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**PSEUDOCOMPACT WHYBURN SPACES OF
COUNTABLE TIGHTNESS NEED NOT BE
FRÉCHET**

ANGELO BELLA AND PETR SIMON*

ABSTRACT. We construct under [CH] a Tychonoff pseudo-compact Whyburn space of countable tightness which is not Fréchet.

Tkachuk and Yaschenko in [7] established that every countably compact regular Whyburn space is Fréchet and asked whether every pseudocompact Whyburn space must also be Fréchet. This question was later answered in [6] by Pelant, Tkachenko, Tkachuk and Wilson. The authors constructed a ZFC example of a scattered pseudocompact zero-dimensional Tychonoff Whyburn space which is not Fréchet. They also provided a separable space with the same properties under $\mathfrak{b} = \mathfrak{d}$.

However, these two examples are both spaces of uncountable tightness. So, it is natural to wonder whether it is possible to have the stronger counterexample of a space of countable tightness.

The aim of the present note is just to show that this can actually be done at least assuming the Continuum Hypothesis [CH].

All undefined notions can be found in [4].

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A space X has the *Whyburn property* provided that for any set $A \subseteq X$ and any point $x \in \overline{A} \setminus A$ there exists a set $B \subseteq A$ such that $\overline{B} \setminus A = \{x\}$.

The Whyburn property is an evident generalization of the Fréchet property in the class of Hausdorff spaces (here the convergent sequences play the role of the set B in the previous definition).

In [6] it is shown that such a generalization is proper in the class of pseudocompact Tychonoff spaces. Here we show that this may happen even in the much smaller class of pseudocompact Tychonoff spaces of countable tightness.

Our construction is based on an idea of Oleg Pavlov, originally used to get a pseudocompact space of countable tightness which does not have countable fan tightness [1].

Without any compactness-type restrictions, a Whyburn space of countable tightness may easily fail to be Fréchet. Several interesting examples of this kind can be found in [2]. For instance, this is the case of the topological vector space $C_p([0, 1])$.

As usual, the formula $Q \subseteq^* P$ means that the set $Q \setminus P$ is finite.

A collection $\{f_\beta : \beta < \omega_1\}$ of functions from ω into ω is a *scale of length* ω_1 if $f_\alpha(m) < f_\beta(m)$ for cofinitely many $m \in \omega$ whenever $\alpha < \beta$ and for every map $f : \omega \rightarrow \omega$, there is α such that $f(m) < f_\alpha(m)$ for cofinitely many $m \in \omega$.

[CH] implies the existence of a scale of length ω_1 (see [3]).

Theorem 1 (CH). *There exists a pseudocompact zero-dimensional T_1 Whyburn space of countable tightness which is not Fréchet.*

Proof. As we are assuming [CH], we may fix a P-point $p \in \omega^*$ [5] and we may further suppose that the ultrafilter p has a base of the form $\{F_\alpha : \alpha < \omega_1\}$, where $F_\alpha \subseteq^* F_\beta$ whenever $\beta \leq \alpha$. Fix a scale $\{f_\beta : \beta < \omega_1\}$ in ${}^\omega\omega$.

Let α, β be two countable ordinals and let $G \subseteq \omega \times \omega$. Let us say that the set G is parametrized by $\langle \alpha, \beta \rangle$ if the following holds:

- (1) G is a partial function $\omega \rightarrow \omega$ and $\text{dom}(G)$ is infinite;
- (2) $\text{dom}(G) \cap F_\alpha = \emptyset$;
- (3) for each $\gamma < \alpha$, $\text{dom}(G) \subseteq^* F_\gamma$;
- (4) $G < f_\beta$, i.e. $G(n) < f_\beta(n)$ for each $n \in \text{dom}(G)$;
- (5) for all $\gamma < \beta$, the set $\{n : f_\gamma(n) > G(n)\}$ is finite.

If a set $G \subseteq \omega \times \omega$ is parametrized by $\langle \alpha, \beta \rangle$ then we will write $\phi_1(G) = \alpha$ and $\phi_2(G) = \beta$.

Now, we show that there exists a maximal almost disjoint collection \mathcal{G} of parametrized subsets of $\omega \times \omega$ having the additional property that any set $A \subseteq \omega \times \omega$ is either covered by finitely many members of \mathcal{G} together with a set of the form $n \times \omega$ or contains a member of \mathcal{G} . Let $\{A_\alpha : \alpha < \omega_1\}$ be an enumeration of all subsets of $\omega \times \omega$.

