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# ATSUJI SPACES: EQUIVALENT CONDITIONS

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*Dedicated to Professor S. A. Naimpally*

ABSTRACT. A metric space $(X, d)$ is called an Atsuji space if every real-valued continuous function on $(X, d)$ is uniformly continuous. In this paper, we study twenty-five equivalent conditions for a metric space to be an Atsuji space. These conditions have been collected from the works of several mathematicians spanning nearly four decades.

## 1. Introduction

The concept of continuity is an old one, but this concept is central to the study of analysis. On the other hand, the concept of uniform continuity was first introduced for real-valued functions on Euclidean spaces by Eduard Heine in 1870. The elementary courses in analysis and topology normally include a proof of the result that every continuous function from a compact metric space to an arbitrary metric space is uniformly continuous. But the compactness is clearly not a necessary condition since any continuous function from a discrete metric space $(X, d)$ to an arbitrary metric space is uniformly continuous where $d$ is the discrete metric : $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0,\ x, y \in X$. The goal of this paper is to present, in a systematic and comprehensive way, conditions, lying

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scattered across the literature, which are necessary as well as sufficient for a metric space $(X, d)$ so that every real-valued continuous function on $(X, d)$ becomes uniformly continuous. Probably Juniti Nagata was the first one to study such metric spaces in 1950 in [19]; while in 1951 in [18], A. A. Monteiro and M. M. Peixoto studied four equivalent characterizations of such metric spaces. In particular, they proved that every real-valued continuous function on a metric space $(X, d)$ is uniformly continuous if and only if every open cover of $X$ has a Lebesgue number. Because of this characterization, such metric spaces have been called *Lebesgue spaces* in [21] and [23]. In 1958, several new equivalent characterizations of such metric spaces were studied by Masahiko Atsuji in [1]. Gerald Beer, in [3], may have been the first to call such metric spaces *Atsuji spaces*. He continued with this term in [4] also. But in [5] and [6], he called these metric spaces *UC spaces* as several mathematicians did so while studying such metric spaces; (see, for example, [25]). In [17], S. G. Mrówka has shown that every real-valued continuous function on a metric space $(X, d)$ is uniformly continuous if and only if for any pair of disjoint nonempty closed sets $A$ and $B$ in $X$, $d(A, B) > 0$. He has called such a metric *normal*. In this paper, we will call a metric space, equipped with a normal metric, an Atsuji space; that is, a metric space on which every real-valued continuous function is uniformly continuous will be called by us an Atsuji space. So the purpose of this paper is to present complete characterizations of Atsuji spaces, more precisely, to present equivalent conditions for a metric space to be an Atsuji space. From the works of several mathematicians spanning nearly four decades, we have collected twenty-five such equivalent conditions.

In order to prove such a large number of equivalent conditions for a metric space to be an Atsuji space, we need to use many results, though most of them are well-known. So first we would like to state these results without proof, except one, the proof of which is essentially due to V. A. Efremovič. The second section, under the heading Basic Tools, includes these results together with some relevant definitions.

In the last section of this paper, we give the complete proofs of twenty-five equivalent conditions for Atsuji spaces. But in order to have easier cycles of the proofs, we split these equivalent conditions into four theorems. This split has been decided as much as possible according to the nature of these conditions. For example, Theorem 3.16 mainly emphasizes the sequential characterizations of an Atsuji space. Since the split has been decided by us, quite often our proofs have been different from the ones available in the literature.

The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and of natural numbers, respectively. Unless mentioned otherwise, $\mathbb{R}$ and its subsets carry the usual distance metric. If $(X, d)$ is a metric space, $x \in X$ and $\delta > 0$, then $B(x, \delta)$ denotes the open ball in $(X, d)$, centered at $x$ with radius $\delta$. Also $X'$ denotes the set of all accumulation points in $(X, d)$.

## 2. Basic tools

A map $f : (X, d) \to (Y, \rho)$ between two metric spaces $X$ and $Y$ is continuous if and only if $f$ sends convergent sequences in $X$ to convergent sequences in $Y$. As this characterization is very useful in dealing with continuity, a sequential characterization of uniform continuity in terms of asymptotic sequences is also very useful while dealing with uniform continuity. In the first result of this paper, we give this characterization. But in order to give it, we first need the definition of asymptotic sequences and the Efremovič Lemma.

**Definition 2.1.** Two sequences $(x_n)$ and $(y_n)$ in a metric space $(X, d)$ are said to be asymptotic, written $(x_n) \asymp (y_n)$, if they satisfy the following condition: for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon \Rightarrow d(x_n, y_n) < \epsilon$.

**Lemma 2.2** (Efremovic)**.** *Let $(X, d)$ be a metric space and $\epsilon > 0$. Suppose that $((x_n, y_n))$ is a sequence in $X \times X$ satisfying $d(x_n, y_n) \geq \epsilon$ for all $n \in \mathbb{N}$. Then there exists a subsequence $((x_{n_k}, y_{n_k}))$ such that $d(x_{n_k}, y_{n_l}) \geq \frac{\epsilon}{4}$ for all $k, l \in \mathbb{N}$.*

*Proof:* See [6, Lemma 3.3.1, p. 92]. □

**Remark 2.3.** The origin of Lemma 2.2 lies in the study of proximity spaces, done by Efremovic in [8]; see [9, 8.5.19(a), pp. 467-468]. But the Efremovic Lemma holds more generally for uniform spaces. For that general result as well as its proof, one can see [20, Lemma 12.17, p. 77].

**Theorem 2.4.** *Let $f : (X, d) \longrightarrow (Y, \rho)$ be a function between two metric spaces $X$ and $Y$. Then the following statements are equivalent:*

($i$) *$f$ is uniformly continuous.*
($ii$) *$f$ preserves asymptotic sequences; that is, if $(x_n)$ and $(z_n)$ are asymptotic sequences in $X$, then $(f(x_n))$ and $(f(z_n))$ are asymptotic sequences in $Y$.*
($iii$) *For every pair of nonempty subsets $A$ and $B$ of $X$, $d(A, B) = 0 \Longrightarrow \rho(f(A), f(B)) = 0$.*

*Proof:* ($i$) $\Longrightarrow$ ($ii$): Assume ($i$) and let $(x_n)$ and $(z_n)$ be asymptotic sequences in $X$.

Let $\epsilon > 0$. Let $\delta$ be a positive number such that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. Since $(x_n) \asymp (z_n)$, there exists $n_\delta \in \mathbb{N}$ such that $d(x_n, z_n) < \delta$ for all $n > n_\delta$. Hence, $(f(x_n))$ and $(f(z_n))$ are asymptotic sequences in $Y$.

($ii$) $\Longrightarrow$ ($iii$): Since $d(A, B) = 0,$ for all $n \in \mathbb{N}$ there exists $x_n \in A$ and $z_n \in B$ such that $d(x_n, z_n) < \frac{1}{n}$. But this means $(x_n) \asymp (z_n)$. Hence by ($ii$), $(f(x_n)) \asymp (f(z_n))$ and consequently, $\rho(f(A), f(B)) = 0$.

($iii$) $\Longrightarrow$ ($i$): If possible, suppose that $f$ is not uniformly continuous. So there exists $\epsilon > 0$ such that for all $\delta > 0,$ there exists $x_\delta$ and $z_\delta \in X$ such that $d(x_\delta, z_\delta) < \delta$, and $\rho(f(x_\delta), f(z_\delta)) \geq \epsilon$.

Take $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then we get two sequences, $(x_n)$ and $(z_n)$, which are asymptotic because $d(x_n, z_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. But $\rho(f(x_n), f(z_n)) \geq \epsilon$ for all $n \in \mathbb{N}$. By Lemma 2.2, there exists a subsequence $((f(x_{n_k}), f(z_{n_k})))$ of the sequence $((f(x_n), f(z_n)))$ in $Y \times Y$ such that $\rho(f(x_{n_k}), f(z_{n_l})) \geq \frac{\epsilon}{4}$ for all $k,\ l \in \mathbb{N}$. Let $A = \{x_{n_k} : k \in \mathbb{N}\}$ and $B = \{z_{n_k} : k \in \mathbb{N}\}$. Then $d(A, B) = 0$, but $\rho(f(A), f(B)) \geq \epsilon/4 > 0$. We arrive at a contradiction. Hence, $f$ must be uniformly continuous. $\square$

Next we state some results related to compact subsets of a metric space.

