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A FINE STRUCTURE CONSTRUCTION OF A PERFECTLY NORMAL NON-REALCOMPACT SPACE

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Abstract. Assuming $V = L$, we shall construct a perfectly normal, non-realcompact space of size $\aleph_1$ which is very different from an Ostaszewski space.

0. Introduction

In [5], the author built a forcing extension in which there exists a locally countable, locally compact, perfectly normal, non-realcompact space of size $\aleph_1$ none of whose uncountable subspaces are separable. In this paper, we shall show that $V = L$ implies the existence of such a space. While this will provide an easy way to see its consistency with other axioms, the technique used here has an intrinsic interest and the author believes that it will have other applications.

There are very few examples of perfectly normal non-realcompact spaces. The easiest example is a discrete space of size equal to or larger than the first measurable cardinal. In [7], Ostaszewski built such a space from the club principle ($\clubsuit$). A distinguished property of the space is that every closed set is either countable or co-countable. A regular Hausdorff space with this property is

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called a sub-Ostaszewski space. Assuming ♣, he gave a construction of a perfectly normal, locally countable, locally compact, sub-Ostaszewski space. It is known that other assumptions, such as the existence of $\aleph_1$-many Cohen reals, suffice to build such a space.

In [4], Hernández-Hernández and the author used forcing to build another example of size $\aleph_1$ which has no uncountable separable subspace, which implies that it has no sub-Ostaszewski subspace. In the same paper, they also showed that this space can exist even under $\mathrm{MA} + \neg\,\mathrm{CH}$, which rejects sub-Ostaszewski spaces.

The space in [4] does not share good properties of Ostaszewski's example, such as first countability and local compactness. In [5], the author showed that by modifying the construction, we can construct (a) a locally countable, locally compact, perfectly normal, non-realcompact space and (b) a first-countable, perfectly normal, non-realcompact space consistent with $\mathrm{MA} + \neg\,\mathrm{CH}$.

In this paper, we shall give another proof of the existence of a space as in (a) from the Axiom of Constructibility, $V = L$. The construction is based on the idea of [5]. We apply the fine structure theory in this paper, instead of forcing, to coherently work through the inductive construction. The fine structure theory, developed by Jensen in [6], gives a detailed picture on the structure of L. While it was essentially used in our arguments, we do not know if some of the combinatorial properties satisfied in L already imply the existence of such a space.

The key ingredient is the tree order $\lhd$ defined in Lemma 2.3. It is a modification of the structure built in [2]. Proper forcing can be viewed as a method to coherently extend partial solutions to the total one. The tree order is used to accomplish the coherent extension without forcing. The author expects that this technique can be applied to construct different spaces.

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1. Topology

We say that a Tychonoff space is *realcompact* if and only if it is homeomorphic to a closed subspace of the product of copies of the real line. We shall use the following characterization. Let X be a topological space. We say that $Y \subseteq X$ is a *zero-set* if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $Y = f^{\leftarrow}\{0\}$. A *z-filter* is a filter consisting of zero-sets. A maximal z-filter is called a *z-ultrafilter*. A filter $\mathcal{F}$ is said to be *fixed* if and only if there exists a singleton in $\mathcal{F}$. Otherwise, it is called *free*. Then, a Tychonoff space is realcompact if and only if every z-ultrafilter with countable intersection property is fixed. It is known that a Hausdorff space is perfectly normal if and only if every closed set is a zero-set. Thus, for a perfectly normal space, a z-filter just means a filter on the closed sets.

A topology associated with a guessing sequence was defined by Eklof, Mekler, and Shelah in [3]. It was fruitfully used by Herández-Herández and the author in [4] and further exploited in [5]. In this section, the definition and necessary lemmas from these papers are presented.

Definition 1.1. A sequence $\vec{C} = \langle C_\gamma : \gamma \in \alpha \cap \mathrm{Lim}\rangle$ is called a *guessing sequence on an ordinal* α if and only if each C_γ is an unbounded subset of γ.

The most typical case is $\alpha = \omega_1$. In this case, a guessing sequence is a candidate for a $\diamondsuit$-sequence, a $\clubsuit$-sequence, and a club guessing sequence on ω_1. However, a bounded version is often used in this paper.

We define a topology associated with a guessing sequence as follows. When x and y are sets of ordinals, we say that x is almost contained in y and write $x \subseteq^* y$ if and only if there exists a $\zeta < \sup(x)$ such that $x \setminus \zeta \subseteq y$, i.e. an end-segment of x is contained in y. We write $x =^* y$ if and only if $x \subseteq^* y$ and $y \subseteq^* x$. Lim stands for the class of limit ordinals.

