Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

TOPOLOGY PROCEEDINGS
Volume 30, No.2, 2006
Pages 523-532

THE ROLE OF NORMALITY IN THE METRIZATION OF MOORE SPACES

H.H. HUNG

Abstract. We take away the property of normality from Bing's collectionwise normality in his celebrated theorem on the metrization of Moore Spaces, and we weaken the notion of normality itself in the equally celebrated theorem of F.B. Jones for the separable case under the assumption of Lusin's. Normality conjectured to be sufficient, if not indispensible, for the metrization of Moore spaces, its role in the theory is hereby questioned. Traylor's theorem that separable metacompact Moore spaces are metrizable is also improved on. There are also two generalizations of Bing's general metrization theorem.

It has been conjectured that if a Moore space fails to be *metrizable*, it cannot be *normal*, an opinion no doubt encouraged by the near-conclusive result for the separable case: Normal separable Moore spaces are metrizable, if $2^\omega < 2^{\omega_1}$ (Theorem 5 of [10]). Strengthening the property of normality to that of *collectionwise normality* in the conjecture, Bing [1] was able to give an affirmative result, arguably (see e.g. §3 of [13]) the most important theorem on the subject. *Taking* the property of normality itself *out* of that of collectionwise normality, I was able in [5] to generalize Bing.

2000 *Mathematics Subject Classification.* 54E30, 54E35, 54D10.
Key words and phrases. Bing's metrization of Moore spaces: Denormalization of collectionwise normality. Jones' Lemma: Separation of two disjoint closed sets with open set and interior of its closure. Traylor's metrization of separable Moore spaces: Point-countable family of open neighbourhoods. Cushioning. Closure-preserving. Assignments of neighbourhoods, Symmetric, Tufted. Metrization.

In this paper, we pursue the idea in [5] to a logical conclusion and have a property, much weaker than collectionwise normality and its generalization in [5], implying by itself no separation axioms of any kind, but sufficient, by itself, nonetheless, to make metrizable spaces of Moore spaces (Theorem 2.1). In the near-conclusive case cited above, where normality alone makes metrizable spaces of separable Moore spaces ($2^\omega < 2^{\omega_1}$), not all of *normality* is indispensible, and we dispose of a good part of it without compromising the conclusion (Cor. 2.3). Thus, the centrality of normality in the metrization of Moore spaces becomes questionable.

Indeed, there has always been the conclusive theorem of Traylor [14] that offers metacompactness instead of normality, as the property that makes separable Moore spaces metrizable. Here too, we have an improvement (Cor. 2.3).

There are also two metrization theorems, qualitatively different from earlier ones in [4] and [7]. The first (Theorem 3.1) follows from our general result on the metrization of Moore spaces in §2 and the second (Theorem 3.3) is included here because of some aspects it shares with the first and the fact that it makes use of the notion of *symmetric neighbourhoods*, one that is of considerable current interest to Nagata [12]. And, it was this second theorem that led me into the research that produced the other results.

Notations, Terminology and Preliminaries

1. A *Moore space* is a T_3 developable space. On an ω_1-*compact* space, every uncountable subset has a limit point.

We recall a characterization of developable spaces.

Theorem 0.1 (Theorem 1 of [5]). *A topological space X is developable if there is a σ-*discrete *family $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ of* closed *sets, each member C of which is contained in a neighbourhood U_C such that if U is a neighbourhood of $\xi \in X$, there is such a $\Gamma \in \mathcal{C}$ that $\xi \in \Gamma \subset U_\Gamma \subset U$. (And conversely.)*

2. A topological space X is *collectionwise normal* if, given a *discrete* family $\mathcal{C}$ of *closed* subsets and, for every $C \in \mathcal{C}$, given a neighbourhood U_C, disjoint from $C' \in \mathcal{C}$ when $C' \neq C$, there is such a family $\mathcal{V} \equiv \{V_C : C \in \mathcal{C}\}$ of neighbourhoods of members of $\mathcal{C}$ that

- *i*) $V_C \subset U_C$ for every $C \in \mathcal{C}$, and
- *ii*) $\{V_C : C \in \mathcal{C}\}$ is *discrete.*