We will construct by induction for any $\alpha < \omega_1$ an almost disjoint family of parametrized sets \mathcal{G}_α in such a way that for any $\beta < \alpha$ the set A_β is either covered by finitely many members of \mathcal{G}_α together with a set of the form $n \times \omega$ or contains a member of \mathcal{G}_α . Put $\mathcal{G}_0 = \emptyset$ and assume to have already defined the families $\{\mathcal{G}_\alpha : \alpha < \gamma\}$ and let $\mathcal{G}'_\gamma = \bigcup\{\mathcal{G}_\alpha : \alpha < \gamma\}$. If γ is a limit ordinal then put $\mathcal{G}_\gamma = \mathcal{G}'_\gamma$. If $\gamma = \alpha + 1$ and the set A_α can be covered by finitely many members of $\mathcal{G}'_\gamma \cup \{n \times \omega : n < \omega\}$ then put $\mathcal{G}_\gamma = \mathcal{G}'_\gamma$. In the remaining case, let $\mathcal{G}'_\gamma = \{H_n : n < \omega\}$ and for each $n < \omega$ pick a point $a_n \in A_\alpha \setminus (H_0 \cup \dots \cup H_n \cup n \times \omega)$ and let $A' = \{a_n : n < \omega\}$. Clearly, $\text{dom}(A')$ is infinite. Since p is an ultrafilter, we cannot have $\text{dom}(A') \subseteq^* F_\alpha$ for each $\alpha < \omega_1$ and therefore we may fix the smallest ordinal α such that $\text{dom}(A') \setminus F_\alpha$ is infinite. Put $B = \text{dom}(A') \setminus F_\alpha$ and denote by C a partial function $\omega \rightarrow \omega$ satisfying $\text{dom}(C) = B$ and $C \subseteq A'$. Since the family $\{f_\beta : \beta < \omega_1\}$ is a scale, we can find an ordinal $\beta < \omega_1$ which is the smallest ordinal for which the set $D = \{n \in B : C(n) < f_\beta(n)\}$ is infinite. Then let $G = C \cap D \times \omega$. It is immediate to check that the set G is parametrized by the pair $\langle \alpha, \beta \rangle$ and obviously $G \subseteq A_\alpha$. To complete the inductive construction, it is now enough to put $\mathcal{G}_\gamma = \mathcal{G}'_\gamma \cup \{G\}$. Finally, let $\mathcal{G} = \bigcup\{\mathcal{G}_\alpha : \alpha < \omega_1\}$. Notice that \mathcal{G} is already a maximal family of almost disjoint parametrized sets because any parametrized set is either covered by finitely many members of \mathcal{G} or contains a member of it.

For any $\alpha < \omega_1$, let $R_\alpha = \{(n, x) \in \omega \times (\omega + 1) : n \notin F_\alpha, x \geq f_\alpha(n)\} \cup \{G \in \mathcal{G} : \phi_1(G) \leq \alpha < \phi_2(G)\}$.

For any $G \in \mathcal{G}$ satisfying the relation $\phi_1(G) \geq \phi_2(G)$, let us fix a partition $G = G' \cup G''$ in two infinite sets. If G satisfies $\phi_1(G) < \phi_2(G)$ then we put $G' = G$.

Let $X = \omega \times (\omega + 1) \cup \mathcal{G} \cup \{a\}$, where a is a point not belonging to $\omega \times (\omega + 1) \cup \mathcal{G}$ and define a topology on the set X as follows:

- $\omega \times (\omega + 1)$ is the usual product of a discrete space with a convergent sequence and it is open in X ;
- A neighbourhood base at a point $G \in \mathcal{G}$ consists of all sets $\{G\} \cup G' \setminus D$, where D is finite;
- A neighbourhood subbase at a consists of all sets $X \setminus R_\alpha$ for $\alpha < \omega_1$, of all sets $X \setminus (n \times (\omega + 1))$, for $n \in \omega$, and of all sets $X \setminus (G' \cup \{G\})$, where $G \in \mathcal{G}$.