Definition 1.2. Let $\vec{C} = \langle C_\gamma : \gamma \in \alpha \cap \mathrm{Lim}\rangle$ be a guessing sequence on a limit ordinal α. Then the *topology* $\tau(\vec{C})$ *associated with* $\vec{C}$ is a topology on α defined as: U is $\tau(\vec{C})$-open if and only if for every $\gamma \in U \cap \mathrm{Lim}$, $C_\gamma \subseteq^* U$.

It is easy to see that if $\vec{C}$ is a guessing sequence on a limit ordinal α, $(\alpha, \tau(\vec{C}))$ is Hausdorff and locally countable. Moreover, if $\alpha = \omega_1$, every unbounded subset of ω_1 is not separable. As every sub-Ostaszewski space is hereditarily separable, it means that $(\omega_1, \tau(\vec{C}))$ cannot have a sub-Ostaszewski subspace.

The following equivalent conditions between combinatorial properties of $\vec{C}$ and properties of $(\omega_1, \tau(\vec{C}))$ were established in [5].

Lemma 1.3. *Let $\vec{C} = \langle C_\gamma : \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ be a guessing sequence on ω_1. Set $\tau = \tau(\vec{C})$. Then, the following statements hold.*

- *(ω_1, τ) is regular if and only if for each $\gamma \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta < \gamma$ such that $\{\gamma\} \cup (C_\gamma \setminus \zeta)$ is τ-closed.*
- *Whenever (ω_1, τ) is regular,*
  - *(ω_1, τ) is first-countable if and only if for every $\delta \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta < \omega_1$ such that for every $\gamma \in (C_\delta \setminus \zeta) \cap \mathrm{Lim}$, $C_\gamma \subseteq^* C_\delta$, and*
  - *(ω_1, τ) is locally compact if and only if for every $\delta \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta < \omega_1$ such that $C_\delta \setminus \zeta$ is closed in the order topology, and for every $\gamma \in (C_\delta \setminus \zeta) \cap \mathrm{Lim}$, $C_\gamma =^* C_\delta \cap \gamma$.*

For proving the normality and perfectness of $(\omega_1, \tau(\vec{C}))$, the following lemmas proved in [5] are useful.

Lemma 1.4. *Let $\vec{C}$ be a guessing sequence such that $(\omega_1, \tau(\vec{C}))$ is regular. If F and H are pairwise disjoint non-stationary $\tau(\vec{C})$-closed sets, then there exist pairwise disjoint $\tau(\vec{C})$-open sets U_0 and U_1 with $F \subseteq U_0$ and $H \subseteq U_1$.*

Lemma 1.5. *Let $\vec{C}$ be a guessing sequence on ω_1. Suppose that $(\omega_1, \tau(\vec{C}))$ is normal and every stationary $\tau(\vec{C})$-closed set contains a club subset of ω_1. Then $(\omega_1, \tau(\vec{C}))$ is perfect if and only if for every club subset D of ω_1, there exists a club subset E of ω_1 such that $E \subseteq D$ and E is $\tau(\vec{C})$-G_δ.*

2. Basic facts in L

To establish the main results, we depend on the fine structure theory developed by Jensen in [6]. We refer the readers to [2] for further information. In this section, we shall present some basic facts from the theory which will be needed in the coming proofs.

Let $\langle J_\alpha : \alpha \in \mathbf{ON} \rangle$ be the Jensen hierarchy. See [2, VI] for the definition and basic properties. This covers all constructible sets and has some advantages to the L-hierarchy. As in the case of L-hierarchy, we can define a well-ordering $<_J$ on L.

An *amenable set* is a transitive set M such that

(i) for every $x, y \in M$, $\{x, y\} \in M$,
(ii) for every $x \in M$, $\bigcup x \in M$,
(iii) $\omega \in M$,
(iv) for every $x, y \in M$, $x \times y \in M$,
(v) if $R \subseteq M$ is $\Sigma_0(M)$, then for every $x \in M$, $R \cap x \in M$,

A structure $\langle M, A \rangle$ is said to be *amenable* if and only if M is an amenable set and for every $u \in M$, $u \cap A \in M$.

An *admissible set* is defined as an amenable set such that if $R \subseteq M \times M$ is a $\Sigma_0(M)$-relation such that $(\forall x \in M)(\exists y \in M) R(x, y)$, then for every $u \in M$, there exists a $v \in M$ such that $(\forall x \in u)(\exists y \in v) R(x, y)$. An ordinal α is *admissible* if and only if there exists an admissible set M such that $M \cap \mathbf{ON} = \alpha$. The following lemma is standard.