**Theorem 2.5.** *For a subset $A$ of the metric space $(X, d)$, the following statements are equivalent.*

($i$) *$A$ is a compact set.*
($ii$) *Every infinite subset of $A$ has an accumulation point in $A$.*
($iii$) *Every sequence in $A$ has a subsequence which converges to a point of $A$.*

For the next result we need the following definition.

**Definition 2.6.** Let $(X,d)$ be a metric space, $A$ be a subset of $X$, and $\delta$ be a positive number. Further, let $\mathcal{U}$ be an open cover of $A$. Suppose that for each $x \in A$, we have $B(x,\delta) \subseteq U$ for some $U \in \mathcal{U}$; that is, the cover $\{B(x,\delta)\}_{x\in A}$ is a refinement of $\mathcal{U}$. Then the number $\delta$ is called a *Lebesgue number* of $A$ for the open cover $\mathcal{U}$.

If $A$ is nonempty, then the $\delta$-ball $B(A,\delta)$ about $A$ is defined as $B(A,\delta) = \{x \in X : d(x,A) < \delta\} = \bigcup_{x\in A} B(x,\delta)$. If $A$ is empty, we define $B(A,\delta)$ to be empty as well.

**Theorem 2.7.** *Let $(X,d)$ be a metric space and $A$ and $B$ be two nonempty closed subsets of $X$ such that $A$ is compact in $X$. Then the following assertions hold.*

(*a*) *Every open cover of $A$ has a Lebesgue number.*
(*b*) *If $A \cap B = \emptyset$, then $d(A,B) > 0$.*
(*c*) *For any open set $U$ containing $A$, there exists a $\delta > 0$ such that $B(A,\delta) \subseteq U$.*

Next we state some results related to uniform continuity and uniform spaces.

**Theorem 2.8.** *Let $f : (X,d) \longrightarrow (Y,\rho)$ be a continuous function. If $(X,d)$ is compact, then $f$ is uniformly continuous.*

For the relevant definitions and results related to topological and uniform spaces, which are needed in this paper, one may see [26] or [9].

If $X$ is a uniformizable topological space, there is a finest uniformity on $X$ compatible with the topology on $X$; (see [26, Theorem 36.12, p. 248]). This uniformity is called the *fine uniformity* on X, and when X is provided with this uniformity, it is called a *fine space*.

Recall that a map from a uniform space $X$ to a uniform space $Y$ is said to be *uniformly continuous* if, for each entourage $V$ of $Y$, there is an entourage $U$ of $X$ such that the relation $(x,x') \in U$ implies $(f(x), f(x')) \in V$.

It can be easily verified that a map $f : (X,d) \longrightarrow (Y,\rho)$ between two metric spaces is uniformly continuous if and only if the map $f : (X,\mathcal{U}_d) \longrightarrow (Y,\mathcal{U}_\rho)$ is uniformly continuous where $\mathcal{U}_d$ and $\mathcal{U}_\rho$ are uniformities generated by $d$ and $\rho$, respectively. Also it is easy to verify that if $f : X \longrightarrow Y$ is a uniformly continuous map between two uniform spaces $X$ and $Y$, then $f$, as a map between the corresponding topological spaces $X$ and $Y$, is continuous. The converse need not be true; that is, a continuous map between two uniformizable topological spaces need not be uniformly continuous. But if the domain is a fine space, the converse is also true. The precise statement follows.

**Theorem 2.9.** *Every continuous function on a fine space to some uniform space is uniformly continuous.*

*Proof:* See [26, Theorem 36.18, p. 249]. □

## 3. Equivalent conditions for Atsuji spaces

Finally, we deal with the goal of this paper. The goal is to present complete characterizations of Atsuji spaces, more precisely to present equivalent conditions for a metric space to be an Atsuji space.

**Definition 3.1.** A metric space $(X, d)$ is called an Atsuji space or a UC space if every real-valued continuous function on $(X, d)$ is uniformly continuous.

Before proceeding to the equivalent conditions for an Atsuji space, we look at a couple of necessary conditions for a metric space to be an Atsuji space.

**Theorem 3.2.** *Let a metric space $(X, d)$ be an Atsuji space. Then $(X, d)$ is complete and $X'$, the set of all accumulation points in $X$, is compact in $(X, d)$.*

*Proof:* If $(X, d)$ is not complete, then there exists a Cauchy sequence $(x_n)$ of distinct points in $X$ such that it does not converge in $X$. Consider the set $A = \{x_n : n \in \mathbb{N}\}$. Then $A$ is closed and discrete in $X$.

Now define $f : A \longrightarrow \mathbb{R}$ as follows: $f(x_n) = n$ for all $n \in \mathbb{N}$.

Since $A$ is discrete, $f$ is continuous. By Tietze's extension theorem, $f$ can be extended to a continuous function on $X$. We denote this extension also by $f$. But $f$ is not uniformly continuous, since $(x_n)$ is Cauchy in $(X, d)$, but $(f(x_n)) = (n)$ is not Cauchy in $\mathbb{R}$.

To prove that $X'$ is compact in $(X, d)$, we show that every sequence in $X'$ has a convergent subsequence. Note that the limit of such a convergent subsequence must belong to $X'$, since $X'$ is closed in $X$. If possible, suppose that there exists a sequence $(x_n)$ in $X'$ which has no convergent subsequence. Of course, we can assume that $x_n \neq x_m$ for $n \neq m$. Let $Z_n = \{x_m : m \neq n\}$. Then $c_n = d(x_n, Z_n) > 0$ for every $n$ since $Z_n$ is closed in $X$. Again since each $x_n$ is an accumulation point of $X$, for each $n$ we can find $y_n$ in $X$ such that $0 < d(x_n, y_n) < \min\{\frac{1}{n}, c_n\}$.

Note that if $m \neq n$, then $d(x_m, y_n) \geq d(x_m, x_n) - d(x_n, y_n) \geq d(x_n, Z_n) - d(x_n, y_n) = c_n - d(x_n, y_n) > 0$. Hence, $x_m \neq y_n$ for $m \neq n$ and consequently, $\{x_n : n \in \mathbb{N}\} \cap \{y_n : n \in \mathbb{N}\} = \emptyset$.

By the normality of $X$, there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x_n) = 0$ for all $n$ and $f(y_n) = 1$ for all $n$. Clearly, $f$ is not uniformly continuous since $\lim_{n\to\infty} d(x_n, y_n) = 0$, but $|f(x_n) - f(y_n)| = 1$ for all $n$. Hence, every sequence in $X'$ must have a convergent subsequence and consequently $X'$ is compact in $(X, d)$. □

Now we give a counter example to show that even the completeness of $X$ and the compactness of $X'$, taken together, are not sufficient to ensure that $(X, d)$ is an Atsuji space.

**Example 3.3** (due to A. B. Raha)**.** Let $X = [0, \frac{1}{2}] \cup \mathbb{N} \cup \{n + \frac{1}{2n} : n \in \mathbb{N}\}$ and consider the usual distance metric on $X$. Clearly, $X$ is complete and $X' = [0, \frac{1}{2}]$ is compact in $X$. Define $f : X \longrightarrow \mathbb{R}$ as follows:

$f(x) = 0$ for all $x \in [0, \frac{1}{2}]$, $f(n) = 1$ for all $n \in \mathbb{N}$, and $f(n + \frac{1}{2n}) = 2$ for all $n \in \mathbb{N}$. Clearly, $f$ is continuous. Since $f(n) = 1$ for all $n \in \mathbb{N}$ and $f(n + \frac{1}{2n}) = 2$ for all $n \in \mathbb{N}$, it is easy to see that $f$ is not uniformly continuous.

**Remark 3.4.** There are several proofs in the literature for Theorem 3.2. The proof for the compactness of $X'$ given here has been taken from [17].

In the next result, we give ten equivalent conditions for an Atsuji space. But in order to do this, we need the following definition.

**Definition 3.5.** Let $(X, d)$ be a metric space and $x \in X$. We define $I(x)$ as follows: $I(x) = d(x, X \setminus \{x\})$; that is, $I(x) = \inf\{d(x, y) : y \in X \setminus \{x\}\}$.

