Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 30, No. 1, 2006
Pages 417-421

ON ω-BOUNDEDNESS OF THE VIETORIS HYPERSPACE

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ABSTRACT. It is shown that the Vietoris hyperspace of a locally compact, ω-bounded space is ω-bounded; furthermore, a Tychonoff, ω-bounded, non-normal, non-locally compact space with an ω-bounded hyperspace is constructed.

1. INTRODUCTION

For a Hausdorff space X, let $CL(X)$ denote the collection of nonempty closed subsets of X (the so-called hyperspace of X). The Vietoris topology τ_V on $CL(X)$ has subbase elements of the form

$U^+ = \{A \in CL(X) : A \subset U\}$ and $U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}$,

where U runs through the nonempty open subsets of X; thus, a typical base element of τ_V is

$$\langle U_0, \ldots, U_n \rangle = (\bigcup_{i \leq n} U_i)^+ \cap \bigcap_{i \leq n} U_i^-,$$

with $U_0, \ldots, U_n \subseteq X$ open, $n \in \omega$.

The Vietoris hyperspace has been thoroughly studied, and some of the results belong to stock theorems of general topology ([6], [7], [8]). In particular, for our immediate purposes, note that $(CL(X), \tau_V)$ is Hausdorff iff X is regular, and $(CL(X), \tau_V)$ is compact iff X is compact [8].

In an attempt to shed more light on countable compactness of the Vietoris topology, a stronger property, ω-boundedness, was also

2000 *Mathematics Subject Classification.* Primary 54B20; Secondary 54D20.
Key words and phrases. ω-bounded space, Vietoris topology.

investigated in the hyperspace setting. Recall that a Hausdorff space is *ω-bounded*, provided every countable subset is contained in a compact subset [9].

The following theorem was first proved by James Keesling [5, Theorem 5]; for other proofs see [2, Corollary 2.2] and [4, Theorem 5.3].

Theorem 1.1. *$(CL(X), \tau_V)$ is ω-bounded, if X is normal and ω-bounded.*

In [5], Keesling also asked if the assumption of normality in the previous theorem can be reduced. Partially answering this, Ľ. Holá and H. P. Künzi showed that the Tychonoff plank $(\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$ is a non-normal, ω-bounded space with an ω-bounded hyperspace [4, Example 5.1].

The purpose of this paper is to generalize Keesling's theorem, as well as the above example and, as a byproduct, obtain a Tychonoff, ω-bounded, non-normal, non-locally compact space with an ω-bounded hyperspace. In this respect, it would be interesting to find a Tychonoff ω-bounded space with a non-ω-bounded hyperspace. Note that this cannot be done using the techniques of Jiling Cao, Tsugunori Nogura, and A. H. Tomita [1] for constructing non-countably compact hyperspaces, as the Vietoris hyperspace of an ω-bounded space is always countably compact [4, propositions 3.4 and 3.7].

2. Main results

In what follows, $\overline{S}$ is the closure of $S \subseteq X$ in X. Also, if we refer to topological properties without specifying the underlying space, it is always in X.

Theorem 2.1. *$(CL(X), \tau_V)$ is ω-bounded, if X is locally compact and ω-bounded.*

Proof: Assume that X is not compact, and let $\alpha X = X \cup \{\infty\}$ be the Alexandroff one-point compactification of X. Let

$$\mathscr{A} = \{A_n : n \in \omega\}$$

be a sequence of elements of the hyperspace $(CL(X), \tau_V)$, and denote by $\mathscr{K}$ the $CL(X)$-closure of $\mathscr{A}$. Note that if B is compact, then $B \in CL(\alpha X)$, and if $B \in CL(X)$ is not compact, then $B \cup \{\infty\}$ is the αX-closure of B. Consequently,

$$\alpha\mathscr{A} := \{A_n : A_n \text{ is compact}\} \cup \\ \cup \{A_n \cup \{\infty\} : A_n \text{ is not compact}\} \subseteq CL(\alpha X).$$

If $\alpha\mathscr{K}$ is the $CL(\alpha X)$-closure of $\alpha\mathscr{A}$, then

$$(\star) \quad \alpha\mathscr{K} = \{A \in \mathscr{K} : A \text{ is compact}\} \cup \\ \cup \{A \cup \{\infty\} : A \in \mathscr{K}, A \text{ is not compact}\} :$$

$\boxed{\supseteq}$ If $A \in \mathscr{K}$ is compact, then $A \in CL(\alpha X)$. By local compactness of X, a typical $CL(\alpha X)$-neighborhood of A is of the form $\mathscr{U} = \langle U_0, \dots, U_k\rangle$, where $\overline{\bigcup_{i\le k} U_i}$ is compact. Then $\mathscr{U}$ is also $CL(X)$-open, and if $A_n \in \mathscr{U}$, then A_n is compact. It follows that $A_n \in \alpha\mathscr{A} \cap \mathscr{U}$ and $A \in \alpha\mathscr{K}$.

If $A \in \mathscr{K}$ is not compact, then a typical $CL(\alpha X)$-neighborhood of $A \cup \{\infty\}$ is of the form $\alpha\mathscr{U} = \langle U_0, U_1, \dots, U_k\rangle$, where $U_1, \dots, U_k$ are open with compact closure, $\infty \in U_0$, and $\alpha X \setminus U_0$ is compact. Then $\mathscr{U} = \langle X \cap U_0, U_1, \dots, U_k\rangle$ is a $CL(X)$-neighborhood of A. Let $A_n \in \mathscr{U} \cap \mathscr{A}$. If A_n is compact, then $A_n \in \alpha\mathscr{A} \cap \alpha\mathscr{U}$, if not, then $A_n \cup \{\infty\} \in \alpha\mathscr{A} \cap \alpha\mathscr{U}$; thus, $A \cup \{\infty\} \in \alpha\mathscr{K}$.