3. We recall a characterization of the property of collectionwise normality.

Theorem 0.2. (Theorem 1 of [6], Lemma 2.5 of [15]). *A topological space X is* collectionwise normal *if, and only if, given a* discrete *family $\mathcal{C}$ of* closed *subsets and, for every $C \in \mathcal{C}$, given a neighbourhood U_C, disjoint from $C' \in \mathcal{C}$ when $C' \neq C$, there is, for every $n \in \mathbb{N}$, such a family $\mathcal{V}_n \equiv \{V_{C,n} : C \in \mathcal{C}\}$ of open subsets that*

- *i) $C \subset \bigcup\{V_{C,n} : n \in \mathbb{N}\} \subset U_C$ for every $C \in \mathcal{C}$, and*
- *ii) for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is* cushioned *in $\{U_C : C \in \mathcal{C}\}$, i.e., for every $\mathcal{B} \subset \mathcal{C}$, $Cl \bigcup\{V_{C,n} : C \in \mathcal{B}\} \subset \bigcup\{U_C : C \in \mathcal{B}\}$.*

4. We recall a result on the metrization of Moore Spaces.

Theorem 0.3 (Theorem 4 of [5]). *A Moore space is metrizable if (and only if) on it (∗) the union of any* discrete *family of* closed *sets can be covered by a σ-*discrete *family of open sets so that every member of the latter family intersects exactly one member of the former family.*

5. We recall a general metrization theorem.

Theorem 0.4 (Theorem 5 of [8], Theorem 3.3 of [9]). *A regular T_0-space X is metrizable if (and only if) there is a neighbourhood $g(n,x)$ of x for every $x \in X$ and every $n \in \mathbb{N}$ such that*

- *i) (ss) for every $\xi \in X$ and an open neighbourhood U, there is an $n \in \mathbb{N}$ such that $\xi \notin g(n,x)$ when $x \notin U$, and*
- *ii) for each $n \in \mathbb{N}$, there is some* closure preserving *partition $\mathcal{P}_n$ of X such that $x \in P \in \mathcal{P}_n \Rightarrow ClP \subset g(n,x)$.*

1. The De-normalization of Collectionwise Normality and the Weakening of Normality

We generalize collectionwise normality, taking the normality part out of it, and define a property designated (†), for want of a better name.

Definition 1.1. A topological space X is said to have property (†) if, given a(n uncountable) *discrete* family $\mathcal{C}$ of *closed* subsets and, for every $C \in \mathcal{C}$, given a neighbourhood U_C, disjoint from $C' \in \mathcal{C}$ when $C' \neq C$, there is, for every $n \in \mathbb{N}$, such a family $\mathcal{V}_n \equiv \{V_{C,n} : C \in \mathcal{C}\}$ of open subsets that

- *i)* $C \subset \bigcup\{V_{C,n} : n \in \mathbb{N}\} \subset U_C$ for every $C \in \mathcal{C}$, and

- *ii*) for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is *cushioned* in $\{ClU_C : C \in \mathcal{C}\}$, i.e., for every $\mathcal{B} \subset \mathcal{C}$, $Cl \bigcup\{V_{C,n} : C \in \mathcal{B}\} \subset \bigcup\{ClU_C : C \in \mathcal{B}\}$.

Remarks. 1. If, in ii), we require that $\mathcal{V}_n$ be cushioned in $\{U_C : C \in \mathcal{C}\}$ as opposed to $\{ClU_C : C \in \mathcal{C}\}$, we would have collectionwise normality again (see Theorem 0.2). The difference apparently small, nonetheless, we see, for one thing, if $|\ \mathcal{C}\ |= 2$, (†) is trivial while the condition in Theorem 0.2 is certainly not.

2. For ω_1-compact X, $\mathcal{C}$ is at most countable and (†) is trivially satisfied.

3. The property (†), unlike collectionwise normality, does not imply normality. It does not imply any of the separation axioms (let alone the property of being collectionwise Hausdorff) and it does not imply the property (∗) in Theorem 0.3 as the following examples show.