**Remark 3.6.** Note that $I(x)$ denotes the degree of isolation of the point $x$ in $X$ and $I(x) = 0$ if and only if $x$ is a non-isolated point; that is, $x$ is an accumulation point. Also note that for an isolated point $x$ in $X$, $I(x) = \sup\{r > 0 : \ B(x, r) = \{x\}\}$.

**Theorem 3.7.** *For a metric space $(X, d)$, the following conditions are equivalent.*

- $(a)$ *Let $(x_n)$ be a sequence in $X$ without a cluster point and $B = \{n \in \mathbb{N} : x_n$ is not an isolated point in $X\}$. Then $B$ is finite and $\inf\{I(x_n) : n \notin B\} > 0$.*
- $(b)$ *Given any uniform space $S$, any continuous function $f : X \longrightarrow S$ is uniformly continuous. (Here obviously on $X$, we consider the uniformity generated by $d$).*
- $(c)$ *Given any metric space $(M, \rho)$, any continuous function $f : (X, d) \longrightarrow (M, \rho)$ is uniformly continuous.*
- $(d)$ *$(X, d)$ is an Atsuji space.*
- $(e)$ *Let $f : X \longrightarrow \mathbb{R}$ be any real valued continuous function. Then there exists a positive integer $n$ such that every point of the set $A = \{x : |f(x)| \geq n\}$ is isolated in $X$ and $\inf\{I(x) : x \in A\} > 0$.*
- $(f)$ *Let $(A_n)$ be a sequence of subsets of $X$ such that $\bigcap_{n=1}^{\infty} \overline{A_n} = \emptyset$. Then there exists $r > 0$ such that $\bigcap_{n=1}^{\infty} B(A_n, r) = \emptyset$.*
- $(g)$ *Let $A_1$ and $A_2$ be two subsets of $X$ such that $\overline{A_1} \cap \overline{A_2} = \emptyset$. Then there exists $r > 0$ such that $B(A_1, r) \cap B(A_2, r) = \emptyset$.*
- $(h)$ *For every closed nonempty subset $A$ of $X$ and any open subset $G$ of $X$ containing $A$, there exists $r > 0$ such that $A \subseteq B(A, r) \subseteq G$.*
- $(i)$ *If $A_1$ and $A_2$ are two disjoint nonempty closed sets in $X$, then $d(A_1, A_2) > 0$.*
- $(j)$ *Let $A$ be a subset of $X$ without an accumulation point in $X$. Then $A \cap X'$ is finite and $\inf\{I(x) : x \in A \setminus X'\} > 0$.*
- $(k)$ *The metric $d$ generates the fine uniformity on $X$.*

*Proof:* $(a) \Longrightarrow (b)$: Let $S$ be a uniform space. If possible, suppose that there exists some continuous function from $X$ to $S$ which is not uniformly continuous. Hence, there is an entourage $V$ of $S$ such that

(3.1) for all $n \in \mathbb{N}$, there exists $x_n, y_n \in X$ with $d(x_n, y_n) < 1/n$, but $(f(x_n), f(y_n)) \notin V$.

Now if the sequence $(x_n)$ has a cluster point, then there exists a strictly increasing sequence $(n_k)$ in $\mathbb{N}$ such that the subsequences $(x_{n_k})$ and $(y_{n_k})$ of $(x_n)$ and $(y_n)$, respectively, converge to some $x$ in $X$. Then by continuity of $f$, for an entourage $W$ of $S$ satisfying $W = W^{-1}$ and $W \circ W \subseteq V$, there exists $n_0 \in \mathbb{N}$ such that $(f(x_{n_k}), f(x)) \in W$ and $(f(y_{n_k}), f(x)) \in W$ for all $k \geq n_0$. So $(f(x_{n_k}), f(y_{n_k})) \in V$ for all $k \geq n_0$, which is a contradiction to (3.1). Thus, $(x_n)$ has no cluster point. By $(a)$, there exists some $m \in \mathbb{N}$ such that for every $n \geq m$, $x_n$ is isolated in $X$, and $\inf\{I(x_n) : n \geq m\} = r > 0$. But this contradicts the inequality in (3.1): $d(x_n, y_n) < \frac{1}{n}$ for all $n > 1/r$. Hence, $(b)$ holds.

$(b) \Longrightarrow (c)$: This holds because a metric space $(M, \rho)$ can be considered as a uniform space with respect to the uniformity generated by $\rho$.

$(c) \Longrightarrow (d)$: This is obvious.

$(d) \Longrightarrow (e)$: Suppose that $(X, d)$ is an Atsuji space and there is a real valued continuous function $f$ on $X$ such that for every $n$ in $\mathbb{N}$, there exists an accumulation point $x_n$ in $X$ with $|f(x_n)| \geq n$.

Since $(X, d)$ is an Atsuji space, by Theorem 3.2, the set $X'$ of all accumulation points in $X$ is compact and so by Theorem 2.5, the sequence $(x_n)$ has a convergent subsequence $(x_{n_k})$. But $(f(x_{n_k}))$ is not bounded and hence not convergent, which is contrary to the continuity of $f$.

Hence, there exists $n_0 \in \mathbb{N}$ such that every point of the set $A = \{x \in X : |f(x)| \geq n_0\}$ is isolated.

Now we just need to show that $\inf\{I(x) : x \in A\} > 0$. Note that $I(x) > 0$ for all $x \in A$ and we can assume that $A$ is infinite. If possible, suppose that $\inf\{I(x) : x \in A\} = 0$. So there exists a sequence $(x_n)$ in $A$ with $\lim\limits_{n\to\infty} I_n = 0$, where $I_n = I(x_n)$. Note that $I(x_n) > 0$ for all $n \in \mathbb{N}$. If needed, by passing to a subsequence we may assume that $(x_n)$ has distinct terms and for each $n$, $I_n < \frac{1}{2n}$. If $(x_n)$ has a convergent subsequence $(x_{n_k})$ converging to some $x$, then since $(x_n)$ has distinct terms, $x \in X'$ and by continuity of

$f$, $|f(x)| = |\lim_{k\to\infty} f(x_{n_k})| = \lim_{k\to\infty} |f(x_{n_k})| \geq n_0$. This means $x$ is an isolated point. But already we have observed that $x \in X'$. Hence, $(x_n)$ has no convergent subsequence and consequently, the set $B = \{x_n : n \in \mathbb{N}\}$ is an infinite closed set in $X$. By Theorem 3.2, $(X, d)$ is complete and hence $B$ is complete. But $B$, being an infinite set of isolated points, is not compact. Hence, $B$ is not totally bounded and consequently, there exist an $\epsilon > 0$ and a subsequence $(x_{n_k})$ of $(x_n)$ of distinct points with $d(x_{n_k}, x_{n_l}) \geq \epsilon\ \forall k, l \in \mathbb{N}$, $k \neq l$.

Now since $I_{n_k} \to 0$, there exists $m$ in $\mathbb{N}$ such that $4I_{n_k} < \epsilon$ for all $k \geq m$.

Also for all $k \in \mathbb{N}$, there exists $y_{n_k}$ in $X$ such that $y_{n_k} \neq x_{n_k}$ and $d(x_{n_k}, y_{n_k}) < 2I_{n_k} < 1/k$. Now for $k, l \geq m$, $k \neq l$, we have $d(x_{n_k}, y_{n_l}) \geq d(x_{n_k}, x_{n_l}) - d(x_{n_l}, y_{n_l}) \geq \epsilon - 2I_{n_l} > 4I_{n_l} - 2I_{n_l} > 0$. This implies $x_{n_k} \neq y_{n_l}$ for all $k, l \geq m$. The sets $\{x_{n_k} : k \geq m\}$ and $\{y_{n_k} : k \geq m\}$ are disjoint and closed. So by normality of $X$, there exists a continuous function $g : X \longrightarrow \mathbb{R}$ such that $g(x_{n_k}) = 0$ and $g(y_{n_k}) = 1$ for every $k \geq m$. By Theorem 2.4, $g$ is not uniformly continuous since $\lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = 0$, but $|g(x_{n_k}) - g(y_{n_k})| = 1$ for all $k \geq m$. So $\inf\{I(x) : x \in A\}$ must be positive.