Lemma 2.1 (Jensen [6]). *An ordinal α is admissible if and only if J_α is an admissible set.*

For each $n > 0$ and $\alpha > 0$, the Σ_n*-projectum* ρ^n_α *of* α is defined as the least ordinal $\rho \leq \alpha$ such that there exists a $\Sigma_n(J_\alpha)$ map f such that $f''\omega\rho = \omega\alpha$. We stipulate $\rho^0_\alpha = \alpha$. When $\langle J_\alpha, A \rangle$ is an amenable structure, we define the Σ_n*-projectum* $\rho^n_{\alpha,A}$ *of* $\langle J_\alpha, A \rangle$ to be the largest ordinal $\rho \leq \alpha$ such that $\langle J_\rho, B \rangle$ is amenable for all $B \in \Sigma_n(\langle J_\alpha, A \rangle) \cap \mathcal{P}(J_\rho)$. For $\alpha > 0$ and $n \geq 0$, we can define the *standard code* $A^n_\alpha \subseteq J_{\rho^n_\alpha}$. For the definition, see [2, VI]. We shall use the following facts.

Fact 2.2. Let $\alpha > 0$ and $n \geq 0$.

- For every $m > 0$, $\Sigma_{n+m}(J_\alpha) \cap \mathcal{P}(J_{\rho^n_\alpha}) = \Sigma_m(\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle)$.
- $\rho^{n+1}_\alpha = \rho^1_{\rho^n_\alpha, A^n_\alpha}$.

We say that ν is Σ_n*-singular over* J_β if and only if there exists a cofinal $\Sigma_n(J_\beta)$-function from some $\varepsilon < \nu$ into ν. We say that ν is *singular over* J_β if and only if there exists an $n < \omega$ such that ν is Σ_n-singular over J_β.

The property needed for the construction is extracted in the following lemma.

Lemma 2.3. *Assume $V = L$. Then there exist S^1 and $\lhd$ such that*

(i) *S^1 is a set of admissible ordinals below ω_2,*
(ii) *$\lhd$ is a tree ordering on S^1,*
(iii) *for every $\nu \in S^1$, there exists an ordinal $\alpha(\nu)$ such that $J_\nu \models$ '$\alpha(\nu)$ is the only uncountable cardinal',*
(iv) *if $\sigma \lhd \nu$, then $\alpha(\sigma) < \alpha(\nu)$, and there exists an elementary embedding $\pi_{\sigma,\nu} : J_\sigma \prec J_\nu$ such that $\pi_{\sigma,\nu} \restriction \alpha(\sigma)$ is an identity and $\pi_{\sigma,\nu}(\alpha(\sigma)) = \alpha(\nu)$,*
(v) *for every $\nu \in S_1$, $\{\alpha(\sigma) : \sigma \lhd \nu\}$ is closed in $\alpha(\nu)$,*
(vi) *for every $\nu \in S_1$, if there exists an $\eta \in S_1$ such that $\alpha(\nu) = \alpha(\eta)$ and $\nu < \eta$, then $\{\alpha(\sigma) : \sigma \lhd \nu\}$ is unbounded in $\alpha(\nu)$,*
(vii) *for every $\eta \in S^1$, $S^1 \cap \eta$, $\langle \pi_{\sigma,\nu} : \sigma \lhd \nu, \nu < \eta\rangle$, and $\pi_{\sigma,\nu} \restriction (S_1 \cap \eta)$ are uniformly Σ_1-definable without parameters in J_η (i.e. there are Σ_1-formulas which define these objects in J_η for every $\eta \in S^1$),*
(viii) *there exists a $\mu \in S^1$ such that*
  - *$\alpha(\mu) = \omega_1$,*
  - *there exists an elementary embedding $\pi_\mu : J_\mu \prec J_{\omega_2}$, and*
  - *$\{\alpha(\nu) : \nu \lhd \mu\}$ is club in ω_1.*

In this case, we let $S^0 = \{\alpha(\nu) : \nu \in S^1\}$, and $S_\alpha = \{\nu \in S^1 : \alpha(\nu) = \alpha\}$.

Proof. Let $\mathcal{I}_0$ be the set of all pairs $\langle \alpha, \nu\rangle$ such that $\alpha < \nu < \omega_2$, $\omega < \alpha \leq \omega_1$, ν is an admissible ordinal, and J_ν satisfies 'α is the largest regular cardinal'. Let $\nu < \omega_2$ be an admissible ordinal such that there exists an α such that $\langle \alpha, \nu\rangle \in \mathcal{I}_0$. Define $\beta(\nu)$ to be the least $\beta \geq \nu$ such that ν is singular over J_β, $n(\nu)$ the least $n \geq 1$ such that ν is Σ_n-singular over $J_{\beta(\nu)}$, $\rho(\nu) = \rho_{\beta(\nu)}^{n(\nu)-1}$, and $A(\nu) = A_{\beta(\nu)}^{n(\nu)-1}$. Let $\mathcal{I}$ be the set of all $\langle \alpha, \nu\rangle \in \mathcal{I}_0$ such that $\nu < \rho(\nu)$ and $\rho(\nu)$ is a limit ordinal. Let $S^0 = \{\alpha : \exists \nu(\langle \alpha, \nu\rangle \in \mathcal{I})\}$, $S^1 = \{\nu : \exists \alpha(\langle \alpha, \nu\rangle \in \mathcal{I})\}$, $S_\alpha = \{\nu : \langle \alpha, \nu\rangle \in \mathcal{I}\}$, and for $\nu \in S^1$, let $\alpha(\nu)$ be the unique α such that $\langle \alpha, \nu\rangle \in \mathcal{I}$. Then by a standard argument, we can show that $\nu \leq \rho(\nu) \leq \beta(\nu)$. If $\sigma \in S_{\alpha(\nu)} \cap \nu$, then we also have $\beta(\sigma) < \nu$. For every $\nu \in S^1$, we can show that $\rho_{\beta(\nu)}^{n(\nu)} \leq \alpha(\nu)$.