It is clear that the space $\omega \times (\omega + 1) \cup \mathcal{G}$, defined according to the first two conditions, is T_1 and zero-dimensional. Taking this into account, it is evident that the above neighbourhood assignment gives to the set X a zero-dimensional T_1 -topology upon the verification that any set R_α is clopen in the space $\omega \times (\omega + 1) \cup \mathcal{G}$.

If $n \in \omega$ then the set $\{n\} \times (\omega + 1)$ satisfies either $\{n\} \times (\omega + 1) \cap R_\alpha = \emptyset$ (case $n \in F_\alpha$) or $\{n\} \times (\omega + 1) \setminus R_\alpha = \{n\} \times f_\alpha(n)$. So, we see that a point of the form (n, ω) is in the interior either of $X \setminus R_\alpha$ or of R_α .

Let us try now to do the same for a point $G \in \mathcal{G}$. We have to distinguish three cases.

Case 1: $\alpha < \phi_1(G)$. Because of (3), we have $\text{dom}(G) \subseteq^* F_\alpha$ and consequently $G \cap R_\alpha \subseteq \{(n, G(n)) : n \in \text{dom}(G) \setminus F_\alpha\}$. Since the latter set is finite, we see that the point G is in the interior of $X \setminus R_\alpha$.

Case 2: $\phi_2(G) \leq \alpha$. Because of (4), there is an integer n_0 such that for any $n \geq n_0$ we have $G(n) < f_\alpha(n)$. Consequently $G \cap R_\alpha \subseteq \{(n, G(n)) : n < n_0\}$ and so G is in the interior of $X \setminus R_\alpha$.

Case 3: $\phi_1(G) \leq \alpha < \phi_2(G)$, i.e. $G \in R_\alpha$. Because of (2) and (5), we have that both sets $\text{dom}(G) \cap F_\alpha$ and $\{n \in \text{dom}(G) : G(n) < f_\alpha(n)\}$ are finite. Now, if $k \in \text{dom}(G) \setminus (\text{dom}(G) \cap F_\alpha \cup \{n : G(n) < f_\alpha(n)\})$, we immediately see that $(k, G(k)) \in R_\alpha$ and hence $G \setminus R_\alpha$ is finite. So, the point G is in the interior of R_α .

Now that we have proved that X is a zero-dimensional T_1 -space, we may proceed to the verification of the other required properties.

- X is pseudocompact. For this, it suffices to show that any infinite subset $A \subseteq \omega \times \omega$ has an accumulation point in X . If $\text{dom}(A)$ is finite, then there is some $n \in \omega$ such that the set $B = A \cap \{n\} \times \omega$ is infinite. So, we have $(n, \omega) \in \overline{B} \subseteq \overline{A}$.

If $\text{dom}(A)$ is infinite then, according to the way we have constructed \mathcal{G} , it follows that A is either covered by finitely many members of \mathcal{G} or contains a member of \mathcal{G} . In both cases, there exists some $G \in \mathcal{G}$ such that either $G' \cap A$ is infinite and consequently $G \in \overline{G' \cap A}$ or $G'' \cap A$ is infinite and consequently $a \in \overline{G'' \cap A}$. The latter case happens because the relation $\phi_1(G) \geq \phi_2(G)$ implies that $G \cap R_\alpha$ is finite for each $\alpha < \omega_1$. Take also into account that for any $H \in \mathcal{G}$ the set $G'' \cap H'$ is finite. The proof that X is pseudocompact is then complete.

- X is not Fréchet. Let us consider the subspace $Y = \omega \times \{\omega\} \cup \{a\} \subseteq X$. Each point of the form (n, ω) is obviously isolated in Y and the trace of the neighbourhood system given at a on Y is the family $\{(F_\alpha \setminus n) \times \{\omega\} : \alpha < \omega_1, n < \omega\}$. Thus, Y is actually homeomorphic to $\omega \cup \{p\}$ as a subspace of $\beta\omega$. It is then clear that no sequence in $\omega \times \{\omega\}$ can converge to a .

- X is a Whyburn space of countable tightness. The space X is first countable at each point different than a . Therefore, we need only to check Whyburn property and countable tightness at the point a . Let $a \in \overline{T}$.

We are reduced to consider three cases.

Case 1: $T \subseteq \mathcal{G}$. Since $\mathcal{G} \cup \{a\}$ is a closed subspace of X with only one accumulation point, it is enough to show that there is a countable $S \subseteq T$ such that $a \in \overline{S}$.