$(e) \Longrightarrow (f)$: Let, if possible, there exist a family of subsets $\{A_n : n \in \mathbb{N}\}$ of $X$ with $\bigcap_{n=1}^{\infty} \overline{A_n} = \emptyset$ but $\bigcap_{n=1}^{\infty} B(A_n, 1/m) \neq \emptyset$ for every $m$ in $\mathbb{N}$. Then for each $m$, let $x_m \in \bigcap_{n=1}^{\infty} B(A_n, 1/m)$. So $d(x_m, A_n) < \frac{1}{m}$ for all $n \in \mathbb{N}$ and hence for each $n \in \mathbb{N}$, there exists $z_n \in A_n$ such that $d(x_m, z_n) < \frac{1}{m}$. If $z_n = x_m$ for all $n \in \mathbb{N}$, then $x_m \in \bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} \overline{A_n}$. But we have assumed $\bigcap_{n=1}^{\infty} \overline{A_n} = \emptyset$. Hence, there exists $z_n$ such that $z_n \neq x_m$. We call this $z_n$ by $y_m$. So there exists $y_m \in X$ such that $0 < d(x_m, y_m) < \frac{1}{m}$.

Now we claim that the sequence $(x_n)$ has no cluster point in $X$. If possible, suppose that $(x_n)$ has a cluster point $x$ in $X$. Let $\epsilon > 0$ be arbitrarily chosen. Since $x$ is a cluster point of $(x_n)$, there exists $k \in \mathbb{N}$ such that $k > \frac{2}{\epsilon}$ and $d(x, x_k) < \frac{\epsilon}{2}$. Then for each $n \in \mathbb{N}$, $d(x, \overline{A_n}) \leq d(x, x_k) + d(x_k, \overline{A_n}) \leq d(x, x_k) + \frac{1}{k} < \epsilon$. Hence, $d(x, \overline{A_n}) = 0$ and consequently, $x \in \overline{A_n}$ for all $n \in \mathbb{N}$; that is,

$x \in \bigcap_{n=1}^{\infty} \overline{A_n}$. But this contradicts our assumption that $\bigcap_{n=1}^{\infty} \overline{A_n} = \emptyset$. Hence, the sequence $(x_n)$ has no cluster point in $X$. Now we can also assume that $(x_n)$ consists of distinct points.

The set $D = \{x_n : n \in \mathbb{N}\}$ is closed and discrete in $X$. Now as in the proof of Theorem 3.2, we can construct a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x_n) = n$, for all $n \in \mathbb{N}$. Let $n_0$ be the positive integer given by the hypothesis in $(e)$ so that every point of the set $A = \{x \in X : |f(x)| \geq n_0\}$ is isolated. Note that $x_n \in A$ for all $n \geq n_0$ and consequently, $\inf\{I(x) : x \in A\} \leq \inf\{I(x_n) : n \geq n_0\} = 0$. But by $(e)$, $\inf\{I(x) : x \in A\} > 0$. We arrive at a contradiction. Hence, there exists $m \in \mathbb{N}$ such that $\bigcap_{n=1}^{\infty} B(A_n, \frac{1}{m}) = \emptyset$.

$(f) \implies (g)$: This is immediate.

$(g) \implies (h)$: Let $A$ be a nonempty closed subset of $X$ and $G$ be an open subset of $X$ containing $A$. Take $A_1 = X \setminus G$. Clearly, $A \cap A_1 = \emptyset$. By $(g)$, there exists $r > 0$ such that $B(A, r) \cap A_1 \subseteq B(A, r) \cap B(A_1, r) = \emptyset$; that is, $B(A, r) \subseteq G$.

$(h) \implies (i)$: Suppose that $A_1, A_2$ are two disjoint nonempty closed sets in $X$. We shall show that $d(A_1, A_2) > 0$.

Clearly $A_1 \subseteq X \setminus A_2$. By $(h)$, there exists some $r > 0$ such that $B(A_1, r) \cap A_2 = \emptyset$. So for all $x \in A_1$, $y \in A_2$, $d(x, y) \geq r$.

Hence, $d(A_1, A_2) = \inf\{d(x, y) : x \in A_1,\ y \in A_2\} \geq r > 0$.

$(i) \implies (j)$: Let $A$ be a subset of $X$ without an accumulation point in $X$. Suppose, if possible, that $A \cap X'$ is infinite. So there exists a sequence $(x_n)$ of distinct points in $A \cap X'$. Since each $x_n \in X'$, there exists $y_n$ in $X$ such that $y_n \neq x_n$ and $d(x_n, y_n) < 1/n$. Since $A$ does not have an accumulation point in $X$, the sequence $(x_n)$ has no cluster point. Consequently, the sequence $(y_n)$ also has no cluster point.

We can also assume that the points of the sequence $(y_n)$ are all distinct. Now we will construct two sequences, $(x'_{n_k})$ and $(y'_{n_k})$, such that $x'_{n_k} \neq y'_{n_l}$ for all $k, l \in \mathbb{N}$ and $d(x'_{n_k}, y'_{n_k}) < 1/k$.

We proceed by induction. For $n = 1$, let $x'_{n_1} = x_1$ and $y'_{n_1} = y_1$. Let $S_1 = \{x'_{n_1}\}$ and $T_1 = \{y'_{n_1}\}$. Then, $S_1 \cap T_1 = \emptyset$. Suppose that we have chosen $x'_{n_1}, \ldots, x'_{n_k}$ and $y'_{n_1}, \ldots, y'_{n_k}$ and $S_k =$

$\{x'_{n_1}, \ldots, x'_{n_k}\}$, $T_k = \{y'_{n_1}, \ldots, y'_{n_k}\}$ are such that $S_k \cap T_k = \emptyset$. Then choose $n_{k+1} = \min\{m > n_k : x_m \notin S_k \cup T_k \text{ or } y_m \notin S_k \cup T_k\}$.

Suppose that $n_{k+1}$ does not exist; that is, for all $n > n_k$, $x_n, y_n \in S_k \cup T_k$. This implies that $\{d(x_n, y_n) : n > n_k\}$ is finite and hence, $\lim d(x_n, y_n) > 0$, which contradicts our choice of $(x_n)$ and $(y_n)$.

Now let $x'_{n_{k+1}} = y_{n_{k+1}}$ and $y'_{n_{k+1}} = x_{n_{k+1}}$, if $x_{n_{k+1}} \in T_k$ or $y_{n_{k+1}} \in S_k$; otherwise, let $x'_{n_{k+1}} = x_{n_{k+1}}$ and $y'_{n_{k+1}} = y_{n_{k+1}}$.

Define the sets $B = \{x_{n'_k} : k \in \mathbb{N}\}$ and $C = \{y_{n'_k} : k \in \mathbb{N}\}$. Since the set $\{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ has no accumulation point in $X$, neither $B$ nor $C$ has any accumulation point in $X$. Therefore, $B$ and $C$ are closed in $X$. Also by construction of $(x_{n'_k})$ and $(y_{n'_k})$, $B \cap C = \emptyset$. So by $(i)$, $d(B, C) > 0$. But since $d(x_{n'_k}, y_{n'_k}) = d(x_{n_k}, y_{n_k}) < \frac{1}{k}$, $d(B, C) = 0$. We arrive at a contradiction. Hence, $A \cap X'$ must be finite.

If $A \setminus X'$ is finite, then $\inf\{I(x) : x \in A \setminus X'\} = \min\{I(x) : x \in A \setminus X'\} > 0$. So we can assume that $A \setminus X'$ is infinite. Now if $\inf\{I(x) : x \in A \setminus X'\} = 0$, then there exists a sequence $(x_n)$ of distinct terms in $A$ and a sequence $(y_n)$ in $X$, $y_n \neq x_n$ with $d(x_n, y_n) < 1/n$. Again as argued earlier, neither of the sequences $(x_n)$ and $(y_n)$ has any cluster point. But then, as seen above, we can have two disjoint closed sets with distance zero between them, contradicting $(i)$. Hence, $\inf\{I(x) : x \in A \setminus X'\} > 0$.

$(j) \Longrightarrow (a)$: This is immediate.

$(b) \Longrightarrow (k)$: Taking $S$ to be the space $X$ with its finest uniformity and considering the identity map from $X$ to $X$, we can conclude that the uniformity on $X$ induced by the metric $d$ is finest.

