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CORRECTION TO THE PAPER "ON THE HYPERSPACES $\mathcal{C}_n(X)$ OF A CONTINUUM X, II"

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Abstract. In this note, we show that Theorem 3.10 in [Sergio Macías, On the Hyperspaces $C_n(X)$ of a Continuum X, II, Topology Proc., 25 (2000), Spring, 255–276] is incorrect as it is stated. We prove the correct version of this result and present and prove the correct version of a consequence of it.

The purpose of this note is to show that Theorem 3.10 of [4] is false. (This result is also stated incorrectly in [6, Theorem 6.5.12].) The notation and definitions of concepts we are using can be found in [4] and [5].

In the proof of [4, Theorem 3.10], it was claimed, without proof, that $\sum_{j=1}^{\ell} m_j = n$, where

$m_j = \max\{\text{number of components of } A \text{ contained in } \kappa_j \mid A \in \mathcal{A}\}$.

This equality is not true. Hence, the function $f\colon \prod_{j=1}^{\ell} \mathcal{C}_{m_j}(\kappa_j) \to \mathcal{A}$ is not well defined. To see that the equality is false, let X be any indecomposable continuum, suppose $n = 5$ and $\ell = 2$. It is easy to see that $m_1 = m_2 = 4$. Thus, $m_1 + m_2 = 8$.

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Let X be an indecomposable continuum, let n be a positive integer greater than one, and let $\mathcal{A}$ be an arc component of $\mathcal{C}_n(X)\setminus\{X\}$ which is not of the form $\mathcal{C}_n(\kappa)$, where κ is a composant of X. Let A_0 be a point of $\mathcal{A}$, and let $\kappa_1,\ldots,\kappa_\ell$ be the composants of X such that $A_0\subset\bigcup_{j=1}^{\ell}\kappa_j$ and $A_0\cap\kappa_j\neq\emptyset$ for each $j\in\{1,\ldots,\ell\}$. In [4, Theorem 3.10] (see also Theorem 1 below), it is shown that any element A of $\mathcal{A}$ is contained in $\bigcup_{j=1}^{\ell}\kappa_j$ and $A\cap\kappa_j\neq\emptyset$ for each $j\in\{1,\ldots,\ell\}$. Since for each positive integer r and each $j\in\{1,\ldots,\ell\}$, $\mathcal{C}_r(\kappa_j)$ is arcwise connected [4, Theorem 3.9], each element A of $\mathcal{A}$ has at least ℓ components, and at least one component of A is contained in each κ_j. Hence, if $m_j=\max\{\text{number of components of } A \text{ contained in } \kappa_j \mid A\in\mathcal{A}\}$, then $m_j=n-\ell+1$.

Let

$$\mathcal{N}=\left\{\{r_1,\ldots,r_\ell\}\ \Big|\ 1\leq r_1,\ldots,r_\ell\leq n-\ell+1 \text{ and } \sum_{j=1}^{\ell}r_j=n\right\}.$$

Now we can state and prove the correct version of Theorem 3.10 of [4].

Theorem 1. *Let n be an integer greater than one, and let X be an indecomposable continuum. If $\mathcal{A}$ is an arc component of $\mathcal{C}_n(X)\setminus\{X\}$, which is not of the form $\mathcal{C}_n(\kappa)$, where κ is a composant of X, then there exist finitely many composants $\kappa_1,\ldots,\kappa_\ell$ of X and there exists a one–to–one map from*

$$\mathcal{R}=\bigcup\left\{\prod_{j=1}^{\ell}\mathcal{C}_{r_j}(\kappa_j)\ \Big|\ \{r_1,\ldots,r_\ell\}\in\mathcal{N}\right\}\subset\prod_{j=1}^{\ell}\mathcal{C}_{n-\ell+1}(\kappa_j)$$

(with the "max" metric ρ_1) onto $\mathcal{A}$.

Proof: Let A_0 be a point of $\mathcal{A}$, and let $\kappa_1,\ldots,\kappa_\ell$ be the composants of X which intersect A_0. Since $\mathcal{A}$ is not of the form $\mathcal{C}_n(\kappa)$, $\ell\geq 2$.

First, we show that every element of $\mathcal{A}$ intersects each κ_j for each $j\in\{1,\ldots,\ell\}$. To see this, suppose that there is a point B