For each $\nu \in S^1$, let $q(\nu)$ be the $<_J$-least $q \in J_{\rho(\nu)}$ such that every $x \in J_{\rho(\nu)}$ is Σ_1-definable from parameters in $\alpha(\nu) \cup \{q\}$ in $\langle J_{\rho(\nu)}, A(\nu)\rangle$. This is possible since $\rho^1_{\rho(\nu),A(\nu)} = \rho^{n(\nu)}_{\beta(\nu)} \leq \alpha(\nu)$. Define $p(\nu) = \langle q(\nu), \nu, \alpha(\nu)\rangle$.

Define a partial order $\lhd$ on S^1 by: $\sigma \lhd \nu$ if and only if $\alpha(\sigma) < \alpha(\nu)$ and there exists a Σ_1-embedding $\tilde{\pi} : \langle J_{\rho(\sigma)}, A(\sigma)\rangle \prec_1 \langle J_{\rho(\nu)}, A(\nu)\rangle$ such that $\tilde{\pi} \restriction \alpha(\sigma)$ is an identity and $\tilde{\pi}(p(\sigma)) = p(\nu)$. We write $\tilde{\pi}_{\sigma,\nu}$ for the Σ_1-embedding witnessing $\sigma \lhd \nu$ and let $\pi_{\sigma,\nu} = \tilde{\pi}_{\sigma,\nu} \restriction J_\sigma$.

Everything except (viii) was proved in [2, VIII]. To show (viii), set $\gamma = \omega_2 + \omega$ and let X be the Σ_1-Skolem hull of $\omega_1 \times \{\omega_2\}$ in J_γ. Let $\tilde{\pi} : J_\beta \to X$ be the inverse of the collapsing map. Let μ be the unique ordinal such that $J_\beta \models$'$\mu = \omega_2$'. Then it is easy to see that $\beta = \mu + \omega$. Note that every element of X is Σ_1-definable from parameters in $\omega_1 \times \{\omega_2\}$ in J_γ. Thus, every element of J_β is Σ_1-definable from parameters in $\omega_1 \times \{\mu\}$. Therefore, we have $\rho^1_\beta = \omega_1$. It follows that $\beta(\mu) = \beta$ and $n(\mu) = 1$. Thus, $\rho(\mu) = \rho^0_\beta = \beta > \mu$. Then clearly $\mu \in S^1$ and $\alpha(\mu) = \omega_1$. It is easy to find a $\mu' \in S^1$ with $\mu' > \mu$ and $\alpha(\mu') = \omega_1$. Thus, by (v) and (vi), $\{\alpha(\nu) : \nu \lhd \mu\}$ is club in ω_1. Let $\pi_\mu = \tilde{\pi} \restriction J_\mu$. Then clearly $\pi_\mu : J_\mu \prec J_{\omega_2}$. $\square$ (Lemma 2.3)

3. Construction

In this section, we shall prove the following theorem.

Theorem 3.1. *Assume $V = L$. Then there exists a guessing sequence $\vec{C}$ on ω_1 such that $(\omega_1, \tau(\vec{C}))$ is a locally countable, locally compact, perfectly normal, non-realcompact space.*

Proof. We shall build a $\vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ by induction. For notational simplicity, we stipulate $C_\alpha = \varnothing$ for a successor ordinal α.

Suppose that we have defined $\vec{C} \restriction \alpha$ for $\alpha \in (\omega_1 + 1) \cap \mathrm{Lim}$. Let $\tau_\alpha = \tau(\vec{C} \restriction \alpha)$ and $\eta = \sup S_\alpha$. We shall define a filter $\mathcal{F}_\alpha$ on α as follows. Let $\langle F_\gamma : \gamma < \varepsilon\rangle$ be the $<_J$-increasing enumeration of τ_α-closed subsets of α lying in J_η. Inductively, we shall define a function $f : \varepsilon \to 2$ as follows. Suppose that $f \restriction \gamma$ has been defined. If there exists a finite subset s of γ such that for every $\xi \in s$, $f(\xi) = 1$ and $F_\gamma \cap \bigcap_{\xi \in s} F_\xi$ is bounded in α, then set $f(\gamma) = 0$.