Example 1.1. Let $X = \{0\} \cup \Xi$, where Ξ is an arbitrary set. Let $\{\{\xi, 0\} : \xi \in \Xi\} \cup \{\{0\}\}$ be a base for a topology $\mathcal{T}$. $(X, \mathcal{T})$ is not normal or T_1. It does not have property (∗), if Ξ is uncountable. But it does have property (†).

Example 1.2. The *Tychonoff Plank*, $(\omega_1 + 1) \times (\omega + 1)\backslash\{(\omega_1, \omega)\}$, Tychonoff and not normal, has property (†).

We also have the following generalization of collectionwise normality, designated (♯) for want of a better name.

Definition 1.2. A topological space X is said to have property (♯) if, given a(n uncountable) closed discrete subset $Z \subset X$, there is an open neighbourhood V_z for every $z \in Z$ such that

- *i*) $V_z \cap Z = \{z\}$, and
- *ii*) $\mathcal{V} \equiv \{V_z : z \in Z\}$ is point-countable.

Theorem 1.1. *Separable spaces X are ω_1-compact, if they have property* (♯).

Proof. Let Y be a countable dense subset of X. Suppose X is not ω_1-compact. Suppose there is an uncountable closed discrete subset $Z \subset X$. Let there be a V_z for every $z \in Z$ as described in Definition 1.2 above. For each $y \in Y$, the subset $\{z \in Z : y \in V_z\}$ and therefore the subset $W \equiv \bigcup\{\{z \in Z : y \in V_z\} : y \in Y\}$ is countable. But then, for every $z \in Z$, $V_z \cap Y \neq \emptyset$, and $W = Z$ and Z is countable after all. X is therefore ω_1-compact. □

Remarks. Notice there is no mention at all in the proof that X is a Moore Space.

Corollary 1.2. *On separable spaces,* $(\sharp) \Rightarrow (\dagger)$.

We can have a weakening of the property of normality as follows.

Definition 1.3. A topological space X is said to have property $(\ddagger)$ if given a closed subset A contained in an open neighbourhood U_A, there is an open neighbourhood V_A such that $A \subset V_A \subset \text{Int } ClV_A \subset U_A$.

Example 1.3. The *bull-nosed* Tychonoff Plank. Every well-ordered space can be extended to a *transfinite line* (Problem L of Chapter 5 of Kelley [11]). Let the *edges* $\omega_1 \times \{\omega\}$ and $\{\omega_1\} \times \omega$ of the Tychonoff Plank (of Example 1.2 above) be so extended. It is not difficult to see that the resulting space, Tychonoff and not normal, has properties $(\ddagger)$ and $(\dagger)$. The Tychonoff Plank does not have $(\ddagger)$.

Theorem 1.3 *Separable spaces X are ω_1-compact, if they have property $(\ddagger)$, provided $2^\omega < 2^{\omega_1}$.*

Proof. Let Y be a countable dense subset of X. Suppose X is not ω_1-compact. Suppose there is an uncountable closed discrete subset $Z \subset X$. For every $A \subset Z$, we let $U_A = X\backslash(Z\backslash A)$ and there is V_A as described in Definition 1.3. Clearly, if $B \subset Z$, $B \neq A$ and if $x \in (B\backslash A)$, then $x \notin \text{Int } ClV_A$, and $(V_B\backslash ClV_A) \cap Y \neq \emptyset$, i.e., $V_A \mid Y \neq V_B \mid Y$. $\therefore$ $\mid 2^Z \mid \leq \mid 2^Y \mid = 2^\omega$, and $\mid Z \mid < \omega_1$. □

Corollary 1.4. *On separable spaces,* $(\ddagger) \Rightarrow (\dagger)$*, if* $2^\omega < 2^{\omega_1}$.

Theorem 1.5 *A topological space X is collectionwise normal if (and only if) it is normal and has property* $(\dagger)$.