$(k) \Longrightarrow (b)$: This follows from Theorem 2.9. □

**Remark 3.8.** With the exception of conditions $(c)$, $(h)$, $(i)$, and $(k)$, Theorem 3.7 is due to Atsuji [1, Theorem 1]. According to Atsuji, condition $(g)$ appeared as Lemma 1 in [19]. Conditions $(c), (i)$, and $(k)$ have been taken from [25], and condition $(h)$ has been taken from [27].

In the next result, we give seven more equivalent conditions for an Atsuji space. This result, in particular, emphasizes the role of $X'$ in determining when a metric space $(X, d)$ becomes an Atsuji space. But before stating the result, we need the following definition.

**Definition 3.9.** A subset $A$ of a metric space (X,d) is called *discrete* if for all $x \in A$ there exists $\delta > 0$ such that $d(x,y) \geq \delta$ for all $y \in A - \{x\}$; that is, $A \cap B(x,\delta) = \{x\}$. It is called *uniformly discrete* if $\delta$ does not depend on $x$; that is, there exists $\delta > 0$ such that $d(x,y) \geq \delta$ for all $x, y \in A, x \neq y$. If $X$ is itself uniformly discrete, then $(X,d)$ is called a uniformly discrete space.

**Theorem 3.10.** *For a metric space $(X,d)$, the following statements are equivalent.*

- $(a)$ *$(X,d)$ is an Atsuji space.*
- $(b)$ *Every bounded real-valued continuous function on $X$ is uniformly continuous.*
- $(c)$ *Every closed discrete subset of $X$ is uniformly discrete in $X$.*
- $(d)$ *The set $X'$ of all accumulation points in $X$ is compact and for each $\epsilon > 0$, the set $X \setminus B(X',\epsilon)$ is uniformly discrete.*
- $(e)$ *The set $X'$ of all accumulation points in $X$ is compact and for any sequence $(x_n)$ in $X \setminus X'$ without a cluster point, $\inf_n I(x_n) > 0$.*
- $(f)$ *The set $X'$ of all accumulation points in $X$ is compact and for all $\delta_1 > 0$, there exists $\delta_2 > 0$ such that for all $x \in X$ with $d(x,X') \geq \delta_1$ we have $I(x) > \delta_2$.*
- $(g)$ *Every open cover of $X$ has a Lebesgue number.*
- $(h)$ *Every open cover of $X$ by two sets has a Lebesgue number.*

*Proof:* $(a) \Longrightarrow (b)$: This is immediate.

$(b) \Longrightarrow (c)$: Suppose on the contrary that there exists a closed discrete subset $T$ of $X$ which is not uniformly discrete. Obviously, $T$ is infinite. For each $n$, we can find $x_n$ and $y_n$ in $T$, $x_n \neq y_n$, such that $d(x_n,y_n) < \frac{1}{n}$. Moreover, we can assume that $x_n$ and $y_n$ are distinct from $2n-2$ preceding points $x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}$. Suppose this is not possible. Then for all $m \geq n$ and for all $x,y \in T$ satisfying $0 < d(x,y) < \frac{1}{m}$, we have either $x \in \{x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}\}$ or $y \in \{x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}\}$. Since $\{x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}\}$ is a finite set, we can find some $z \in \{x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}\}$ and a strictly increasing sequence $(n_k)$ in $\mathbb{N}$ such that for all $k \in \mathbb{N}$, there exists some $z_k \in T$ such that $0 < d(z_k,z) < \frac{1}{n_k}$. This implies that $z$ is an accumulation point of $T$. But since $T$ is discrete and $z \in T$, $z$ cannot be an accumulation point of $T$. We arrive at a contradiction. Hence, $T$ must be uniformly discrete.

The sets $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$ are disjoint. Also, $A$ and $B$ are closed in $X$ as they are subsets of a closed and discrete set $T$. Then by the normality of $X$, there exists a bounded continuous function $f : X \longrightarrow [0,1]$ with $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Since $d(x_n, y_n) \to 0$ and $|f(x_n) - f(y_n)| = 1$, by Theorem 2.4, $f$ is not uniformly continuous. But this contradicts $(b)$.

$(c) \Longrightarrow (d)$: Assume that $(c)$ holds.

Let, if possible, $X'$ be not compact. Then by Theorem 2.5, there exists a sequence $(x_n)$ in $X'$ having no cluster point. Since each $x_n$ is an accumulation point in $X$, there exists $y_n$ in $X$, $y_n \neq x_n$ with $d(x_n, y_n) < 1/n$. Since $(x_n)$ has no cluster point in $X$ and $(x_n) \asymp (y_n)$, $(y_n)$ has no cluster point in $X$. Hence, the set $\{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ has no accumulation point in $X$ and thus is a closed and discrete subset of $X$ which is not uniformly discrete. This is a contradiction to $(c)$. Hence, $X'$ is compact.

Now for every $\epsilon > 0$, the set $X \setminus B(X', \epsilon)$ is clearly closed and discrete. Therefore by $(c)$, $X \setminus B(X', \epsilon)$ is uniformly discrete.

$(d) \Longrightarrow (e)$: We just need to show that if $(x_n)$ is a sequence in $X \setminus X'$ without a cluster point, then $\inf\limits_{n\in\mathbb{N}} I(x_n)$ is positive.

Let, if possible, $\inf\limits_{n\in\mathbb{N}} I(x_n) = 0$. The set $A = \{x_n : n \in \mathbb{N}\}$ has no accumulation point and hence is closed. Also, $A \cap X' = \emptyset$ and $X'$ is compact. By Proposition 2.7, we have $d(A, X') = \delta > 0$. Now, since $\inf\limits_{n\in\mathbb{N}} I(x_n) = 0$, we can get a subsequence $(x_{n_k})$ of $(x_n)$ and a sequence $(y_k)$ in $X$ such that $0 < d(x_{n_k}, y_k) < \min\{1/k, \delta/2\}$. Now for every $k$ in $\mathbb{N}$ and $x$ in $X'$, $d(y_k, x) \geq d(x_{n_k}, x) - d(x_{n_k}, y_k) > \delta - \delta/2 = \delta/2$.

Consider the set $C = \{x_{n_k} : k \in \mathbb{N}\} \cup \{y_k : k \in \mathbb{N}\}$. Then $C \subseteq X \setminus B(X', \delta/2)$ and consequently, it is uniformly discrete, contradicting the choice of $(x_{n_k})$ and $(y_k)$. Hence, $\inf\limits_{n\in\mathbb{N}} I(x_n) > 0$.

$(e) \Longrightarrow (f)$: Suppose on the contrary that there exists $\delta_1 > 0$ for which no $\delta_2 > 0$ exists. That is, for all $n \in \mathbb{N}$, we can get $x_n \in X$ such that $d(x_n, X') \geq \delta_1$ but $I(x_n) < 1/n$.

We can assume that the sequence $(x_n)$ consists of distinct points. Since $(x_n)$ is a sequence in the closed set $X \setminus B(X', \delta_1)$ which is disjoint from $X'$, it has no cluster point. By $(e)$, we get $\inf_{n\in\mathbb{N}} I(x_n) > 0$ contrary to the inequality $I(x_n) < 1/n$. Hence, $(f)$ holds.

$(f) \Longrightarrow (g)$: Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of $X$. Since $X'$ is a compact subset of $X$, by Theorem 2.7, there exists $\delta > 0$ such that for all $x \in X'$, there is some $\lambda_x \in \Lambda$ such that $B(x, \delta) \subseteq O_{\lambda_x}$.

Consider the set $B(X', \delta/2)$. Now for all $x \in B(X', \delta/2)$, there exists $x' \in X'$ such that $d(x, x') < \delta/2$. It can be easily seen that $B(x, \delta/2) \subseteq B(x', \delta) \subseteq O_{\lambda_{x'}}$. Now by $(f)$, for $\delta/2$ there exists $\eta > 0$ such that for all $x \in X \setminus B(X', \delta/2)$, $B(x, \eta) = \{x\} \subseteq O_\lambda$ for some $\lambda$.

Thus, $\epsilon = \min(\delta/2, \eta)$ is a Lebesgue number for the open cover.

$(g) \Longrightarrow (h)$: This is immediate.