Otherwise, let $f(\gamma) = 1$. Then let $\mathcal{F}_\alpha$ be the filter on α generated by $\{F_\gamma : f(\gamma) = 1\}$. Note that for every τ_α-closed set $F \in J_\eta$, $F \in \mathcal{F}_\alpha$ is uniformly $\Delta_1^{J_\beta}$ for every β.

For each $\nu \in S_\alpha$, we say that ν is good for normality if $J_\nu \models$'for every disjoint pair of τ_α-closed sets $K^0 \in \mathcal{F}_\alpha$ and K^1, there exists a τ_α-open set U such that $K^0 \subseteq U$ and $K^1 \cap U = \varnothing$'. Otherwise ν is bad for normality. Similarly, we say that ν is good for perfectness if $J_\nu \models$'every τ_α-closed set $F \in \mathcal{F}_\alpha$ is τ_α-G_δ', and otherwise ν is bad for perfectness. Notice that for $\sigma \lhd \nu$, σ is good for normality (resp. perfectness) if and only if ν is good for normality (resp. perfectness).

In the course of inductive construction, we shall also define for every $\nu \in S^1$, F_ν, $W_{\nu,n}$ $(n < \omega)$, K^0_ν, K^1_ν, and U_ν so that for every $\alpha \in \omega_1 \cap \mathrm{Lim}$,

- (i) C_α is a club subset of α such that for every $\xi \in C_\alpha \cap \mathrm{Lim}$, $C_\xi =^* C_\alpha \cap \xi$,
- (ii) for every τ_α-closed $F \in \mathcal{F}_\alpha$, $C_\alpha \subseteq^* F$,
- (iii) for every $\nu \in S_\alpha$,
  - (a) if ν is bad for perfectness, F_ν is the $<_J$-least τ_α-closed set which belongs to $\mathcal{F}_\alpha$ and is not τ_α-G_δ in J_ν; $F_\nu = \alpha$ otherwise,
  - (b) $\langle W_{\nu,n} : n < \omega\rangle$ is a decreasing sequence of τ_α-open sets such that $\bigcap_{n<\omega} W_{\nu,n} = F_\nu$,
  - (c) if ν is bad for normality, $\langle K^0_\nu, K^1_\nu\rangle$ is the $<_J$-least disjoint pair of τ_α-closed sets such that $K^0_\nu \in \mathcal{F}_\alpha$ and there is no τ_α-clopen set $U \in J_\nu$ such that $K^0_\nu \subseteq U$ and $K^1_\nu \cap U = \varnothing$; otherwise $K^0_\nu = \alpha$ and $K^1_\nu = \varnothing$,
  - (d) U_ν is a τ_α-clopen set such that $K^0_\nu \subseteq U_\nu$ and $K^1_\nu \cap U_\nu = \varnothing$,
  - (e) for every $\sigma \lhd \nu$, $K^i_\sigma = K^i_\nu \cap \alpha(\sigma)$ for $i = 0, 1$, $U_\sigma = U_\nu \cap \alpha(\sigma)$, $F_\sigma = F_\nu \cap \alpha(\sigma)$, and $W_{\sigma,n} = W_{\nu,n} \cap \alpha(\sigma)$ for every $n < \omega$.

Suppose that $\langle C_\xi : \xi \in \alpha \cap \mathrm{Lim}\rangle$ has been defined. When $\alpha \notin S^0$, let C_α be the $<_J$-least unbounded subset of α consisting of successor ordinals. Assume that $\alpha \in S^0$. Fix $\nu \in S_\alpha$. We shall define $F_\nu, W_{\nu,n}, K^0_\nu, K^1_\nu$, and U_ν. If there exists a $\sigma \lhd \nu$, define $\bar{\alpha} = \sup\{\alpha(\sigma) : \sigma \lhd \nu\}$. Otherwise, let $\bar{\alpha} = 0$. When $0 < \bar{\alpha} < \alpha$, let $\bar{\nu} \lhd \nu$ be such that $\alpha(\bar{\nu}) = \bar{\alpha}$.

For the definition of K^0_ν, K^1_ν, and U_ν, there are two cases.

Case 1. ν is good for normality.

Let $K^0_\nu = \alpha$, $K^1_\nu = \varnothing$, and $U_\nu = \alpha$. Since every $\sigma \lhd \nu$ is good for normality, it trivially satisfies the coherence condition.

Case 2. ν is bad for normality.

