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TOPOLOGIES ON POSETS AND FUNCTION SPACES

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Abstract. In this paper, using a modification of the definition of the Scott topology, we define some topologies on posets. Through these topologies, by a standard way, we define topologies on function spaces and generalize known results in a straightforward manner.

1. Preliminaries

In this paper, we denote by $(L, \leq)$ or simply L a fixed poset (that is, a partially ordered set). For every $x \in L$ and $H \subseteq L$ we denote by $\downarrow x$, $\uparrow x$, and $\uparrow H$ the following subsets of L:

$$\downarrow x = \{y \in L : y \leq x\},$$
$$\uparrow x = \{y \in L : x \leq y\},$$

and

$$\uparrow H = \cup\{\uparrow x : x \in H\}.$$

We start given the well known topologies on L: the Scott topology, lower topology, and Lawson topology (see, for example, [6]).

The *Scott topology* on L, denoted here by τ_{Sc}, is defined as follows. A subset H of L is an element of τ_{Sc} if

(α) $H = \uparrow H$ and

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(β) for every directed subset D of L the condition $\sup D \in H$ implies $D \cap H \neq \emptyset$.

The *lower topology* on L, denoted here by τ_{lo}, is the topology for which the sets

$$L \setminus \uparrow x, \ x \in L,$$

form a subbase.

The *Lawson topology* on L, denoted here by τ_{La}, is the topology for which the set

$$\tau_{Sc} \cup \tau_{lo}$$

form a subbase.

Suppose that L is a complete lattice. The *strong Scott topology* on L (see [8]), denoted here by τ_{sSc}, is defined as follows. A subset H of L is an element of τ_{sSc} if:

(1) $H = \uparrow H$.

(2) For every directed subset D of L with $\sup D = 1$, where 1 is the maximum element of L, we have $D \cap H \neq \emptyset$.

A subset B of a space X is called *bounded* (see, for example, [8]) if every open cover of X contains a finite subcover of B.

For every space X we denote by $\mathcal{O}(X)$ the complete lattice consisting of all open subsets of X with inclusion as the partial order.

A space X is called *corecompact* (see, for example, [6]) if for every open neighborhood U of a point $x \in X$ there exists an open neighborhood $V \subseteq U$ of x such that V is bounded in the space U.

In what follows, we denote by Y and Z two fixed topological spaces, by $C(Y,Z)$ the set of all continuous maps of Y into Z, and by $C_t(Y,Z)$ the set $C(Y,Z)$ equipped with a topology t.

By $\mathcal{O}_Z(Y)$ we denote the set of all non empty elements $f^{-1}(V)$ of $\mathcal{O}(Y)$, where $f \in C(Y,Z)$ and $V \in \mathcal{O}(Z)$.

For a continuous map $F : X \times Y \to Z$, where X is a space, we denote by F_x the continuous map of Y into Z such that $F_x(y) = F(x,y)$, $y \in Y$, and by $\widehat{F}$ the map of X into the set $C(Y,Z)$ such that $\widehat{F}(x) = F_x$, $x \in X$.

For a given map $G : X \to C(Y,Z)$ we denote by $\widetilde{G}$ the map of $X \times Y$ into Z such that $\widetilde{G}(x,y) = G(x)(y)$, $(x,y) \in X \times Y$.

A topology t on $C(Y,Z)$ is called *splitting* if for every space X, the continuity of a map $F : X \times Y \to Z$ implies that of the map $\widehat{F} : X \to C_t(Y,Z)$. A topology t on $C(Y,Z)$ is called *admissible* if for every space X, the continuity of a map $G : X \to C_t(Y, Z)$ implies that of the map $\widetilde{G} : X \times Y \to Z$ (see [1], [2], and [5].)

The *Isbell topology* (see, for example, [7], [9], and [4]) (respectively, the *strong Isbell topology* (see [8])) on $C(Y, Z)$, denoted here by t_{Is} (respectively, by t_{sIs}), is the topology for which the family of all sets of the form

$$(H, U) = \{f \in C(Y, Z) : f^{-1}(U) \in H\},$$

where H is an element of the topology τ_{Sc} (respectively, τ_{sSc}) on $\mathcal{O}(Y)$ and U is an open subset of Z, form a subbase.

The *compact open topology* (see [5]) on $C(Y, Z)$, denoted here by t_{co}, is the topology for which the family of all sets of the form

$$(K, U) = \{f \in C(Y, Z) : f(K) \subseteq U\},$$

where K is a compact subset of Y and U is an open subset of Z, form a subbase.

It is known that $t_{co} \subseteq t_{Is}$ (see, for example, [9] (page 49)).

In this paper, using a modification of the definition of the Scott topology, we define some topologies on posets. Through these topologies, by a standard way, we define topologies on function spaces and generalize known results in a straightforward manner.

2. Topologies on Posets and Function Spaces

Definition 1. Let A be a non empty subset of L. By τ_{Sc}^{A} we denote the family of all subsets H of L such that:

(1) $H = \uparrow H$.