$(h) \Longrightarrow (a)$: Let $A$ be a nonempty closed subset of $X$ and $G$ be an open subset of $X$ containing $A$. Then $X = G \cup (X \setminus A)$. By $(h)$, there exists $\delta > 0$ such that for every $x \in X$, $B(x, \delta) \subseteq G$ or $B(x, \delta) \subseteq X \setminus A$. If $x \in A$, then clearly $B(x, \delta) \subseteq G$ and hence, $B(A, \delta) \subseteq G$. So by Theorem 3.7, $(X, d)$ is an Atsuji space. $\square$

**Remark 3.11.** Conditions $(b)$, $(g)$, and $(h)$ have been taken from [18]. Condition $(b)$ has also been given in [22], while condition $(g)$ has been studied in [7] and [27], as well. Conditions $(c)$, $(d)$, and $(f)$ have been taken from [22], [3], and [11], respectively. Condition $(e)$ is the third condition of Theorem 1 in [1]. This condition, according to Atsuji, appeared as Theorem 2 in [13].

**Remark 3.12.** Condition $(c)$ in Theorem 3.10 is alternatively stated in [7] as follows:

> If $A$ is an infinite subset of $X$ without any accumulation point in $X$, then $A$ is uniformly discrete in $X$.

The next theorem gives mainly the sequential characterizations of an Atsuji space. In order to state this result, we need the following definitions.

**Definition 3.13.** A sequence $(x_n)$ of distinct isolated points in a metric space $(X, d)$ is called a sequence of *paired isolated points* if $\lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = 0$.

**Definition 3.14.** A sequence $(x_n)$ in a metric space $(X, d)$ is called *pseudo-Cauchy* if it satisfies the following condition: for all $\epsilon > 0$ and for all $n \in \mathbb{N}$, there exist $j, k \in \mathbb{N}$ such that $j \neq k$, $j, k > n$, and $d(x_j, x_k) < \epsilon$.

Intuitively, a pseudo-Cauchy sequence is one in which pairs of points are arbitrarily close frequently rather than eventually as in a Cauchy sequence.

**Definition 3.15.** Let $(X, d)$ be a metric space. A continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) \neq f(1)$ is called a *path* in $X$. In this case, we say that the metric space $(X, d)$ contains a path.

**Theorem 3.16.** *For a metric space $(X, d)$, the following statements are equivalent.*

$(a)$ *There exists a metric space $(M, \rho)$ containing a path such that any continuous function $f : (X, d) \longrightarrow (M, \rho)$ is uniformly continuous.*

$(b)$ *If $(x_n)$ and $(y_n)$ are two asymptotic sequences in $X$ such that $x_n \neq y_n$ for each $n$, then the sequence $(x_n)$ (equivalently $(y_n)$) has a cluster point in $X$.*

$(c)$ *Every sequence of paired isolated points in $X$ has a cluster point and every sequence in $X'$ has a cluster point.*

$(d)$ *Every pseudo-Cauchy sequence with distinct terms in $X$ has a cluster point.*

$(e)$ *$(X, d)$ is an Atsuji space.*

*Proof:* $(a) \implies (b)$: Suppose on the contrary that $(b)$ does not hold; that is, there exist two asymptotic sequences $(x_n)$ and $(y_n)$ in $X$, $x_n \neq y_n$ for each $n$ such that $(x_n)$ (equivalently $(y_n)$ ) has no cluster point. By the method used in the proof of Theorem 3.7 $(i) \implies (j)$, we can get two disjoint closed sets $B$ and $C$ from sequences $(x_n)$ and $(y_n)$ such that $d(B, C) = 0$, and by the normality of $X$, we have a real valued continuous function, $f : X \longrightarrow [0, 1]$ such that $f(B) = \{0\}$ and $f(C) = \{1\}$.

Now since the metric $(M, \rho)$ contains a path, there exists a continuous function $\phi : [0, 1] \longrightarrow M$ such that $a = \phi(0) \neq \phi(1) = b$, where $a, b \in M$.

Consider the mapping $g : X \longrightarrow M$ as $g(x) = \phi(f(x))$.

Being a composition of two continuous functions, $g$ is continuous. But $g$ is not uniformly continuous as $d(B, C) = 0$ and

$\rho(g(B), g(C)) = \rho(a, b) > 0$. Therefore, we get a contradiction to $(a)$ and hence, $(b)$ holds.

$(b) \Longrightarrow (c)$: Let $(x_n)$ be a sequence of paired isolated points in $X$. Then the sequence $(z_n) = (x_{2n-1})$ and $(y_n) = (x_{2n})$ are asymptotic and thus $(z_n)$ and $(y_n)$ have a common cluster point. Hence, $(x_n)$ also has a cluster point.

Now if $(x_n)$ is a sequence in $X'$, then there exists a sequence $(y_n)$ in $X$ such that $0 < d(x_n, y_n) < \frac{1}{n}$ for all $n$. Then $(x_n)$ and $(y_n)$ are asymptotic sequences and hence, $(x_n)$ has a cluster point.

$(c) \Longrightarrow (d)$: Suppose that $(x_n)$ is a pseudo-Cauchy sequence with distinct terms. Now, if for infinitely many $n$, $x_n \in X'$, then $(x_n)$ has a subsequence $(x_{n_k})$ in $X'$. Hence by $(c)$, $(x_{n_k})$ has a cluster point and consequently, $(x_n)$ also has a cluster point.

Now suppose that $x_n \in X'$ for only finitely many $n$. Then there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $x_n$ is isolated in $X$. Again since $(x_n)$ is pseudo-Cauchy and has distinct terms, there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that for each $k$, $x_{n_k}$ is isolated and $d(x_{n_{2k-1}}, x_{n_{2k}}) < \frac{1}{k}$; that is, $(x_{n_k})$ is a sequence of paired isolated points in $X$. Again by $(c)$, as argued earlier, $(x_n)$ has a cluster point.

$(d) \Longrightarrow (e)$: If possible, suppose that there exists an open cover $\{O_\lambda : \lambda \in \Lambda\}$ of $X$ having no Lebesgue number. Then for each $n \in \mathbb{N}$, $1/n$ is not a Lebesgue number for the covering. Hence for each $n \in \mathbb{N}$, there exists $x_{2n-1} \in X$ such that $B(x_{2n-1}, 1/n) \nsubseteq O_\lambda$ for all $\lambda \in \Lambda$. Since $\{O_\lambda : \lambda \in \Lambda\}$ is an open cover of $X$, there exists $\lambda_n \in \Lambda$ such that $x_{2n-1} \in O_{\lambda_n}$. But since $B(x_{2n-1}, 1/n) \nsubseteq O_{\lambda_n}$, there exists $x_{2n} \in B(x_{2n-1}, 1/n) \setminus O_{\lambda_n}$; that is, $d(x_{2n-1}, x_{2n}) < 1/n$ and $x_{2n} \notin O_{\lambda_n}$.

Now we have a sequence $(x_n)$ in $X$ which is clearly pseudo-Cauchy. Also since $x_{2n-1} \neq x_{2n}$ for all $n \in \mathbb{N}$, we can find a pseudo-Cauchy subsequence $(x_{n_k})$ of $(x_n)$ with distinct terms. Hence by $(d)$, $(x_{n_k})$, and consequently $(x_n)$, has a cluster point, say $a$. Let $a \in O_\alpha$ for some $\alpha \in \Lambda$ and $\delta > 0$ be such that $B(a, \delta) \subseteq O_\alpha$. Since $a$ is a cluster point of the sequence $(x_n)$, there exists $m \in \mathbb{N}$ such that $m > 2/\delta$ and $x_{2m-1} \in B(a, \delta/2)$. Then $B(x_{2m-1}, 1/m) \subseteq B(a, \delta) \subseteq O_\alpha$. But this contradicts the fact that for each $n \in \mathbb{N}$, $B(x_{2n-1}, \frac{1}{n}) \nsubseteq O_\lambda$ for all $\lambda \in \Lambda$. Consequently, every open cover of $X$ must have a Lebesgue number. Hence by Theorem 3.10, $(e)$ holds.

$(e) \Longrightarrow (a)$: This is immediate. $\square$

**Remark 3.17.** Conditions $(b)$ and $(d)$ have been taken from [25] and [24], respectively. Condition $(d)$ has also been mentioned in [3] and [4]. Condition $(c)$ has been taken from [3], and a condition similar to $(a)$ has been given in [25].

