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A NOTE ON TRANSFINITE COHESIONS IN TOPOLOGICAL SPACES

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ABSTRACT. An example is given to show that there is a T_1-space of transfinite cohesion. It answers P. A. Cairns' question posed in 1994.

1. Introduction

P. A. Cairns introduced the concept of cohesion in [1] and proved that there is no regular space of transfinite cohesion. Finally, he raised an open problem: Is there any space of transfinite cohesion? In this note, we give an example of a T_1-space of transfinite cohesion. Let N be the set of all natural numbers.

Definition 1.1 ([1]). For a topological space X, the cohesion of X, abbreviated $cohX$, is defined by transfinite recursion as follows: $cohX = -1$ if and only if $X = \emptyset$. For any ordinal α, $cohX \leq \alpha$ if for every nowhere dense subset $C \subset X$, $cohC < \alpha$. For a space X and an ordinal α, $cohX = \alpha$ if $cohX \leq \alpha$ and for every $\beta < \alpha$, it is not the case that $cohX \leq \beta$.

Definition 1.2 ([1]). A space X is said to be scattered if every subset F of X has an isolated point. Taking X^d and denoting the set of non-isolated points of X, we make the following definitions: $X^{(0)} = X, X^{(\alpha+1)} = (X^{(\alpha)})^d$ and $X^{(\lambda)} = \cap\{X^{(\alpha)} : \alpha < \lambda\}$ for a limit ordinal number λ. The scattered length of X, denoted by $sl(X)$, is taken to be the smallest λ, such that $X^{(\lambda)} = \emptyset$.

Lemma 1.3 ([1]). *For a scattered space X and $n \in \omega$, $sl(X) = n$ if and only if $cohX = n - 1$.*

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Example 1.4. *To illustrate that there is a T_1-space X of transfinite cohesion, let $X = N \times N$. For any $i \in N, j \in N$, we let $\mathcal{W}_{ij} = \{W_{iq} \cup \{(i,j)\} : q \geq j+1\}$, where $W_{iq} = \{(m,n) : m \geq i+1, n \geq q\}, q \in N$. $\mathcal{W}_{ij}$ is a base of the point (i,j) in X.*

Claim 1.5. *Let $M \subset X$. If there is some $n \in N$, such that $max\{i : (i,j) \in M\} \leq n$, then the set M is a scattered subset of X.*

Proof: Let N_1 be any subset of M and let $m = max\{i : (i,j) \in N_1\}$; then $m \leq n$. Take a point $x \in N_1$, such that $x = (m,j)$ for some j. Then $W \cap N_1 = \{x\}$ for any $W \in \mathcal{W}_{mj}$. So the point x is an isolated point of N_1. Thus, M is a scattered subset of X. □

Claim 1.6. *Let $n \in N$. If $M_n = \{(i,j) : i \leq n, j \in N\}$, then $cohM_n = n - 1$.*

Proof: By Claim 1.5, we know that M_n is a scattered subset of X for $n \in N$. We may easily see that $M_n^d = M_{n-1}$, $M_0 = \emptyset$. So $M_n^{(n)} = \emptyset$. Thus, $sl(M_n) = n$. So we know that $cohM_n = n - 1$ by Lemma 1.3. □

Claim 1.7. *Let $M \subset X$. If there is some $n \in N$, such that $max\{j : (i,j) \in M\} \leq n$, then $cohM = 0$.*

Proof: For any $(i,j) \in M$, if we let $W \in \mathcal{W}_{ij}$ and $W = W_{in+1}$, then $W \cap M = \{(i,j)\}$. So M is a discrete subset of X. Thus, $cohM = 0$. □

Claim 1.8. *Let $C \subset X$. If C is a nowhere dense subset of X, then there is some $i \in N$, such that $cohC \leq i$.*

Proof: The set $C \subset X$, and C is a nowhere dense subset of X. So $\overline{C}^{\circ} = \emptyset$. Hence, $\overline{C} \neq X$. So we have some point $(i,j) \notin \overline{C}$. Thus, we have some $W_{iq} \in \mathcal{W}_{ij}$, such that $C \cap W_{iq} = \emptyset$, where $q \geq j+1$. Thus, any nowhere dense subset C has to be confined to an L shape; more specifically, C has to be a subset of $A = N \times \{1, 2, \ldots, q-1\} \cup \{1, 2, \ldots, i\} \times N$ for some q and i. For any $(p,k) \in A$, if $p \geq i+1$, then the point (p,k) is an isolated point of A. So $A^d \subset \{1, 2, \ldots, i\} \times N$. By Claim 1.5, we know that $\{1, 2, \ldots, i\} \times N$ is a scattered subset of X. Thus, A is a scattered set and $sl(A) \leq i+1$. So $sl(C) \leq i+1$ following from $C \subset A$. Thus, $cohC \leq i$ by Lemma 1.3. □

Claim 1.9. *The cohesion of the space X is ω.*

Proof: From the above discussion, we know that the cohesion of any nowhere dense subset of X is finite, and for each $n \in N$, there is a nowhere dense set M_{n+1} satisfying $cohM_{n+1} = n$. So $cohX = \omega$ by the definition of cohesion. We may easily know that the space X is a T_1-space. □

2. About cohesions of product spaces

Finally, we will say that, even if $cohX_s$ exists for each $s \in S$, $cohX$ may not exist for $X = \prod_{s\in S} X_s$.

Let $C_1 = \{0\}$, so $cohC_1 = 0$.

Let $C_2 = \{0\} \cup \{\frac{1}{n} : n \in N \text{ and } n \geq 2\}$. This gives a sequence of line converging down to point of C_1. We know that $C_2^d = C_1$ and $cohC_2 = 1$.

Suppose we have defined C_n, $n \geq 2$. Next we define C_{n+1}. For each $x \in C_n \setminus C_{n-1}$, we can choose a sequence A_x of line, such that A_x converges to the point x and $A_x \cap C_n = \emptyset$. And also A_x and A_y have no common points for different x and y of $C_n \setminus C_{n-1}$.

We let $C_{n+1} = (\cup\{A_x : x \in C_n \setminus C_{n-1}\}) \cup C_n$. So we have $C_{n+1}^d = C_n$ and $cohC_{n+1} = n$.

Let $X = \prod_{n\in N} X_n$, where $X_n = C_n, n \in N$. We know that

$$X_1 \times \prod_{n\geq 2} X_n^d$$

is homeomorphic to X and is a nowhere dense closed set of X. So $cohX$ doesn't exist.

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References

[1] P. A. Cairns, *Cohesion in topological spaces*, Topology Proc. **19** (1994), 37–61.

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