(2) For every directed subset D of L the conditions $\sup D \in H$ and $A \cap \downarrow \sup D \neq \emptyset$ imply $D \cap H \neq \emptyset$.

It is easy to verify that τ_{Sc}^{A} is a topology on L.

Definition 2. Let A be a non empty subset of L. We denote by τ_{La}^{A} the topology on L for which the set

$$\tau_{Sc}^{A} \cup \tau_{lo}.$$

form a subbase.

Remark 2.1. (1) If $x_1, x_2 \in L$ and $x_1 \leq x_2$, then

$$\tau_{Sc}^{\{x_1\}} \subseteq \tau_{Sc}^{\{x_2\}} \text{ and } \tau_{La}^{\{x_1\}} \subseteq \tau_{La}^{\{x_2\}}.$$

(2) For every $d \in L$ we have

$$\tau_{Sc}^{\{d\}} = \cap\{\tau_{Sc}^{\{x\}} : x \in L,\ x \leq d\}$$

and

$$\tau_{La}^{\{d\}} = \cap\{\tau_{La}^{\{x\}} : x \in L,\ x \leq d\}.$$

(3) Let L be a complete lattice, 0 the minimal element of L, and 1 the maximum element of L. Then, we have:
   (a) $\tau_{Sc}^{\{0\}} = \tau_{Sc}$ and $\tau_{La}^{\{0\}} = \tau_{La}$.
   (b) $\tau_{Sc}^{\{1\}} = \tau_{sSc}$.
   (c) $\tau_{Sc} \subseteq \tau_{Sc}^{\{x\}} \subseteq \tau_{sSc}$, for every $x \in L$.
   (d) If $0 \in A \subseteq L$, then $\tau_{Sc} = \tau_{Sc}^{A}$.

(4) If A is a non empty subset of L, then

$$\begin{array}{ccc} \tau_{La} & \subseteq & \tau_{La}^{A} \\ \cup & & \cup \\ \tau_{Sc} & \subseteq & \tau_{Sc}^{A} \end{array}$$

(5) If A_1, A_2 are two subsets of L such that $A_1 \subseteq A_2$, then $\tau_{Sc}^{A_2} \subseteq \tau_{Sc}^{A_1}$ and $\tau_{La}^{A_2} \subseteq \tau_{La}^{A_1}$.

In the case, where L is the complete lattice $\mathcal{O}(Y)$, the topology τ_{Sc}^{A} can be defined as following.

Remark 2.2. Let Y be a space and $\emptyset \neq A \subseteq \mathcal{O}(Y)$. A subset H of $\mathcal{O}(Y)$ is an element of τ_{Sc}^{A} if:

(1) The conditions $U \in H$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in H$.

(2) For every collection $\{U_i : i \in I\}$ of open sets of Y whose union belongs to H and $\downarrow \cup\{U_i : i \in I\} \cap A \neq \emptyset$, there are finitely many elements U_{i_1},..., U_{i_n} of this collection such that:

$$\cup\{U_{i_m} : m \in \{1, ..., n\}\} \in\ H.$$

Definition 3. Let A be a non empty subset of $\mathcal{O}(Y)$. The *topology* t_{Is}^{A} on $C(Y, Z)$ is the topology, which has as a subbase the family of all sets of the form:

$$(H, U) = \{f \in C(Y, Z) : f^{-1}(U) \in H\},$$

where H is an element of the topology τ_{Sc}^{A} on $\mathcal{O}(Y)$ and $U \in \mathcal{O}(Z)$.

Remark 2.3. Using Remark 2.1 it is easy to verify that:

(1) If $V, U \in \mathcal{O}(Y)$ and $V \subseteq U$, then

$$t_{Is}^{\{V\}} \subseteq t_{Is}^{\{U\}}.$$

(2) For every $U \in \mathcal{O}(Y)$ we have

$$t_{Is}^{\{U\}} = \cap\{t_{Is}^{\{V\}} : V \in \mathcal{O}(Y),\ V \subseteq U\}.$$

(3) If A_1, A_2 are two non empty subsets of $\mathcal{O}(Y)$ such that $A_1 \subseteq A_2$, then $t_{Is}^{A_2} \subseteq t_{Is}^{A_1}$.

(4) If A is a subset of $\mathcal{O}(Y)$ and $\emptyset \in A$, then $t_{Is} = t_{Is}^{A}$.

(5) If A is a non empty subset of $\mathcal{O}(Y)$, then $t_{Is} \subseteq t_{Is}^{A}$.

(6) $t_{Is}^{\{\emptyset\}} = t_{Is}$.

(7) $t^{\{Y\}} = t_{sIs}$.

(8) $t_{Is} \subseteq t_{Is}^{\{U\}} \subseteq t_{sIs}$, for every $U \in \mathcal{O}(Y)$.