**Remark 3.18.** Condition $(b)$ in Theorem 3.16 is alternatively stated in [7] as follows:

> For every sequence $(x_n)$ in $X$ which has no convergent subsequence, the only sequences $(x'_n)$ in $X$ such that $\lim d(x_n, x'_n) = 0$ are those which are almost equal to $(x_n)$, in the sense that $x_n = x'_n$ for all but a finite set of indices.

We have already seen that a compact metric space is an Atsuji space while an Atsuji space is complete. So for a metric space $(X, d)$, the property of being an Atsuji space lies in between the completeness and compactness. Since a metric space is compact if and only if it is countably compact, for a metric space $(X, d)$, the following conditions are equivalent: $(a)$ $(X, d)$ is compact; $(b)$ every family of closed subsets of $X$ with the finite intersection property has a nonempty intersection; and $(c)$ every decreasing sequence of nonempty closed subsets of $X$ has a nonempty intersection. Note that a family $\mathcal{F}$ of sets is said to have the finite intersection property if every finite intersection of sets in $\mathcal{F}$ is nonempty. On the other hand, Cantor's theorem says that a metric space $(X, d)$ is complete if and only if for every decreasing sequence $F_1 \supseteq F_2 \supseteq \ldots$ of nonempty closed subsets of $X$, such that $\lim\limits_{n\to\infty} \delta(F_n) = 0$, the intersection $\bigcap\limits_{n=1}^{\infty} F_n$ is nonempty. Here $\delta(F_n) =$ the diameter of $F_n = \sup\{d(x, y) : \ x, y \in F_n\}$. In 1930, Casimir Kuratowski in [14] gave an analog, actually a generalization, of Cantor's theorem in terms of a notion of measure of noncompactness. In order to state Kuratowski's result, first we need to explain this notion in the following definition.

**Definition 3.19.** Let $(X, d)$ be a metric space and $A$ be a nonempty subset of $X$. Then we define $\alpha(A)$ and $\chi(A)$ as follows:

(a) $\alpha(A) = \inf\{\epsilon > 0 :\ A$ can be covered with a finite number of sets of diameter less than $\epsilon\}$;
(b) $\chi(A) = \inf\{\epsilon > 0 :\ A \subseteq B(F, \epsilon)$ for some finite subset $F$ of $X\}$.

We also set $\alpha(\emptyset) = \chi(\emptyset) = 0$. The first one $\alpha$ is known as the *Kuratowski measure of noncompactness*, while the second one $\chi$ is known as the *Hausdorff or ball measure of noncompactness*.

Note that a subset $A$ of $X$ is totally bounded if and only if $\alpha(A) = 0 = \chi(A)$. Actually, both $\alpha$ and $\chi$ measure nontotal boundedness. Also $\alpha$ and $\chi$ are in some sense equivalent, since $\chi(A) \le \alpha(A) \le 2\chi(A)$ for all $A \subseteq X$. For more details on $\alpha$ and $\chi$, one can see [2], [12], and [6, Exercise 1.1.4, p. 7]. Now we are ready to state Kuratowski's theorem:

Let $(X, d)$ be a metric space. Then the following statements are equivalent.
(a) $(X, d)$ is complete.
(b) For every decreasing sequence $F_1 \supseteq F_2 \supseteq \ldots$ of nonempty closed subsets of $X$ such that $\lim\limits_{n\to\infty} \alpha(F_n) = 0$, the intersection $\bigcap\limits_{n=1}^{\infty} F_n$ is nonempty and compact.
(c) For every decreasing sequence $F_1 \supseteq F_2 \supseteq \ldots$ of nonempty closed subsets of $X$ such that $\lim\limits_{n\to\infty} \chi(F_n) = 0$, the intersection $\bigcap\limits_{n=1}^{\infty} F_n$ is nonempty and compact.

In view of the discussion and observation made above, in the next result, we present analogous characterizations for a metric space to be an Atsuji space in terms of a variant of finite intersection property and Cantor-like conditions. But in order to state this result, first we need the following definition and lemma.

**Definition 3.20.** Let $(X, d)$ be a metric space and $A$ be a nonempty subset of $X$. We define $\overline{d}(A)$ and $\underline{d}(A)$ as follows:

(a) $\overline{d}(A) = \sup\{d(a, X \setminus \{a\}) : a \in A\}$ and
(b) $\underline{d}(A) = \inf\{d(a, X \setminus \{a\}) : a \in A\}$.

Note that $\underline{d}(A) \leq \overline{d}(A)$ and $\overline{d}(A) \leq \overline{d}(B)$ if $\emptyset \neq A \subseteq B \subseteq X$. It can be easily shown that $\overline{d}(A) = \overline{d}(\overline{A})$. If $A = \{x\}$, then $\overline{d}(A) = \underline{d}(A)$ and in this case, the common value is simply $I(x)$.

**Lemma 3.21.** *Let $(X, d)$ be a metric space and let $(x_n)$ be a sequence in $X$ with $\lim\limits_{n\to\infty} I(x_n) = 0$. If $(x_n)$ has a cluster point $x$ in $X$, then $x$ is an accumulation point of $X$.*

**Theorem 3.22.** *For a metric space $(X, d)$, the following statements are equivalent.*

- $(a)$ *$(X, d)$ is an Atsuji space.*
- $(b)$ *Let $(x_n)$ be a sequence in $X$ such that the sequence $(I(x_n))$ converges to $0$. Then $(x_n)$ has a cluster point.*
- $(c)$ *Let $\mathcal{F}$ be a family of closed subsets of $X$ with the finite intersection property. If there exists a sequence $(A_n)$ in $\mathcal{F}$ such that $\lim\limits_{n\to\infty} \overline{d}(A_n) = 0$, then $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$.*
- $(d)$ *For every decreasing sequence $F_1 \supseteq F_2 \supseteq \dots$ of nonempty closed subsets of $X$, such that $\lim\limits_{n\to\infty} \overline{d}(F_n) = 0$, the intersection $\bigcap\limits_{n=1}^{\infty} F_n$ is nonempty.*
- $(e)$ *For every decreasing sequence $F_1 \supseteq F_2 \supseteq \dots$ of nonempty closed subsets of $X$, such that $\lim\limits_{n\to\infty} \underline{d}(F_n) = 0$, the intersection $\bigcap\limits_{n=1}^{\infty} F_n$ is nonempty.*

*Proof:* $(a) \implies (b)$: Let $(X, d)$ be an Atsuji space and $(x_n)$ be a sequence in $X$ such that $\lim\limits_{n\to\infty} I(x_n) = 0$. If possible, assume that $(x_n)$ does not have a cluster point. Then by Theorem 3.7, the set $B = X' \cap \{x_n : n \in \mathbb{N}\}$ is finite and $\inf\{I(x_n) : x_n \notin B\} > 0$. Obviously then, $\lim\limits_{n\to\infty} I(x_n) > 0$ and we arrive at a contradiction. Hence, the sequence $(x_n)$ must have a cluster point.

$(b) \implies (c)$: Let $\mathcal{F}$ be a family of closed subsets of $X$ with the finite intersection property. Suppose that there exists a sequence $(A_n)$ in $\mathcal{F}$ such that $\lim\limits_{n\to\infty} \overline{d}(A_n) = 0$. We need to show that $\cap\{F : F \in \mathcal{F}\}$ is nonempty.

Note that if $(b)$ holds, then $X'$ is compact, because $x \in X'$ if and only if $I(x) = 0$. So if $(x_n)$ is a sequence in $X'$, then $(x_n)$

has a convergent subsequence. But the limit of this convergent subsequence must belong to $X'$ since $X'$ is closed.

Now let $\Sigma$ be a finite subfamily of $\mathcal{F}$. For each $n \in \mathbb{N}$, let $E_n = (\bigcap_{i=1}^{n} A_i) \cap (\cap\{F : F \in \Sigma\})$. Since $\mathcal{F}$ has the finite intersection property, each $E_n$ is nonempty. For each $n$, pick an element $x_n$ from $E_n$. Since $\overline{d}$ is monotone, $I(x_n) = \overline{d}(\{x_n\}) \leq \overline{d}(E_n) \leq \overline{d}(A_n)$. But $\lim_{n\to\infty} \overline{d}(A_n) = 0$. Hence, $\lim_{n\to\infty} I(x_n) = 0$ and consequently by $(b)$, the sequence $(x_n)$ has a cluster point $x$ in $X$.