Proof. We are going to prove that X satisfies the condition in Theorem 0.2. Given $\mathcal{C}$ and U_C for every $C \in \mathcal{C}$ as described there, we have, by normality, an open W_C such that $C \subset W_C \subset ClW_C \subset U_C$ for every $C \in \mathcal{C}$. With respect to these W_C's, property $(\dagger)$ asserts the existence, for every $n \in \mathbb{N}$, of $\mathcal{V}_n$ *cushioned* in $\{ClW_C : C \in \mathcal{C}\}$ and $\therefore$ in $\{U_C : C \in \mathcal{C}\}$. □

Remarks: Our theorem breaks up the notion of collectionwise normality into two complementary parts, neither implying the other (See Examples 1.1, 1.2 and 1.3 above as well as Example G of Bing [1], and the George of Fleissner [3]). If one, by itself, has the ability to turn Moore Spaces into metrizable spaces, it would be quite remarkable that the other one also does.

2. Metrization of Moore Spaces

Theorem 2.1. *Moore Spaces X are metrizable if (and only if) they have property* (†).

Proof. We apply (†) to every $\mathcal{C}_n$ and U_C for $C \in \mathcal{C}_n$ described in Theorem 0.1 and there is $\mathcal{V}_n$ as described in Definition 1.1. Clearly, we have a σ-cushioned pair-base [2] in

$$\{\{(V_{C,i}, ClU_C) : C \in \mathcal{C}_n\} : i \in \mathbb{N}, n \in \mathbb{N}\}$$

and X is M_3 [2], or stratifiable, and therefore paracompact and metrizable. □

Remarks: For the purpose of this theorem, because of the first countability of our spaces, the property (†) could be weakened so that ii) in Definition 1.1 asks only that $\mathcal{V}_n$ be *countably* cushioned in $\{ClU_C : C \in \mathcal{C}\}$, i.e., for every *countable* $\mathcal{B} \subset \mathcal{C}$,

$$Cl\bigcup\{V_{C,n} : C \in \mathcal{B}\} \subset \bigcup\{ClU_C : C \in \mathcal{B}\}.$$

Indeed, we could have, instead of property (†), the following:

Given a(n uncountable) *discrete* family $\mathcal{C}$ of *closed* subsets and, for every $C \in \mathcal{C}$, given a neighbourhood U_C, disjoint from $C' \in \mathcal{C}$ when $C' \neq C$, there is, for every $x \in \bigcup\mathcal{C}$, an open neighbourhood U_x, and, for every $C \in \mathcal{C}$, a partition $\{C^n : n \in \mathbb{N}\}$ in such a manner that

$$Cl\bigcup\{U_{x_i} : i \in \mathbb{N}\} \subset \bigcup\{ClU_{C_i} : i \in \mathbb{N}\}$$

whenever $x_i \in C_i^n \subset C_i \in \mathcal{C}$, for some $n \in \mathbb{N}$,
and still have the same conclusion.

With our characterization of developable spaces in Theorem 0.1, the following is obvious.

Theorem 2.2. *ω_1-compact Moore Spaces are second countable and therefore metrizable.*

In view of Theorems 1.1 and 1.3, we have,

Corollary 2.3. *Separable Moore Spaces are metrizable if they have property* (♯) *or, provided $2^\omega < 2^{\omega_1}$, if they have property* (‡).

Remarks. 1. Note in the above (Theorems 1.1 and 1.3) how (♯) compels separable spaces to ω_1-compactness, as (‡) (with the assumption of Lusin's) does, trivializing (†), on the way to metrizability, if the separable spaces are Moore. For general (non-separable) spaces, on the other hand, the hurdle on the road to metrizability requires the full force of (†). To be encouraged on the Normal Moore Space Conjecture by the near success on the part of normality (or (‡)) in the separable case is thus, in my opinion, to have been misled. This of course is not an opinion on the Conjecture itself, but merely a comment on the irrelevance of the near success in the special case to the prospect of success in the general one.

2. While the Conjecture cannot be proved within the usual set theorey, a counterexample has yet to be constructed after almost seventy years. However, in contrast, we can offer an example of a non-metrizable Moore Space that has property (‡), bespeaking the difference between normality on the one hand and (‡) on the other.

