preserved as well.

Theorem 8.1. *Let $G : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA. For each $j \in \mathbb{Z}$, let $H_j : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}}$ be the restriction $(H_j x)_i = x_{(i,j)}$. Now define the CA $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ by $(Fx)_{(i,j)} = (G \circ H_j x)_i$. Then F is sensitive if and only if G is sensitive.*

Proof: ($\Leftarrow$) Assume that G is sensitive. Then there is an $\varepsilon_G = 2^{-k}$ such that for all $x \in A^{\mathbb{Z}}$ and $\delta > 0$, there is some $y \in B_\delta(x)$

and $n > 0$ such that $d(G^n x, G^n y) \geq \varepsilon_G$. We claim that $\varepsilon = \varepsilon_G$ will give sensitivity of F also. To see this, let $x \in A^{\mathbb{Z}^2}$ and let $\delta = 2^{-m} > 0$. Let $y_0 \in B_\delta(H_0 x)$ be the one-dimensional sequence and $n > 0$ be the iterate guaranteed by the sensitivity of G to have $d(G^n \circ H_0 x, G^n y_0) \geq \varepsilon$. Now there is a point $y \in A^{\mathbb{Z}^2}$ having both $d(x, y) < \delta$ and $H_0 y = y_0$ since $d(H_0 x, y_0) < \delta$. Then we have $d(G^n \circ H_0 x, G^n \circ H_0 y) = d(G^n \circ H_0 x, G^n y_0) \geq \varepsilon$. That is, after n iterations of F, there are differences in the ε-regions of iterates of x and y in the central row, so that $d(F^n x, F^n y) \geq \varepsilon$. Therefore, F is sensitive.

($\Rightarrow$) Assume G is not sensitive. Let $\varepsilon = 2^{-k} > 0$; then there exists $x \in A^{\mathbb{Z}}$ and $\delta_G = 2^{-m} > 0$ such that for all $y \in B_{\delta_G}(x) \subseteq A^{\mathbb{Z}}$, the iterates of x and y under G stay close; $d(G^n x, G^n y) < \varepsilon$ for all $n \geq 0$. Take $z \in A^{\mathbb{Z}^2}$ such that $H_j z = x$ for all $|j| \leq \max\{k, m\}$. That is, z is a point in the infinitely wide cylinder set

$$\begin{bmatrix} x \\ x \\ \vdots \\ x \\ x \end{bmatrix},$$

having the central $2k+1$ and $2m+1$ rows all equal to the one-dimensional sequence x. Now take $\delta = \min\{\delta_G, \varepsilon\}$.

Claim. For all $y \in B_\delta(z) \subseteq A^{\mathbb{Z}^2}$, $d(F^n y, F^n z) < \varepsilon$ for all $n \geq 0$.

Let $y \in A^{\mathbb{Z}^2}$ be such that $d(y, z) < \delta$. Then for $|j| \leq m,\ k$, we have $d(H_j y, H_j z) = d(H_j y, x) < \delta$. So for these j,

$$d(G^n \circ H_j y, G^n \circ H_j z) < \varepsilon \ \forall\ n \geq 0. \tag{8.1}$$

That is, for $|i| \leq k$, $(G^n \circ H_j y)_i = (G^n \circ H_j z)_i$ for all $n \geq 0$. Thus, for $|i|, |j| \leq k$, $(F^n y)_{i,j} = (F^n z)_{i,j}$ for all $n \geq 0$, or $d(F^n y, F^n z) < \varepsilon$ for all $n \geq 0$. Therefore, F is not sensitive. □

Another topological property of interest is that of transitivity, also referred to as topological transitivity. A dynamical system (Y, T) is *transitive* if there is a point $y \in Y$ with a dense forward orbit, $Y = \overline{\{T^n y : n \geq 0\}}$. As in the one-dimensional case [11], transitive CA's in higher dimensions must be sensitive if the subshift space is infinite.

Theorem 8.2. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton and suppose $X \subseteq A^{\mathbb{Z}^D}$ is an F-invariant subshift. If (X, F) is transitive, then it is either sensitive or consists of a single periodic orbit.*

Proof: Suppose that (X, F) is transitive but not sensitive. By a result of Eli Glasner and Benjamin Weiss (Lemma 1.2, [6]), (X, F) is uniformly rigid. So for $\varepsilon = 1$, there is an $n \geq 0$ so that for all $x \in X$, $d(F^n x, x) < 1$. That is, $(F^n x)_{\vec{0}} = x_{\vec{0}}$ for all x and this n. Now for any integer vector $\vec{\imath} \in \mathbb{Z}^D$ and all $x \in X$ we have

$$(F^n x)_{\vec{\imath}} = (\sigma_{\vec{\imath}} \circ F^n x)_{\vec{0}} = (F^n \circ \sigma_{\vec{\imath}} x)_{\vec{0}} = (\sigma_{\vec{\imath}} x)_{\vec{0}} = x_{\vec{\imath}}. \tag{8.2}$$

Therefore, $F^n = Id$, and so X must consist of a single periodic orbit. □

Corollary 8.3. *Every CA on an infinite subshift which is transitive is also sensitive.*

Proof: This is clear, since the only alternative to a transitive CA being sensitive is that the subshift must consist of a single periodic orbit. However, this would then make the subshift finite. □

The following result is true for general dynamical systems, and appears regularly in the literature without proof. However, we include it for the purpose of stating Corollary 8.5, which tells us more about the structure of transitive cellular automata.

Theorem 8.4. *Let (X, T) be a dynamical system with no isolated points. If T is transitive, it is also surjective.*

Corollary 8.5. *Every CA on an infinite subshift space which is transitive is surjective.*

9. Interaction of topological properties

In [11], Kůrka gives a diagram showing the interaction of topological properties for one-dimensional CA's. We conclude with a similar diagram (see Figure 16) illustrating the interaction of the topological properties for higher dimensional CA's. The letters in boxes correspond to examples given in this paper, and I is the identity map.

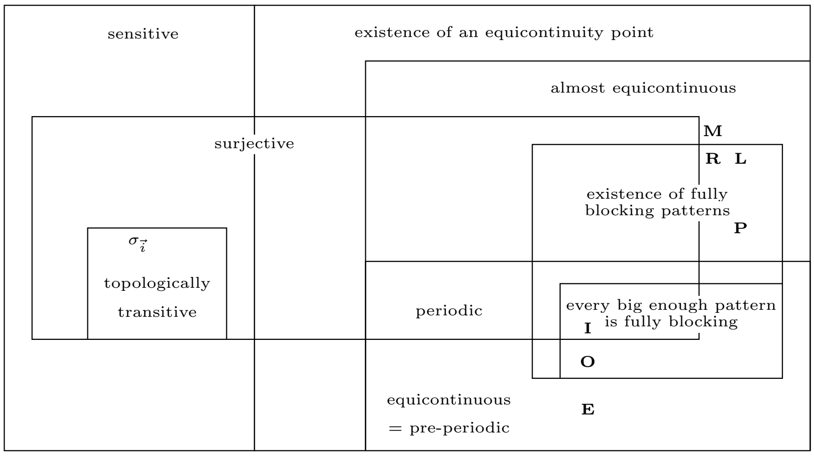

FIGURE 16. Interaction of topological properties

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