Theorem 2.1. *If A is a non empty subset of $\mathcal{O}(Y)$ and Z is a T_i-space, where $i = 0, 1, 2$, then $C_{t_{Is}^{A}}(Y, Z)$ is also a T_i-space.*

Proof. The fact that $C_{t_{co}}(Y, Z)$ is a T_i-space, $i = 0, 1, 2$, follows by [3](page 258). Since $t_{co} \subseteq t_{Is}$, by Remarks 2.3(5) we have $t_{co} \subseteq t_{Is}^{A}$ and, therefore, $C_{t_{Is}^{A}}(Y, Z)$ is also a T_i-space. □

Theorem 2.2. *If Y is corecompact, then t_{Is}^{A} is admissible.*

Proof. Since Y is corecompact, t_{Is} is admissible (see, for example, [8] (Theorem 2.2) and [11] (Theorem 2.16)). Thus, by Remarks 2.3(5), t_{Is}^{A} is admissible. □

Theorem 2.3. *If $A = \mathcal{O}_Z(Y)$, then t_{Is}^{A} is splitting.*

Proof. Let X be an arbitrary space and $F : X \times Y \to Z$ a continuous map. We must prove that $\widehat{F} : X \to C_{t_{Is}^{A}}(Y, Z)$ is continuous.

Let $x \in X$, H be an element of the topology τ_{Sc}^{A} on $\mathcal{O}(Y)$, and $U \in \mathcal{O}(Z)$ such that

$$\widehat{F}(x) \in (H, U)$$

or equivalently

$$(\widehat{F}(x))^{-1}(U) \in H.$$

For every $y \in (\widehat{F}(x))^{-1}(U)$ we have:

$$\widehat{F}(x)(y) = F(x, y) \in U.$$

Since the map F is continuous there exists an open neighborhood V_x^y of x in X and an open neighborhood W_y of y in Y such that

$$F(V_x^y \times W_y) \subseteq U.$$

Then, we have

$$(\widehat{F}(x))^{-1}(U) \subseteq \cup\{W_y : y \in (\widehat{F}(x))^{-1}(U)\}$$

Since $(\widehat{F}(x))^{-1}(U) \in H$ and $H \in \tau_{Sc}^A$. we have

$$\cup\{W_y : y \in (\widehat{F}(x))^{-1}(U)\} \in H.$$

On the other hand $(\widehat{F}(x))^{-1}(U) \in A$ and, therefore,

$$\downarrow \cup\{W_y : y \in (\widehat{F}(x))^{-1}(U)\} \cap A \neq \emptyset.$$

Thus, by Remark 2.2 there exist points y_1,..., y_k of Y such that

$$\cup\{W_{y_i} : i \in \{1, 2, ..., k\}\} \in H.$$

We set

$$V_x = \cap\{V_x^{y_i} : i \in \{1, ..., k\}\}$$

and prove that

$$\widehat{F}(V_x) \subseteq (H, U).$$

Let $v \in V_x$. We need to prove that

$$\widehat{F}(v) \in (H, U).$$

To prove this relation it is sufficient to show that

$$\cup\{W_{y_i} : i \in \{1, ..., k\}\} \subseteq (\widehat{F}(v))^{-1}(U).$$

Let $w \in \cup\{W_{y_i} : i \in \{1, ..., k\}\}$ and i be the element of $\{1, ..., k\}$ such that $w \in W_{y_i}$. Since $v \in V_x^{y_i}$ we have:

$$(v, w) \in V_x^{y_i} \times W_{y_i} \subseteq F^{-1}(U).$$

Thus, $(v, w) \in F^{-1}(U)$ and, therefore, $w \in (\widehat{F}(v))^{-1}(U)$ which completes the proof of the theorem. □

Theorem 2.4. *Let $A = \mathcal{O}_Z(Y)$ and $Z = \mathbf{S}$. Then, $t_{Is} = t_{Is}^A$ and, therefore, t_{Is}^A coincide with the greatest splitting topology on $C(Y, \mathbf{S})$.*

Proof. By Theorem 2.3, t_{Is}^A is splitting. Since t_{Is} is the greatest splitting topology (see [10] and [11]), $t_{Is}^A \subseteq t_{Is}$. On the other hand, by Remarks 2.3(5), $t_{Is} \subseteq t_{Is}^A$. Thus, $t_{Is} = t_{Is}^A$. □

It easy to verify the following theorem.

Theorem 2.5. *Let* $A = \mathcal{O}_Z(Y)$*. Then:*
(1) If Y *is locally compact, then* $t_{co} = t_{Is} = t^A_{Is}$.
(2) If Y *is corecompact, then* $t_{co} \subseteq t_{Is} = t^A_{Is}$.

Question. Let Y and Z be arbitrary spaces. Does $\tau_{Sc} = \tau_{Sc}^{\mathcal{O}_Z(Y)}$ and does $t_{Is} = t_{Is}^{\mathcal{O}_Z(Y)}$?

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