of $\mathcal{A}$ such that $B \cap \kappa_j = \emptyset$, for some $j \in \{1, \ldots, \ell\}$. Since A_0 and B belong to $\mathcal{A}$, there exists an arc $\alpha\colon [0,1] \to \mathcal{A}$ such that $\alpha(0) = A_0$ and $\alpha(1) = B$. Let $\beta\colon [0,1] \to \mathcal{C}_n(X)$ be given by $\beta(t) = \sigma(\alpha([0,t]))$. Then β is well defined [3, Lemma 7.2], β is an order arc, $\beta(0) = \alpha(0) = A_0$, and $\beta(1) = \sigma(\alpha([0,1]))$ is an element of $\mathcal{C}_n(X)$ intersecting κ_j and containing B. On the other hand, the map $\gamma\colon [0,1] \to \mathcal{C}_n(X)$ given by $\gamma(t) = \sigma(\alpha([1-t,1]))$ is also well defined [3, Lemma 7.2], γ is an order arc, $\gamma(0) = \alpha(1) = B$, and $\gamma(1) = \sigma(\alpha([0,1])) = \beta(1)$. Thus, there exists an order arc in $\mathcal{C}_n(X)$ from B to $\beta(1)$, $B \cap \kappa_j = \emptyset$ and $\beta(1) \cap \kappa_j \neq \emptyset$; this contradicts [7, Theorem (1.8)] if $\sigma(\alpha([0,1]))$ is a proper subset of X. Otherwise, a similar argument to the one given in [4, Theorem 3.9] shows that there exists $t_0 \in [0,1]$ such that $\alpha(t_0) = X$, which is also a contradiction. Therefore, every element of $\mathcal{A}$ intersects each κ_j, $j \in \{1, \ldots, \ell\}$. A similar argument proves that if $A \in \mathcal{A}$ and $A \cap \kappa \neq \emptyset$, for some composant κ of X, then $\kappa \in \{\kappa_1, \ldots, \kappa_\ell\}$.

Let $f\colon \mathcal{R} \to \mathcal{A}$ be given by

$$f(A_1, \ldots, A_\ell) = \bigcup_{j=1}^{\ell} A_j.$$

Let us see first that f is well defined. Clearly, $f(A_1, \ldots, A_\ell) \in \mathcal{C}_n(X)$. On the other hand, for each $j \in \{1, \ldots, \ell\}$, both A_j and $A_0 \cap \kappa_j$ belong to $\mathcal{C}_{r_j}(\kappa_j)$, where $\{r_1, \ldots, r_\ell\} \in \mathcal{N}$. Since $\mathcal{C}_{r_j}(\kappa_j)$ is arcwise connected, there is an arc $\alpha_j\colon [0,1] \to \mathcal{C}_{r_j}(\kappa_j)$ such that $\alpha_j(0) = A_0 \cap \kappa_j$ and $\alpha_j(1) = A_j$. Hence, $\alpha\colon [0,1] \to \mathcal{C}_n(X) \setminus \{X\}$ given by $\alpha(t) = \bigcup_{j=1}^{\ell} \alpha_j(t)$ is a path joining A_0 and $\bigcup_{j=1}^{\ell} A_j$. Therefore, $\bigcup_{j=1}^{\ell} A_j \in \mathcal{A}$.

Let $A \in \mathcal{A}$. For each $j \in \{1, \ldots, \ell - 1\}$, let r_j be the number of components of A contained in κ_j. Let $r_\ell = n - \sum_{j=1}^{\ell-1} r_j$. Then $\sum_{j=1}^{\ell} r_j = n$. Hence, $\{r_1, \ldots, r_\ell\} \in \mathcal{N}$, $(A \cap \kappa_1, \ldots, A \cap \kappa_\ell) \in \prod_{j=1}^{\ell} \mathcal{C}_{r_j}(\kappa_j)$, and $f(A \cap \kappa_1, \ldots, A \cap \kappa_\ell) = A$. Therefore, f is surjective.

Let $(A_1,\dots,A_\ell)$ and $(B_1,\dots,B_\ell)$ be two different points of $\mathcal{R}$. Then $A_{j_0}\neq B_{j_0}$ for some $j_0\in\{1,\dots,\ell\}$. Hence, $\bigcup_{j=1}^{\ell}A_j\neq\bigcup_{j=1}^{\ell}B_j$, being a disjoint union. Therefore, f is one–to–one.

To see that f is continuous, let $\varepsilon>0$ be given. Let $(A_1,\dots,A_\ell)$ and $(B_1,\dots,B_\ell)$ be two points of $\mathcal{R}$ such that

$$\rho_1\left((A_1,\dots,A_\ell),(B_1,\dots,B_\ell)\right)<\frac{\varepsilon}{2}\,.$$

Then for each $j\in\{1,\dots,\ell\}$, $\mathcal{H}(A_j,B_j)<\frac{\varepsilon}{2}$. Hence, for every $j\in\{1,\dots,\ell\}$, $A_j\subset\mathcal{V}^d_{\frac{\varepsilon}{2}}(B_j)\subset\mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{r=1}^{\ell}B_r\right)$ and $B_j\subset\mathcal{V}^d_{\frac{\varepsilon}{2}}(A_j)\subset\mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{r=1}^{\ell}A_r\right)$. Thus, we have that $\bigcup_{j=1}^{\ell}A_j\subset\mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{j=1}^{\ell}B_j\right)$ and $\bigcup_{j=1}^{\ell}B_j\subset\mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{j=1}^{\ell}A_j\right)$. This implies that

$$\mathcal{H}\left(\bigcup_{j=1}^{\ell}A_j,\bigcup_{j=1}^{\ell}B_j\right)\leq\frac{\varepsilon}{2}<\varepsilon.$$

Therefore, f is continuous. □

Since [4, Theorem 3.10] is false, we have that Theorem 6.4 of [5] is false too.

