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ON FINEST SPLITTING AND ADMISSIBLE TOPOLOGIES FOR SOME FUNCTION SPACES

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ABSTRACT. It is known (see [5] and [3]) that the intersection of **all** admissible topologies on the set $C(Y, Z)$ of all continuous maps of a space Y into a space Z coincides with the finest splitting topology. In this paper, in the case where Z is a finite space $\mathbf{F}$ we define some **concrete** admissible topologies whose intersection is also the finest splitting topology. These topologies are also used to characterize splitting topologies on $C(Y, \mathbf{F})$.

1. PRELIMINARIES

For two topological spaces Y and Z we denote by $C(Y, Z)$ the set of all continuous maps of Y into Z and by $C_t(Y, Z)$ the set $C(Y, Z)$ equipped with a topology t.

Let X be a space and $F : X \times Y \to Z$ a continuous map. We denote by F_x the (continuous) map of Y into Z, for which $F_x(y) = F(x, y)$ for every $y \in Y$, and by $\widehat{F}$ the map of X into the set $C(Y, Z)$, for which $\widehat{F}(x) = F_x$ for every $x \in X$.

Let G be a map of X into $C(Y, Z)$. By $\widetilde{G}$ we denote the map of $X \times Y$ into Z, for which $\widetilde{G}(x, y) = G(x)(y)$ for every $(x, y) \in X \times Y$.

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A topology t on $C(Y,Z)$ is called *splitting* if for every space X, the continuity of a map $F : X \times Y \to Z$ implies that of the map $\widehat{F} : X \to C_t(Y,Z)$. A topology t on $C(Y,Z)$ is called *admissible* if for every space X, the continuity of a map $G : X \to C_t(Y,Z)$ implies that of the map $\widetilde{G} : X \times Y \to Z$. (See [1], [4], and [2].)

In what follows, we fix a positive integer n and denote by

$$\mathbf{F} = \{0, 1, ..., n\}$$

a fixed (finite) topological space. By $\mathbf{U}_i$, $i = 0, 1, ..., n$, we denote the open set which is the intersection of all open neighborhoods of the point i in $\mathbf{F}$.

Also, we denote by Y a fixed topological space. For every continuous map $f : Y \to \mathbf{F}$ we set

$$\mathbf{U}_i^f = f^{-1}(\mathbf{U}_i),\ i \in \{0, 1, ..., n\}.$$

(It is possible that $\mathbf{U}_i^f = \emptyset$, for some $i \in \{0, 1, ..., n\}$.)

Finally, for every set I we denote by $\mathcal{F}(I)$ the family of all nonempty finite subsets of I.

It is known (see [5] and [3]) that the intersection of **all** admissible topologies on $C(Y,Z)$ coincides with the finest splitting topology. In connection with this result the following problem arises: to find some **concrete** admissible topologies on $C(Y,Z)$ whose intersection is also the finest splitting topology. This general problem seems to be "difficult" in the case, where Z is an arbitrary space. In this paper, we deal with this problem for the case, where Z is the finite space $\mathbf{F}$. More precisely, for each element f of $C(Y,\mathbf{F})$ we construct an admissible topology on this set such that the intersection of all constructed admissible topologies is the finest splitting topology. These topologies are also used to characterize splitting topologies on $C(Y,\mathbf{F})$.

2. On finest splitting and admissible topologies

Definition 1. Let $f \in C(Y,\mathbf{F})$. An $(n+1)$-tuple $s = (V_0, ..., V_n)$ of open subsets of Y is called an *f-family* if

$$V_i \subseteq \mathbf{U}_i^f,\ i \in \{0, 1, ..., n\}.$$

(We note that some of the sets V_0,...,V_n may be empty.)

Definition 2. Let $s = (V_0, ..., V_n)$ be an $(n+1)$-tuple of open subsets of Y. By $<s>$ we denote the set of all elements $f \in C(Y, \mathbf{F})$ for which s is an f-family, that is

$$<s> = \{f \in C(Y, \mathbf{F}) : V_i \subseteq \mathbf{U}_i^f,\ i \in \{0, 1, ..., n\}\}.$$

(It is possible that $<s> = \emptyset$.)

Definition 3. Let $f \in C(Y, \mathbf{F})$. An $(n+1)$-tuple

$$c = (\{O_{0,j} :\ j \in I_0\}, ..., \{O_{n,j} :\ j \in I_n\})$$

of indexed sets $\{O_{i,j} :\ j \in I_i\}$, $i \in \{0, 1, ..., n\}$, of open subsets of Y is called an *f-system* (*of Y corresponding to* $C(Y, \mathbf{F})$) if

$$\mathbf{U}_i^f = \cup\{O_{i,j} : j \in I_i\}, \quad i \in \{0, 1, ..., n\}.$$

Definition 4. Let $f \in C(Y, \mathbf{F})$,

$$c = (\{O_{0,j} :\ j \in I_0\}, ..., \{O_{n,j} :\ j \in I_n\})$$

an f-system, and $K_i \in \mathcal{F}(I_i)$, $i \in \{0, 1, ..., n\}$. The $(n+1)$-tuple

$$s = (\cup\{W_{0,j} : j \in K_0\}, ..., \cup\{W_{n,j} : j \in K_n\}),$$

where $W_{i,j}$ is an open subset of Y such that $W_{i,j} \subseteq O_{i,j}$ for every $i \in \{0, 1, ..., n\}$ and $j \in K_i$, is called an *f-family* (*determined by the f-system* c). (We note that s is really an f-family.)