If we put $V_\alpha = \{G \in \mathcal{G} : \phi_1(G) > \alpha \text{ or } \phi_2(G) \leq \alpha\}$, then the family $\{V_\alpha \setminus A : \alpha < \omega_1, A \in [\mathcal{G}]^{<\omega}\}$ is just the trace of a local base at a on the subspace \mathcal{G} .

Subcase 1: the set T' of all elements $G \in T$ for which $\phi_1(G) \geq \phi_2(G)$ is infinite. In this case, it follows that $T' \subseteq^* V_\alpha$ for every $\alpha < \omega_1$ and so a is in the closure of any countable infinite set $S \subseteq T'$.

If subcase 1 fails, then we may assume that $\phi_1(G) < \phi_2(G)$ holds for each $G \in T$. Let γ be the smallest ordinal such that $a \in \overline{\{G \in T : \phi_2(G) < \gamma\}}$.

Subcase 2: γ is a successor ordinal. Then the minimality of γ implies that $a \in \overline{T'}$, where $T' = \{G \in T : \phi_2(G) = \gamma - 1\}$. Since $V_\alpha \cap T' \neq \emptyset$ for each $\alpha < \gamma - 1$, it follows that $\sup\{\phi_1(G) : G \in T'\} = \gamma - 1$. Therefore, for any $n < \omega$ we may pick $G_n \in T'$ in such a way that $\{\phi_1(G_n) : n < \omega\}$ is an increasing sequence converging to $\gamma - 1$. It is clear that $a \in \overline{S}$ where $S = \{G_n : n < \omega\}$.

Subcase 3: γ is limit. Fix $\alpha < \gamma$ and denote the set $\{G \in T : \alpha < \phi_2(G) < \gamma\}$ by T' . Then $a \in \overline{T'}$ by the minimality of γ . Hence $T' \cap V_\alpha \neq \emptyset$. It follows from the definitions of T' and V_α that $\phi_1(G) > \alpha$ for some $G \in T'$. Also we are assuming that $\phi_1(G) < \phi_2(G)$ for every such G . So for every $\alpha < \gamma$, there is $G \in T$ such that $\alpha < \phi_1(G) < \phi_2(G) < \gamma$. Then there is a sequence $S = \{G_n \in P : n \in \omega\}$ such that $\phi_1(G_n) > \phi_2(G_k)$ whenever $n > k$. Then $|S \cap R_\beta| \leq 1$ for every $\beta < \omega_1$. This means that $S \subseteq^* V_\beta$ for every $\beta < \omega_1$ and therefore $a \in \overline{S}$.

Case 2: $T \subseteq \omega \times \{\omega\}$. This is immediate, as $\omega \times \{\omega\} \cup \{a\}$ is a countable closed subset of X with only one accumulation point.

Case 3: $T \subseteq \omega \times \omega$. As we are assuming $a \in \overline{T}$, $\text{dom}(T)$ is infinite. If there is some $T' \subseteq T$ such that $a \in \overline{T'}$ and the set T' is covered by finitely many members of \mathcal{G} together with a set of the form $m \times \omega$, then for some G in this finite family the set $G'' \cap T'$ must be infinite. The reason is that otherwise the set T' would be almost contained in a finite union of sets of the form G' and consequently $a \notin \overline{T'}$. Now, the definition of the topology on X implies that $G'' \cap T' \subseteq T$ is actually a sequence converging to a .

According to the way we have constructed the family \mathcal{G} , we can assume in the sequel that whenever $a \in \overline{T'}$ there exists some $G \in \mathcal{G}$ such that $G \subseteq T'$.

We will finish by showing that there exists some $G \in \mathcal{G}$ such that $G \subseteq T$ and $\phi_1(G) \geq \phi_2(G)$.