Since $\lim_{n\to\infty} I(x_n) = 0$, by Lemma 3.21, $x \in X'$. Now $x$ is a cluster point of $(x_n)$; therefore, there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \to x$. Note that for any $m \in \mathbb{N}$, $x_{n_k} \in E_m$ for all $k \geq m$. But each $E_m$ is closed. Hence, $x \in E_m$ for all $m \in \mathbb{N}$; that is, $x \in \bigcap_{n=1}^{\infty} E_n$ and consequently, $x \in (\bigcap_{n=1}^{\infty} E_n) \cap X'$. But $(\bigcap_{n=1}^{\infty} E_n) \cap X' \subseteq (\cap\{F : F \in \Sigma\}) \cap X' = \cap\{F \cap X' : F \in \Sigma\}$. Hence, the family of closed sets $\{F \cap X' : F \in \mathcal{F}\}$ has the finite intersection property. But $X'$ is compact. Hence, $\cap\{F \cap X' : F \in \mathcal{F}\}$ is nonempty and consequently, $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$.

$(c) \Longrightarrow (d)$: This is trivial.

$(d) \Longrightarrow (a)$: Let $(x_n)$ be a pseudo-Cauchy sequence in $X$ with distinct terms. If needed, by passing to a subsequence we may assume that for each $n$, $d(x_{2n-1}, x_{2n}) < \frac{1}{n}$. For each $n$, let $B_n = \{x_k : k \geq n\}$ and $A_n = \overline{B_n}$. Since the terms of $(x_n)$ are distinct, $\overline{d}(A_n) = \overline{d}(\overline{B_n}) = \overline{d}(B_n) \leq \frac{2}{n}$ and consequently, $\lim_{n\to\infty} \overline{d}(A_n) = 0$. Obviously $(A_n)$ is a decreasing sequence. Hence by $(d)$, $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} A_n$. Choose any $\epsilon > 0$ and any $n \in \mathbb{N}$. Since $x \in A_n$, there exists $k \geq n$ such that $d(x_k, x) < \epsilon$. Hence, $x$ is a cluster point of the sequence $(x_n)$ and consequently by Theorem 3.16, $(a)$ holds.

$(b) \Longrightarrow (e)$: Let $(F_n)$ be a decreasing sequence of nonempty closed sets in $X$ such that $\lim_{n\to\infty} \underline{d}(F_n) = 0$. For each $n \in \mathbb{N}$, choose $x_n \in F_n$ such that $I(x_n) \leq \underline{d}(F_n) + \frac{1}{n}$. Since $\lim_{n\to\infty} \underline{d}(F_n) = 0$, $\lim_{n\to\infty} I(x_n) = 0$ and consequently by $(b)$, the sequence $(x_n)$ has a cluster point $x$. Since $(F_n)$ is decreasing, for each $k \in \mathbb{N}$, $\{x_n : n \geq k\} \subseteq F_k$. Hence, $x \in \overline{F_k} = F_k$ for all $k \in \mathbb{N}$ and consequently, $x \in \bigcap_{n=1}^{\infty} F_n$.

$(e) \implies (d)$: It immediately follows from the observation that for any nonempty subset $A$ of $X$, $\underline{d}(A) \leq \overline{d}(A)$. $\square$

**Remark 3.23.** The condition $(b)$ of the last theorem appears in [11], while the rest of the result appears in [4].

**Corollary 3.24.** *Every closed subset of an Atsuji space is an Atsuji space.*

*Proof:* Let $A$ be a closed subset of $X$ and $(x_n)$ be a pseudo-Cauchy sequence in $A$. Since $(X, d)$ is an Atsuji space, $(x_n)$ has a cluster point in $X$. But since $A$ is closed in $X$, this cluster point must belong to $A$. Hence, $(A, d)$ is an Atsuji space. $\square$

Now we answer an interesting question. We have already seen that the compactness of $X'$ is a necessary condition for a metric space to be an Atsuji space. Now we would like to query if the converse is also true. More precisely, if $X'$ is compact in a metrizable space $X$, then does $X$ admit a compatible metric $\rho$ so that $(X, \rho)$ becomes an Atsuji space? The answer is affirmative. In fact, there are several equivalent conditions on a metrizable space $X$, each of which ensures that there exists a compatible metric $\rho$ on $X$ such that $(X, \rho)$ becomes an Atsuji space.

In Theorem 2 of [22], John Rainwater has given eight such equivalent conditions and one more in the proof of this result. Here we list four of them. For a metrizable space $X$, the following conditions are equivalent: $(a)$ the set $X'$ of all accumulation points in $X$ is compact; $(b)$ the diagonal in $X \times X$ has a countable basis of neighborhoods; $(c)$ every subset of $X$ has a compact boundary; and $(d)$ every closed set in $X$ has a countable basis of neighborhoods. Due to Theorem 1 of [22], it means that if $X$ is a metrizable space and $X'$ is compact, then there exists a compatible metric $\rho$ on $X$ such that $(X, \rho)$ becomes an Atsuji space. But two constructive proofs of this result are also available in the literature. Out of these two proofs, one was given by Mrówka in [17] and the other one has been given by Beer in [5]. We would also like to mention that the problem of describing those metrizable topological spaces which admit an Atsuji metric has also been considered by Nagata and Levšenko in [19] and [15], respectively. Here we reproduce Beer's proof in a slightly different form, but in detail.

**Theorem 3.25.** *Let $X$ be a metrizable space and $X'$ be compact. Then there exists a compatible metric $\rho$ on $X$ such that $(X, \rho)$ becomes an Atsuji space.*

*Proof (due to Beer):* If $X' = \emptyset$, then consider the uniformly discrete metric $\rho$ on $X$ defined as follows: $\rho(x,y) = 1$ if $x \neq y$ and $\rho(x,x) = 0,\ x, y \in\ X$. The metric $\rho$ is compatible with the discrete topology on $X$ and $(X, \rho)$ is clearly an Atsuji space.

Now suppose that $X' \neq \emptyset$. Let $d$ be a compatible metric for $X$. Define $\rho : X \times X \longrightarrow \mathbb{R}$ as:

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ d(x,y) + \max\{d(x,X'), d(y,X')\} & \text{if } x \neq y. \end{cases}$$

Using the inequality,

$\max\{\alpha,\gamma\} \leq \alpha + \gamma \leq \max\{\alpha,\beta\} + \max\{\beta,\gamma\}\ \forall\ \alpha,\ \beta,\ \gamma \geq 0,$
it can be easily verified that $\rho$ is a metric on $X$.

Clearly, every convergent sequence in $(X, \rho)$ is convergent in $(X, d)$. Now let $(x_n)$ be a sequence of distinct points converging to $x$ in $(X, d)$. Then, $x \in X'$ and $d(x, X') = 0$. Now,

$$\begin{aligned} \rho(x_n, x) &\leq d(x_n, x) + d(x_n, X') \\ &\leq 2d(x_n, x). \end{aligned}$$

So $(x_n)$ is convergent to $x$ in $(X, \rho)$. Therefore, $\rho$ and $d$ are equivalent metrics on $X$.

Let $(x_n)$ and $(y_n)$ be two asymptotic sequences in $(X, \rho)$. Then $0 \leq d(x_n, X') \leq \rho(x_n, y_n) \to 0$ as $n \to \infty$. So there exists a sequence $(p_n)$ in $X'$ asymptotic to $(x_n)$ in $(X, d)$. Since $X'$ is compact, by Theorem 2.5, $(p_n)$ has a cluster point in $(X, d)$. Consequently, $(x_n)$ (equivalently $(y_n)$) has a cluster point in $(X, d)$ and since $d$ and $\rho$ are equivalent, $(x_n)$ (equivalently $(y_n)$) has a cluster point in $(X, \rho)$. By Theorem 3.16, $(X, \rho)$ is an Atsuji space. □

We end this paper with the following remark.

**Remark 3.26.** In [3, Theorem 1] and in [6, Theorem 2.3.4, p. 57], Beer has given four equivalent external characterizations for a metric space $(X, d)$ to be an Atsuji space in terms of relationships between the topology of uniform convergence and two hyperspace topologies on the space of continuous functions from $(X, d)$ to a metric space $(Y, \rho)$. These characterizations have not been included in this paper.

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