Then let $\langle K^0_\nu, K^1_\nu \rangle$ be the $<_J$-least disjoint pair of τ_α-closed subsets such that $K^0_\nu \in \mathcal{F}_\alpha$ and there is no τ_α-open set $U \in J_\nu$ such that $K^0_\nu \subseteq U$ and $K^1_\nu \cap U = \varnothing$. We will define U_ν in the following three subcases.

Subcase 1. $\bar{\alpha} = 0$ (i.e. there is no $\sigma \lhd \nu$)

Let U_ν be the $<_J$-least τ_α-clopen subset of α such that $K^0_\nu \subseteq U_\nu$ and $K^1_\nu \cap U_\nu = \varnothing$. There exists such a set because τ_α is regular and countable, hence normal in V.

Before going to the next subcase, we shall show the following claim.

Claim 1. For every $\sigma \lhd \nu$, $K^0_\sigma = K^0_\nu \cap \alpha(\sigma)$ and $K^1_\sigma = K^1_\nu \cap \alpha(\sigma)$.

Proof of Claim 1. Since $\pi_{\sigma,\nu}$ is an elementary embedding, we have $\pi_{\sigma,\nu}(\langle K^0_\sigma, K^1_\sigma \rangle) = \langle K^0_\nu, K^1_\nu \rangle$. Since $\pi_{\sigma,\nu} \restriction \alpha(\sigma)$ is an identity, we have $K^0_\sigma = K^0_\nu \cap \alpha(\sigma)$ and $K^1_\sigma = K^1_\sigma \cap \alpha(\sigma)$. □(Claim 1)

Subcase 2. $0 < \bar{\alpha} < \alpha$.

By Claim 1, $K^0_\nu \cap \bar{\alpha} = K^0_{\bar{\nu}} \subseteq U_{\bar{\nu}}$ and $K^1_\nu \cap U_{\bar{\nu}} = K^1_{\bar{\nu}} \cap U_{\bar{\nu}} = \varnothing$. Let U'_ν be the $<_J$-least τ_α-clopen subset of $(\bar{\alpha}, \alpha)$ such that $K^0_\nu \cap (\bar{\alpha}, \alpha) \subseteq U'_\nu$ and $K^1_\nu \cap U'_\nu = \varnothing$. Define $U_\nu = U_{\bar{\nu}} \cup \{\bar{\alpha}\} \cup U'_\nu$.

Claim 2. $K^0_\nu \subseteq U_\nu$ and $K^1_\nu \cap U_\nu = \varnothing$.

Proof of Claim 2. Since $K^0_\nu \cap \bar{\alpha} \subseteq U_{\bar{\nu}}$ and $K^0_\nu \cap (\bar{\alpha}, \alpha) \subseteq U'_\nu$, we have $K^0_\nu \subseteq U_\nu$. Since $K^0_\nu \cap \bar{\alpha} = K^0_{\bar{\nu}} \in \mathcal{F}_{\bar{\alpha}}$, we have $C_{\bar{\alpha}} \subseteq^* K^0_\nu \cap \bar{\alpha}$. Since K^0_ν is τ_α-closed, this implies $\bar{\alpha} \in K^0_\nu$. Since K^0_ν and K^1_ν are disjoint, we have $\bar{\alpha} \notin K^1_\nu$. Since $(K^1_\nu \cap \bar{\alpha}) \cap U_\nu = K^1_{\bar{\nu}} \cap U_{\bar{\nu}} = \varnothing$ and $(K^1_\nu \cap (\bar{\alpha}, \alpha)) \cap U'_\nu = \varnothing$, we have $K^1_\nu \cap U_\nu = \varnothing$. □(Claim 2)

To see that U_ν is τ_α-open, it suffices to show that $C_{\bar{\alpha}} \subseteq^* U_\nu$. But this is clear since $C_{\bar{\alpha}} \subseteq^* K^0_\nu$.

Subcase 3. $\bar{\alpha} = \alpha$.

Define $U_\nu = \bigcup\{U_\sigma : \sigma \lhd \nu\}$. By the hypothesis (iii) (e), for every $\sigma \lhd \nu$, we have $U_\sigma = U_\nu \cap \alpha(\sigma)$. Thus, $K^0_\nu \subseteq \bigcup\{K^0_\sigma : \sigma \lhd \nu\} \subseteq \bigcup\{U_\sigma : \sigma \lhd \nu\} = U_\nu$ and $K^1_\nu \cap U_\nu = \bigcup\{K^1_\sigma \cap U_\sigma : \sigma \lhd \nu\} = \varnothing$.

For the definition of F_ν and $W_{\nu,n}$, there are also two cases.

Case 1. ν is good for perfectness.

Let $F_\nu = \alpha$ and $W_{\nu,n} = \alpha$ for every $n < \omega$. Since every $\sigma \lhd \nu$ is good for perfectness, the inductive hypothesis trivially holds.