Example 2.1 The *hirsute* Niemytzki Space. Let X be the set of points above the line $\{(x,-1) : x \in \mathbb{R}\}$ on the Euclidean plane. To describe the topology $\mathcal{T}$, we write $B((x,y),\delta)$ for the usual open ball of radius δ centred at the point (x,y) on the Euclidean plane. For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, let

$$U_n(x,0) \equiv B((x,\frac{1}{n}),\frac{1}{n}) \cup \{(x,\xi) : \frac{-1}{n} < \xi \le 0\}$$

be an open neighbourhood of the point $(x,0)$ and let

$$U_n(x,y) \equiv \{(x,\xi) :\mid \xi - y \mid < \frac{1}{n} min\{\mid -1-y \mid, \mid 0-y \mid\}\}$$

be an open neighbourhood of the point (x,y), when $-1 < y < 0$. Let $\{U_n(x,y) : n \in \mathbb{N}\}$ be a local base at (x,y) when $-1 < y \le 0$ and $x \in \mathbb{R}$. When $0 < y$, let the point (x,y) have $\{B((x,y):\xi):0<\xi<y\}$ as a local base.

Clearly, the (Moore) Space $(X,\mathcal{T})$ has property (‡) and is Tychonoff even though it is still not normal.

3. General Metrization Theorems

Theorem 3.1. *A T_3-space X is metrizable if there is such a σ-*discrete *family $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ of* closed *sets, for each C of which there is an open V_C and a U_C such that*

- *i) $C \subset V_C \subset U_C$,*
- *ii) $\{V_C : C \in \mathcal{C}_n\}$ is* cushioned *in $\{U_C : C \in \mathcal{C}_n\}$, for every $n \in \mathbb{N}$, and*
- *iii) given an open neighbourhood U of a point $\xi \in X$, there is such a $\Gamma \in \mathcal{C}$ that $\xi \in \Gamma \subset V_\Gamma \subset U_\Gamma \subset U$.*

Corollary 3.2. *A T_3-space X is metrizable if there is such a σ-* discrete *family $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ of* closed *sets, for each C of which there is an open V_C such that*

- *i) $C \subset V_C$,*
- *ii) $\{V_C : C \in \mathcal{C}_n\}$ is* closure-preserving*, for every $n \in \mathbb{N}$, and*
- *iii) given an open neighbourhood U of a point $\xi \in X$, there is such a $\Gamma \in \mathcal{C}$ that $\xi \in \Gamma \subset V_\Gamma \subset U$.*

Given a topological space X. If for every point $x \in X$, we assign a neighbourhood $U(x)$ to it, we have $\mathcal{U} \equiv \{U(x) : x \in X\}$, an assignment of neighbourhoods. We emphasize that those neighbourhoods need not be *open* neighbourhoods. If $y \in U(x) \Leftrightarrow x \in U(y)$ for all $x, y \in X$, we say the assignment is *symmetric*. If $\{T(U) \equiv \{x \in X : U(x) = U\} : U \in \mathcal{U}\}$ is *closure-preserving*, we say the assignment is *tufted*, and speak of the collection $\{T(U) : U \in \mathcal{U}\}$ as the *tufting*.

Theorem 3.3. *A regular T_0-space X is metrizable if, and only if, there is, for every $n \in \mathbb{N}$, an assignment $\mathcal{U}_n \equiv \{U_n(x) : x \in X\}$ of neighbourhoods,* symmetric *and* tufted*, such that given $\xi \in X$ and any neighbourhood U there is such a $\nu \in \mathbb{N}$ that $U_\nu(\xi) \subset U$.*

Proof. The sufficiency of our condition is obvious, if we note that the symmetry of $\mathcal{U}_n$ implies that $ClT(U) \subset U$ for every $U \in \mathcal{U}_n$ and Theorem 0.4 can be invoked. To show the necessity of our condition on the other hand, we note that for every $n \in \mathbb{N}$, there is always a locally finite partition $\mathcal{P}_n$ of the metrizable space X, the diameter of every member of which is $< \frac{1}{n}$. For every $x \in P \in \mathcal{P}_n$, we can let $U'_n(x) \equiv \bigcup\{B(y, \frac{1}{n}) : y \in P\}$ and $U_n(x) \equiv \bigcup\{Q \in \mathcal{P}_n : Q \cap U'_n(x) \neq \emptyset\}$. □