In view of Theorem 1, we can state and prove the correct version of Theorem 6.4 of [5].

Theorem 2. *Let X be an indecomposable continuum and let n be a positive integer. If $\mathcal{A}$ is an arc component of $HS_n(X)\setminus\{q^n_X(X),F^n_X\}$, then either $\mathcal{A}$ is homeomorphic to $q^n_X(\mathcal{C}_n(\kappa)\setminus\mathcal{F}_n(\kappa))$ for some composant κ of X, or there exist finitely many composants $\kappa_1,\dots,\kappa_\ell$ of X and there exists a one–to–one map from*

$$\mathcal{T}=\bigcup\left\{\prod_{j=1}^{\ell}\mathcal{C}_{r_j}(\kappa_j)\setminus\prod_{j=1}^{\ell}\mathcal{F}_{r_j}(\kappa_j)\;\middle|\;\{r_1,\dots,r_\ell\}\in\mathcal{N}\right\}$$

onto $\mathcal{A}$.

Proof: Let κ be a composant of X. By [4, Theorem 3.9], $\mathcal{C}_n(\kappa)$ is an arc component of $\mathcal{C}_n(X) \setminus \{X\}$. By [5, Lemma 6.3], $\mathcal{C}_n(\kappa) \setminus \mathcal{F}_n(\kappa) = \mathcal{C}_n(\kappa) \setminus \mathcal{F}_n(X)$ is an arc component of $\mathcal{C}_n(X) \setminus (\{X\} \cup \mathcal{F}_n(X))$. Hence, $q_X^n(C_n(\kappa)\setminus\mathcal{F}_n(\kappa))$ is an arc component of $HS_n(X)\setminus \{q_X^n(X), F_X^n\}$.

Now, suppose $\mathcal{A}$ is an arc component of $HS_n(X) \setminus \{q_X^n(X), F_X^n\}$ and $\mathcal{A}$ is not homeomorphic to $q_X^n(\mathcal{C}_n(\kappa)\setminus\mathcal{F}_n(\kappa))$ for any composant κ of X. Let $\mathcal{B}$ be the arc component of $\mathcal{C}_n(X) \setminus \{X\}$ containing $(q_X^n)^{-1}(\mathcal{A})$. Note that $\mathcal{B} \setminus \mathcal{F}_n(X)) = (q_X^n)^{-1}(\mathcal{A})$. Hence, $(q_X^n)^{-1}(\mathcal{A})$ is an arc component of $\mathcal{C}_n(X) \setminus (\{X\} \cup \mathcal{F}_n(X))$ [5, Lemma 6.3]. By Theorem 1, there exist finitely many composants $\kappa_1, \dots, \kappa_\ell$ of X and a one–to–one map $f\colon \mathcal{R} \to \mathcal{B}$. Let $g = f|_{\mathcal{T}}$. Then g is continuous, one–to–one and its image is contained in $\mathcal{B} \setminus \mathcal{F}_n(X)$. To see g is onto, let $B \in \mathcal{B} \setminus \mathcal{F}_n(X)$. Note that B has at least one nondegenerate component. For each $j \in \{1, \dots, \ell - 1\}$, let r_j be the number of components of B contained in κ_j. Let $r_\ell = n - \sum_{j=1}^{\ell-1} r_j$. Then $\sum_{j=1}^{\ell} r_j = n$ and $\{r_1, \dots, r_\ell\} \in \mathcal{N}$. Hence, $(B \cap \kappa_1, \dots, B \cap \kappa_\ell) \in \prod_{j=1}^{\ell} \mathcal{C}_{r_j}(\kappa_j) \setminus \prod_{j=1}^{\ell} \mathcal{F}_{r_j}(\kappa_j)$ and $g(B \cap \kappa_1, \dots, B \cap \kappa_\ell) = B$. Therefore, $h = q_X^n \circ g$ is the desired map. □

Finally, we thank Professor Sam B. Nadler, Jr. for pointing out to the author that the wording of the proof of Theorem 3.6 of [5] is misleading, specifically in the proof of the following fact.

Theorem 3. *If X is a continuum and n is a positive integer such that* $\dim(\mathcal{C}_n(X)) < \infty$, *then* $\dim(\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)) = \dim(\mathcal{C}_n(X))$.

Proof: Since $\dim(C_n(X)) < \infty$, $\dim(X) = 1$ [2, Theorem 2.1]. Hence, by [1, Lemm 3.1], $\dim(\mathcal{F}_n(X)) \leq n$. Since $\mathcal{C}_n(X)$ contains n–cells [3, Theorem 3.4], $\dim(C_n(X)) \geq n$. Thus, there exists $A \in \mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ such that $\dim_A(\mathcal{C}_n(X)) = \dim(\mathcal{C}_n(X))$. Therefore, $\dim(\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)) = \dim(\mathcal{C}_n(X))$. □

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