Case 2. ν is bad for perfectness.

Then there exists an $F \in J_\nu \cap \mathcal{F}_\alpha$ such that $J_\nu \models$'F is τ_α-closed but not τ_α-G_δ'. Let F_ν be the $<_J$-least such F.

Subcase 1. $\bar{\alpha} = 0$.

Then let $\langle W_{\nu,n} : n < \omega\rangle$ be the $<_J$-least sequence of τ_α-open sets such that $F_\nu = \bigcap_{n<\omega} W_{\nu,n}$. This is possible since τ_α is regular and countable, and hence perfect.

By a similar argument as in Claim 1, we can prove the following claim.

Claim 3. If $\sigma \lhd \nu$, then $F_\sigma = F_\nu \cap \alpha(\sigma)$.

Subcase 2. $0 < \bar{\alpha} < \alpha$.

Then pick the $<_J$-least decreasing sequence $\langle W'_{\nu,n} : n < \omega\rangle$ of τ_α-open subsets of $(\bar{\alpha}, \alpha)$ such that $\bigcap_{n<\omega} W'_{\nu,n} = F_\nu \cap (\bar{\alpha}, \alpha)$. For each $n < \omega$, define $W_{\nu,n} = W_{\bar{\nu},n} \cup \{\bar{\alpha}\} \cup W'_{\nu,n}$. Since $F_\nu \cap \alpha(\bar{\nu}) = F_{\bar{\nu}} \in \mathcal{F}_{\bar{\alpha}}$, we have $C_{\bar{\alpha}} \subseteq^* F_\nu$. Since F_ν is τ_α-closed, it implies $\bar{\alpha} \in F_\nu$. Thus, $\bigcap_{n<\omega} W_{\nu,n} = F_\nu$. It is easy to see that $W_{\nu,n}$ is τ_α-open for all $n < \omega$.

Subcase 3. $\bar{\alpha} = \alpha$.

Let $W_{\nu,n} = \bigcup\{W_{\sigma,n} : \sigma \lhd \nu\}$ for every $n < \omega$. As in the argument for normality, we can show that it satisfies the inductive hypothesis.

Now we shall define C_α. Let $\langle \nu_k : k < \omega\rangle$ be the $<_J$-least enumeration of S_α and $\langle \gamma_k : k < \omega\rangle$ be the $<_J$-least increasing cofinal sequence in α such that $\{\gamma_k : k < \omega\} \subseteq^* F$ for every $F \in \mathcal{F}_\alpha$ and for every $k < \omega$ and $m \leq k$, $\gamma_k \in F_{\nu_m} \cap K^0_{\nu_m}$.

We shall define $\zeta_k < \gamma_k$ as follows. Fix $k < \omega$. Since $\gamma_k \in \bigcap_{m\le k} F_{\nu_m} \cap K^0_{\nu_m} \subseteq \bigcap_{m\le k} W_{\nu_m,k} \cap U_{\nu_m}$, there exists a $\zeta_k < \gamma_k$ such that $C_{\gamma_k} \setminus \zeta_k \subseteq \bigcap_{m\le k} W_{\nu_m,k} \cap U_{\nu_m}$. Without loss of generality, we may assume that ζ_k is a successor ordinal and if $k > 0$, then $\zeta_k > \gamma_{k-1}$. Define

$$C_\alpha = \bigcup_{k<\omega} (\{\gamma_k\} \cup (C_{\gamma_k} \setminus \zeta_k))$$

This finishes the definition of $\vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$. Let $\tau = \tau(\vec{C})$ and $\mathcal{F} = \mathcal{F}_{\omega_1}$. All objects defined through the induction are definable without parameters in J_{ω_2}. By Lemma 2.3 (viii), there exists a $\mu \in S_{\omega_1}$ such that $\pi_\mu : J_\mu \prec J_{\omega_2}$ and $\{\alpha(\nu) : \nu \lhd \mu\}$ is club in ω_1.

Claim 4. Every τ-closed set $F \in \mathcal{F}$ contains a club subset of ω_1.

Proof of Claim 4. Suppose otherwise. Let F be the $<_J$-least counterexample. Then F belongs to J_μ and is the $<_J$-least counterexample in J_μ. We shall show that for every $\nu \lhd \mu$, $\alpha(\nu) \in F$. This suffices since $\{\alpha(\nu) : \nu \lhd \mu\}$ is club in ω_1. Let $\nu \lhd \mu$. Since $\pi_{\nu,\mu}$ is an elementary embedding and F is definable in J_μ, there exists a $\bar{F} \in J_\nu$ such that $\pi_{\nu,\mu}(\bar{F}) = F$. Clearly, we have $\bar{F} \in \mathcal{F}_{\alpha(\nu)}$ and $\bar{F} = F \cap \alpha(\nu)$. By construction, $C_{\alpha(\nu)} \subseteq^* F \cap \alpha(\nu)$. Thus, $\alpha(\nu)$ is a τ-limit point of F and hence $\alpha(\nu) \in F$. □(Claim 4)

Claim 5. Every stationary τ-closed set belongs to $\mathcal{F}$ and hence contains a club subset of ω_1.