Remarks. Given a σ-discrete open base $\bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$. If, for each $m, n \in \mathbb{N}$, we let $U' \equiv Cl \bigcup\{V \in \mathcal{U}_n : ClV \subset U\}$ for every $U \in \mathcal{U}_m$, 1) we can let $\mathcal{V}_{mn} \equiv \{U' : U \in \mathcal{U}_m\}$, and define a *symmetric* and *tufted* assignment of neighbourhoods on X as follows.

$$U_{m,n}(x) = U, \text{ if } x \in U' \text{ for some } U \in \mathcal{U}_m,$$

$$U_{m,n}(x) = X \backslash \bigcup \mathcal{V}_{mn}, \text{ if } x \notin \bigcup \mathcal{U}_m, \text{ and}$$

$$U_{m,n}(x) = X \backslash \bigcup(\mathcal{V}_{mn} \backslash \{U'\}), \text{ if } x \in U \backslash U' \text{ for some } U \in \mathcal{U}_m.$$

2) We can also let $\mathcal{C}_{m,n} \equiv \{U' : U \in \mathcal{U}_m\}$ and $V_C \equiv U$ if $C = U' \in \mathcal{C}_{m,n}$, and have the σ-discrete family $\mathcal{C} = \bigcup_{m,n \in \mathbb{N}} \mathcal{C}_{m,n}$ of the description in Cor. 3.2. Bing's metrization theorem is therefore a corollary to both Theorems 3.1 and 3.3.

References

[1] R.H. Bing, *Metrization of Topological Spaces*, Canad. J. Math. **3** (1951) 175-186.

[2] J.G. Ceder, *Some Generalizations of Metric Spaces*, Pac. J. Math. **3** (1951) 105-126.

[3] W.G. Fleissner, *A Normal Collectionwise Hausdorff, not Collectionwise Normal Space*, Gen. Top. Appl. **6** (1976) 57-64.

[4] H.H. Hung, *A Contribution to the Theory of Metrization*, Can. J. Math. **29** (1977) 1145-1151.

[5] H.H. Hung, *On Developability*, Proc. Amer. Math. Soc. **85** (1982) 275-279.

[6] H.H. Hung, *Collectionwise Normality and Fragmented Collectionwise Normality*, Coll. Math. **49** (1984) 11-14.

[7] H.H. Hung, *Topological Symmetry and the Theory of Metrization*, Papers on General Topology and Applications, Ann. NY Acad. Sci. (1996) 207-213.

[8] H.H. Hung, *Enmeshment of Structures and Characterizations of Generalizations of Metric Spaces*, Q. and A. Gen. Top. **21** (2003) 197-202.

[9] H.H. Hung, *Bases, Networks and g-functions*, Q. and A. Gen. Top. **23** (2005), 55-63.

[10] F.B. Jones, *Concerning Normal and Completely Normal Spaces*, Bull. Amer. Math. Soc. **43** (1937) 671-679.

[11] J.L. Kelley, General Topology, van Nostrand, Princeton, N.J., 1955.

[12] J. Nagata, *Symmetric Neighbourhood Assignments and Metrizability*, Q. and A. Gen. Top. **22** (2004) 181-184.

[13] M.E. Rudin, *The Metrizability of Normal Moore Spaces, Studies in Topology, Eds. Stavrakas, N.M., and Allen, K.R.*, Acad. Press, Inc., New York 1975.

[14] D.R. Traylor, *Concerning Metrizability of Pointwise Paracompact Moore Spaces*, Canad. J. Math. **16** (1964) 407-411.
[15] P. Zenor, *Some Continuous Separation Axioms*, Fund. Math. **90** (1976) 143-158.

Concordia University, Montréal, Québec, Canada H4B 1R6
E-mail address: `juliahung@videotron.ca`