$\boxed{\subseteq}$ If $B \in \alpha\mathscr{K}$ is compact, then, as above, by local compactness the elements of $\alpha\mathscr{A}$ converging to B are compact and, consequently, $B \in \mathscr{K}$.

If $B \in \alpha\mathscr{K}$ is not compact, then $A = X \cap B$ is not compact; otherwise, take an open neighborhood U of A with compact closure. Construct two transfinite sequences, $\{U_\nu : \nu < \omega_1\}$ of αX-open sets and $\{B_\nu : \nu < \omega_1\} \subseteq \alpha\mathscr{A}$, respectively, such that for each $\nu < \omega_1$

- $B \subseteq U_\nu$,
- $(B_\nu \setminus U) \cap X \neq \emptyset$,
- $B_\nu \subseteq U_\nu$,
- $B_\nu \nsubseteq U_{\nu'}$ for all $\nu < \nu' < \omega_1$

as follows: put $U_0 = U \cup (\alpha X \setminus \overline{U})$ and choose some

$$B_0 \in \alpha\mathscr{A} \cap U_0^+ \cap (X \setminus \overline{U})^-.$$

Assuming that U_ν and B_ν have been constructed for all $\nu < \nu' < \omega_1$, choose $b_\nu \in (B_\nu \setminus U) \cap X$ for all $\nu < \nu'$ and a compact set K

containing $\{b_\nu : \nu < \nu'\}$. Put

$$U_{\nu'} = U \cup (\alpha X \setminus (\overline{U} \cup K)),$$

and choose some

$$B_{\nu'} \in \alpha\mathscr{A} \cap U_{\nu'}^{+} \cap (X \setminus \overline{U})^{-}.$$

This yields a contradiction, since the B_ν's are distinct and $\alpha\mathscr{A}$ is countable.

Also, $A \in \mathscr{K}$, since if $\mathscr{U} = \langle U_0, \dots, U_k \rangle$ is a $CL(X)$-neighborhood of A and $U = \bigcup_{i \le k} U_i$, then $\alpha\mathscr{U} = (U \cup \{\infty\})^{+} \cap \bigcap_{i \le k} U_i^{-}$ is a $CL(\alpha X)$-neighborhood of B, and $A_n \cup \{\infty\} \in \alpha\mathscr{U} \cap \alpha\mathscr{A}$ implies $A_n \in \mathscr{U} \cap \mathscr{A}$. It means that $B = A \cup \{\infty\}$, and $(\star)$ follows.

The mapping

$$\varphi : \alpha\mathscr{K} \to \mathscr{K} \text{ defined via } \varphi(B) = B \cap X$$

is bijective, by $(\star)$, and continuous, since if $\mathscr{U} = \langle U_0, \dots, U_k \rangle \cap \mathscr{K}$ and $U = \bigcup_{i \le k} U_i$, then

$$\varphi^{-1}(\mathscr{U}) = \alpha\mathscr{K} \cap (U \cup \{\infty\})^{+} \cap \bigcap_{i \le k} U_i^{-}$$

is open in $\alpha\mathscr{K}$.

In conclusion, φ is a continuous bijection from the compact $\alpha\mathscr{K}$ onto the Hausdorff $\mathscr{K}$; thus, φ is a homeomorphism and $\mathscr{K}$ is compact. □

Theorem 2.2. *If X is a topological sum of a locally compact and of a normal space, then the following are equivalent:*

(1) $(CL(X), \tau_V)$ *is ω-bounded;*
(2) X *is ω-bounded.*

Proof: (1)⇒(2) X is a closed subspace of $(CL(X), \tau_V)$.

(2)⇒(1) Let $X = Y \oplus Z$, where Y is locally compact and Z is normal. Since X is ω-bounded, so is Y and Z and, by Theorem 1.1 and Theorem 2.1, $CL(Y)$ and $CL(Z)$ are ω-bounded. Moreover, $CL(Y) \times CL(Z)$ is ω-bounded, since ω-boundedness is productive. Finally, it suffices to observe that $CL(Y \oplus Z)$ is homeomorphic to $CL(Y) \oplus CL(Z) \oplus [CL(Y) \times CL(Z)]$. □

Example 2.3. *There exists a Tychonoff, ω-bounded, non-normal, non-locally compact space X, such that $(CL(X), \tau_V)$ is ω-bounded.*

Proof: Let $Y = (\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$, which is an ω-bounded, locally compact, non-normal space. Let Z be a Σ-product in $\{0,1\}^I$ with z as base point and with I of size ω_1. Then X is normal ([3], 4.5.12(b)) and ω-bounded (since countable subsets embed into 2^ω). Also, Z is not locally compact, since if

$$[z_0] = \{f \in Z : z_0 \subset f\}$$

is a fixed neighborhood of z, where $z_0 = z \restriction_{I_0}$, $I_0 \subset I$ finite, then $\{[z_i] : i \in I \setminus I_0\}$, with $z_i = z \restriction_{I_0 \cup \{i\}}$, is an open cover of $[z_0]$ without a finite subcover.

Then, by Theorem 2.2, $X = Y \bigoplus Z$ is as required. $\square$

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