Subcase 1: there is some $F \in p$ such that $a \notin \overline{T \cap F \times \omega}$. Let α be the smallest ordinal satisfying $a \notin \overline{T \cap F_\alpha \times \omega}$. Since $a \notin \overline{R_\alpha \cap T}$, we have $a \in \overline{M}$, where $M = (T \setminus R_\alpha) \cap ((\omega \setminus F_\alpha) \times \omega) = \{(m, n) : m \in \omega \setminus F_\alpha, n < f_\alpha(m)\} \cap T$. Observe that any parametrized set G such that $G \subseteq M$ must satisfy both $\phi_1(G) \leq \alpha$ and $\phi_2(G) \leq \alpha$. We claim that there exists some $G \in \mathcal{G}$ such that $G \subseteq M$ and $\phi_1(G) = \alpha$. Of course, such a G will provide us a sequence $G'' \subseteq M$ which converges to a .

The claim is easily verified if α is a successor ordinal. The minimality of α implies that a is in the closure of the set $M' = M \cap F_{\alpha-1} \times \omega$ and any parametrized set $G \subseteq M'$ clearly satisfies $\phi_1(G) = \alpha$. Now, assume that α is a limit ordinal and fix an increasing sequence of ordinals $\{\alpha_n : n < \omega\}$ which is cofinal in α . Put $F_n = \bigcap \{F_{\alpha_i} : 0 \leq i \leq n\}$. Again by the minimality of α , we have $a \in \overline{M \cap F_n \times \omega}$ and we may fix $G_n \in \mathcal{G}$ such that $G_n \subseteq M \cap F_n \times \omega$. Obviously, by passing to a subsequence if necessary, we may assume that the G_n are all distinct. For every $n \in \omega$ choose an infinite set $K_n \subseteq G_n$ with $G_n \setminus K_n$ infinite and such that $K_n \cap K_m = \emptyset$ whenever $n \neq m$. The set $K = \bigcup \{K_n : n < \omega\}$ cannot be covered by finitely many members of \mathcal{G} together with a set of the form $m \times \omega$ and therefore there exists $G \in \mathcal{G}$ such that $G \subseteq K$. The crucial fact here is that $\phi_1(G) = \alpha$. Indeed, if $\phi_1(G) < \alpha$ then there would be some $n < \omega$ such that $G \subseteq K_0 \cup \dots \cup K_n$. And this in turn implies that $|G \cap K_i| = \omega$ for some $i \leq n$. But then the maximality of \mathcal{G} would imply that $G = G_i$, which is impossible since G misses the infinite set $G_i \setminus K$.

Subcase 2: a is in the closure of $T \cap F_\gamma \times \omega$ for every $\gamma < \omega_1$. Choose $G_0 \in \mathcal{G}$ satisfying $G_0 \subseteq T$. Next, choose inductively $G_{n+1} \in \mathcal{G}$ satisfying $G_{n+1} \subseteq T \cap F_{\phi_2(G_n)} \times \omega$. If for some $n < \omega$ we have $\phi_1(G_n) \geq \phi_2(G_n)$, then we are done. In the remaining case, we have $\phi_1(G_n) < \phi_2(G_n) \leq \phi_1(G_{n+1}) < \phi_2(G_{n+1})$ for each $n < \omega$. So, let $\alpha = \sup\{\phi_1(G_n) : n < \omega\} = \sup\{\phi_2(G_n) : n < \omega\}$. For each n we may fix a finite set H_n in such a way that $\text{dom}(G_n \setminus H_n) \cap F_\alpha = \emptyset$, $G_n \setminus H_n < f_\alpha$ and $(G_n \setminus H_n) \cap (G_m \setminus H_m) = \emptyset$ whenever $n \neq m$. Finally, fix an infinite set $K_n \subseteq G_n \setminus H_n$ such that $G_n \setminus K_n$ is infinite and let $K = \bigcup \{K_n : n < \omega\}$. The set K cannot be covered by finitely many members of \mathcal{G} and so there exists $G \in \mathcal{G}$ such that $G \subseteq K$. It is immediate that $\phi_2(G) \leq \alpha$. Moreover, by arguing as at the end of subcase 1, we see that $\phi_1(G) = \alpha$ and again we are done. \square

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DEPARTMENT OF MATHEMATICS – UNIVERSITY OF CATANIA, CITTA' UNIVERSITARIA, VIALE A. DORIA 6, 95125 CATANIA, ITALY
E-mail address: `Bella@dmf.unict.it`

DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICAL LOGIC, CHARLES UNIVERSITY, MALOSTRANSKÉ NÁM. 25, 11800 PRAHA 1, CZECH REPUBLIC
E-mail address: `psimon@ms.mff.cuni.cz`