Theorem 2.1. *Let f be a fixed element of $C(Y, \mathbf{F})$ and c an f-system. Then, the set consisting of:* (α) *all singletons $\{g\}$, where $g \in C(Y, \mathbf{F})$ and $g \neq f$, and* (β) *all sets of the form $<s>$, where s is an f-family determined by c, is a basis for a topology, denoted by t_c, on $C(Y, \mathbf{F})$.*

Proof. Let

$$c = (\{O_{0,j} :\ j \in I_0\}, ..., \{O_{n,j} :\ j \in I_n\}).$$

Also, let

$$s_1 = (\cup\{W_{0,j}^1 : j \in K_0^1\}, ..., \cup\{W_{n,j}^1 : j \in K_n^1\})$$

and

$$s_2 = (\cup\{W_{0,j}^2 : j \in K_0^2\}, ..., \cup\{W_{n,j}^2 : j \in K_n^2\})$$

be two f-families determined by c. It is sufficient to find an f-family

$$s_3 = (\cup\{W_{0,j}^3 : j \in K_0^3\}, ..., \cup\{W_{n,j}^3 : j \in K_n^3\})$$

determined by c such that

$$<s_3> \subseteq <s_1> \cap <s_2> . \qquad (2.1)$$

To define s_3 we set $K_i^3 = K_i^1 \cup K_i^2$, $i \in \{0, 1, ..., n\}$, and

$$W_{i,j}^3 = \begin{cases} W_{i,j}^1 \cup W_{i,j}^2, & \text{if } j \in K_i^1 \cap K_i^2 \\ W_{i,j}^1, & \text{if } j \in K_i^1 \setminus K_i^2 \\ W_{i,j}^2, & \text{if } j \in K_i^2 \setminus K_i^1. \end{cases}$$

It is easy to verify that s_3 is an f-family satisfying relation (2.1). $\square$

Theorem 2.2. *Let $f \in C(Y, \mathbf{F})$ and c be an f-system. Then, the topology t_c on $C(Y, \mathbf{F})$ is admissible.*

Proof. Let X be a space and $G : X \to C_{t_c}(Y, \mathbf{F})$ a continuous map. It is sufficient to prove that the map $\widetilde{G} : X \times Y \to \mathbf{F}$ is continuous.

Let $(x, y) \in X \times Y$ and $\widetilde{G}(x, y) = k \in \mathbf{F}$. We need to find open neighborhoods V_x of x in X and V_y of y in Y such that

$$\widetilde{G}(V_x \times V_y) \subseteq \mathbf{U}_k. \qquad (2.2)$$

Suppose that $G(x) = g \in C_{t_c}(Y, \mathbf{F})$. First, we consider the case, where $g \neq f$. Since G is continuous and $\{g\}$ is an open neighborhood of g in $C_{t_c}(Y, \mathbf{F})$, there exists an open neighborhood V_x of x in X such that

$$G(V_x) \subseteq \{g\}.$$

Since

$$k = \widetilde{G}(x, y) = G(x)(y) = g(y) \in \mathbf{U}_k$$

we have $y \in g^{-1}(\mathbf{U}_k) = \mathbf{U}_k^g$. We set $V_y = \mathbf{U}_k^g$. Then, V_y is an open neighborhood of y in Y and, therefore, $(x, y) \in V_x \times V_y$.

We prove relation (2.2). Let $(x', y') \in V_x \times V_y$. Since $G(V_x) \subseteq \{g\}$ and $g(V_y) \subseteq \mathbf{U}_k$ we have

$$\widetilde{G}(x', y') = G(x')(y') = g(y') \in \mathbf{U}_k,$$

which proves relation (2.2).

Now, we suppose that $G(x) = f$. Then,

$$\widetilde{G}(x, y) = G(x)(y) = f(y) \in \mathbf{U}_k$$

and, therefore, $y \in f^{-1}(\mathbf{U}_k) = \mathbf{U}_k^f$. Let

$$c = (\{O_{0,j} : \ j \in I_0\}, ..., \{O_{n,j} : \ j \in I_n\}).$$

Since $\mathbf{U}_k^f = \cup\{O_{k,j} :\ j \in I_k\}$, there exists $j_k \in I_k$ such that

$$y \in O_{k,j_k} \subseteq \mathbf{U}_k^f.$$

We set $V_y = O_{k,j_k}$.