Proof of Claim 5. Let F be the $<_J$-least counterexample, i.e. F is stationary and τ-closed, but $F \notin \mathcal{F}$. Then there exist $F_0, \cdots, F_{k-1} \in \mathcal{F}$ such that $F_i <_J F$ for every $i < k$, and $F \cap \bigcap_{i<k} F_i$ is bounded in ω_1. But, by Claim 4, each F_i contains a club and hence $\bigcap_{i<k} F_i$ contains a club. Since F is stationary, $F \cap \bigcap_{i<k} F_i$ is stationary. This implies $F \in \mathcal{F}$, contradicting the choice of $\mathcal{F}$. □(Claim 5)

We shall show that τ is normal. Suppose the contrary. Then there exists a disjoint pair $\langle K^0, K^1\rangle$ of τ-closed sets such that there exists no τ-clopen set U such that $K^0 \subseteq U$ and $K^1 \cap U = \varnothing$. By Lemma 1.4, either K^0 or K^1 is stationary. Without loss of generality, we may assume that K^0 is stationary. By Claim 5,

K^0 belongs to $\mathcal{F}$ and contains a club. We may assume that $\langle K^0, K^1\rangle$ is the $<_J$-least such pair. Then $\langle K^0, K^1\rangle \in J_\mu$. By construction, for every $\sigma \lhd \nu \lhd \mu$, we have $U_\sigma = U_\nu \cap \alpha(\sigma)$.

Define $U = \bigcup\{U_\nu : \nu \lhd \mu\}$. Then for every $\nu \lhd \mu$, $U_\nu = U \cap \alpha(\nu)$. Since every U_ν is $\tau_{\alpha(\nu)}$-clopen, U is τ-clopen. For every $\nu \lhd \mu$, since $\pi_{\nu,\mu}$ is an elementary embedding, we have $\pi_{\nu,\mu}(\langle K^0_\nu, K^1_\nu\rangle) = \langle K^0, K^1\rangle$. Since $\pi_{\nu,\mu}$ is identity on $\alpha(\nu)$, we have $K^0_\nu = K^0 \cap \alpha(\nu)$ and $K^1_\nu = K^1 \cap \alpha(\nu)$. Thus, $K^0 = \bigcup\{K^0_\nu : \nu \lhd \mu\} \subseteq \bigcup\{U_\nu : \nu \lhd \mu\} = U$ and $K^1 \cap U = \bigcup\{K^1_\nu \cap U_\nu : \nu \lhd \mu\} = \varnothing$. This is a contradiction.

We shall show that τ is perfect. By Lemma 1.5, it suffices to show that every stationary τ-closed set is τ-G_δ. Let F be the $<_J$-least counterexample. As in the proof of normality, we can show that

- for every $\nu \lhd \mu$, then $F \cap \alpha(\nu) = F_\nu$, and
- if $\sigma \lhd \nu \lhd \mu$, then $W_{\sigma,n} = W_{\nu,n} \cap \alpha(\sigma)$ for every $n < \omega$.

Let $W_n = \bigcup\{W_{\nu,n} : \nu \lhd \mu\}$. Then we can show that $F = \bigcap_{n<\omega} W_n$ and each W_n is τ-open. This is a contradiction.

Since τ is perfectly normal, every τ-closed set is a zero-set. Hence $\mathcal{F}$ is a z-ultrafilter with countable intersection property. It follows that (ω_1, τ) is not realcompact. □ (Theorem 3.1)

4. Concluding remarks

We have observed that there are two typical types of perfectly normal, non-realcompact spaces of size $\aleph_1$, namely sub-Ostaszewski spaces and the spaces of the form $(\omega_1, \tau(\vec{C}))$. This leads to the following natural question, which is considered as a basis problem.

Question 1. For every perfectly normal non-realcompact space X of size $\aleph_1$, does it embed either

- a perfectly normal sub-Ostaszewski space, or
- a perfectly normal, non-realcompact space of the form $(\omega_1, \tau(\vec{C}))$?

These two classes of spaces are very different. For example, every sub-Ostaszewski space is hereditarily separable while $(\omega_1, \tau(\vec{C}))$ has no uncountable separable subspace. Such differences may be used to classify perfectly normal non-realcompact spaces and then solve this problem.

The following question, asked by Blair in [1], still remains open.

Question 2. Does there always exist a perfectly normal, non-realcompact space?

If we can find a canonically defined basis for this class of spaces, it will be a great step toward the solution to this problem.

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