For every $i \in \{0, 1, \dots, n\} \setminus \{k\}$ we take an arbitrary element j_i of I_i and consider the f-family (determined by the f-system c)

$$s = (O_{0,j_0}, ..., O_{k-1,j_{k-1}}, V_y, O_{k+1,j_{k+1}}, ..., O_{n,j_n}).$$

Since the map G is continuous and $<s>$ is an open neighborhood of f in $C_{t_c}(Y, \mathbf{F})$, there exists an open neighborhood V_x of x in X such that

$$G(V_x)\ \subseteq\ <s>\ .$$

Now, we prove relation (2.2). Let $(x', y') \in V_x \times V_y$. By the definition of $\widetilde{G}$, we have $\widetilde{G}(x', y') = G(x')(y')$. Since $x' \in V_x$ and $G(V_x)\ \subseteq\ <s>$,

$$G(x') \in\ <s>\ .$$

Put $g = G(x')$. By Definition 2 and the choice of $<s>$ we have $V_y \subseteq \mathbf{U}_k^g$, that is $g(V_y) \subseteq \mathbf{U}_k$, which means that

$$g(y') = G(x')(y') = \widetilde{G}(x', y') \in \mathbf{U}_k.$$

Thus,

$$\widetilde{G}(V_x \times V_y) \subseteq \mathbf{U}_k,$$

which completes the proof of the theorem. □

Theorem 2.3. *A topology t on $C(Y, \mathbf{F})$ is splitting if and only if for every $f \in C(Y, \mathbf{F})$ and for every f-system c, $t \subseteq t_c$.*

Proof. Let t be a topology on $C(Y, \mathbf{F})$. First, we suppose that t is a splitting topology, f is an arbitrary element of $C(Y, \mathbf{F})$, and c is an arbitrary f-system. By Theorem 3.7 of [2] and Theorem 2.2, $t \subseteq t_c$.

Conversely, suppose that t satisfies the condition of the theorem. We prove that t is splitting.

Let $F : X \times Y \to \mathbf{F}$ be a continuous map. We need to show that the map $\widehat{F} : X \to C_t(Y, \mathbf{F})$ is continuous.

Let $x_0 \in X$, $\widehat{F}(x_0) = f$, and H be an open neighborhood of f in $C_t(Y, \mathbf{F})$. It is sufficient to find an open neighborhood V_{x_0} of x_0 in X such that:

$$\widehat{F}(V_{x_0}) \subseteq H.$$

Since the map F is continuous, for every $i \in \{0, 1, ..., n\}$ and $y \in Y$ with $F(x_0, y) \in \mathbf{U}_i$, there exists an open neighborhood $V^{x_0}_{i,y}$ of x_0 in X and an open neighborhood $O_{i,y}$ of y in Y such that

$$F(V^{x_0}_{i,y} \times O_{i,y}) \subseteq \mathbf{U}_i. \tag{2.3}$$

Therefore,

$$F(\{x_0\} \times O_{i,y}) = \widehat{F}(x_0)(O_{i,y}) = f(O_{i,y}) \subseteq \mathbf{U}_i$$

or equivalently

$$O_{i,y} \subseteq f^{-1}(\mathbf{U}_i) = \mathbf{U}^f_i.$$

Then,

$$\mathbf{U}^f_i \ = \ \cup\{O_{i,y} : y \in \mathbf{U}^f_i\}.$$

Thus, the $(n+1)$-tuple

$$c = (\cup\{O_{0,y} : y \in \mathbf{U}^f_0\}, ..., \cup\{O_{n,y} : y \in \mathbf{U}^f_n\})$$

is an f-system. Since $H \in t$ and $t \subseteq t_c$, by Theorem 2.1 there exists an f-family

$$s = (\cup\{W_{0,y} : y \in K_0 \in \mathcal{F}(\mathbf{U}^f_0)\}, ..., \cup\{W_{n,y} : y \in K_n \in \mathcal{F}(\mathbf{U}^f_n)\})$$

determined by c such that

$$f \in <s> \subseteq \ H.$$

Let

$$V_{x_0} = \cap\{V^{x_0}_{i,y} : i \in \{0, 1, ..., n\}, \ y \in K_i\}.$$

Clearly, $x_0 \in V_{x_0}$. We prove that

$$\widehat{F}(V_{x_0}) \ \subseteq <s> .$$

Indeed, let $x \in V_{x_0}$ and $\widehat{F}(x) = g$. By Definition 2 it is sufficient to prove that

$$\cup\{W_{i,y} : y \in K_i\} \subseteq \mathbf{U}^g_i, \ i \in \{0, ..., n\}.$$

Let

$$z \in \cup\{W_{i,y} : y \in K_i\}.$$

Then, there exists $y \in K_i$ such that

$$z \in W_{i,y} \ \subseteq \ O_{i,y}.$$

Therefore, by relation 2.3,

$$(x, z) \in V^{x_0}_{i,y} \times O_{i,y} \subseteq F^{-1}(\mathbf{U}_i)$$

which means that

$$F(x,z) = \widehat{F}(x)(z) = g(z) \in \mathbf{U}_i$$

or equivalently

$$z \in g^{-1}(\mathbf{U}_i) = \mathbf{U}_i^g,$$

which complete the proof of the theorem. □

Corollary 2.4. *The finest splitting topology on $C(Y, \mathbf{F})$ (which is defined as the union of all splitting topologies) is the intersection of all topologies t_c, where c is an f-system and $f \in C(Y, \mathbf{F})$.*

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