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ABOUT LINDELÖF DK–LIKE SPACES AND RELATED CONCLUSIONS

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Abstract. We denote by $\mathbf{K}$ a class of spaces which are hereditary with respect to closed subspaces. Let $\mathbf{1}$ and $\mathbf{W}$ be the classes of all one point spaces and all countable spaces, respectively. We denote by $\mathbf{DK}$ the class of all spaces which are discrete unions of spaces from $\mathbf{K}$. In 1975, R. Telgársky raised the following problem: Is every regular Lindelöf $\mathbf{DK}$–like space a $\mathbf{K}$–like space? The positive answer to the problem was given by R. Telgársky and Y. Yajima in 1983, independently. In this paper, we discuss the non-regular case of Telgársky's problem. We get the following result: There exists a non–regular Lindelöf space X (under CH) which is a $\mathbf{D1}$–like space, but X is not a $\mathbf{1}$–like space. To obtain this result, the concept of weak $\mathbf{K}$–like spaces is introduced. By proving the equivalence of weak $\mathbf{K}$–likeness and $\mathbf{K}$–likeness, we also get the following results: If $(X, \mathcal{T})$ is a $\mathbf{W}$–like space, then $(X, \mathcal{T}_\omega)$ is a $\mathbf{W}$–like space; If $(X, \mathcal{T})$ is a Hausdroff $\mathbf{1}$–like space, then $b(X, \mathbf{T})$ is a $\mathbf{1}$–like space.

Introduction

The concept of the topological games $G(\mathbf{K}, X)$ was introduced and studied by R. Telgársky in 1975(cf.[1]). Recall from [1] that a space X is said to be $\mathbf{K}$–like if player one has a winning strategy in $G(\mathbf{K}, X)$. In [1], R. Telgársky raised the following problem: Is every regular Lindelöf $\mathbf{DK}$–like space a $\mathbf{K}$–like space? The positive answer to this problem was given by R. Telgársky [2] and Y. Yajima [3]. Some other results on $\mathbf{K}$–like spaces may be found in [4]. One part of this paper is to discuss the non–regular case of the above

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problem. We point out that (under CH) there exists a non–regular Lindelöf space X which is **D1**–like, but not **1**–like. So the answer to non-regular case of the Telgársky's problem is negative. To get the above conclusion, the concept of weak **K**–like spaces is introduced. It is proved that weak **K**–like spaces and **K**–like spaces are equivalent.

If $(X, \mathcal{T})$ is a topological space, and $\mathcal{B}$ is a base of $\mathcal{T}$, then we let $\mathcal{T}_\omega$ denote the topology generated by a base $\mathcal{B}_\omega = \{U \setminus C : U \in \mathcal{T}, C \subset X, |C| \leq \omega\}$, and $b(X, \mathcal{T})$ denote the G_δ–topology on X (cf. [5]). We also denote $b(X, \mathcal{T})$ by $(X, \mathcal{T}_b)$. So every open set of $(X, \mathcal{T}_b)$ is unions of G_δ– set of $(X, \mathcal{T})$. As an application of weak **K**–like spaces, we also obtain the following results: If $(X, \mathcal{T})$ is a **W**–like space, then $(X, \mathcal{T}_\omega)$ is a **W**–like space; If $(X, \mathcal{T})$ is a Hausdroff **1**–like space, then $b(X, \mathbf{T})$ is a **1**–like space.

The set of all natural numbers is denoted by N, and natural numbers are denoted by n, m, i, j, k etc. ω is $N \cup \{0\}$. Throughout this paper, except where specifically noted, all spaces are assumed to be T_1. 2^X denotes the collection of all closed subspaces of a space X. In notation and terminology we will follow [6].

1. Non-regular Lindelöf **DK**–like spaces

In order to solve the non–regular case of Telgársky's problem, next we will introduce the concept of weak **K**–like spaces. For a class **K** and a topological space $(X, \mathcal{T})$, let $\mathcal{B}$ be a base of $\mathcal{T}$. We denote by $\mathcal{B}(x) = \{B : x \in B, B \in \mathcal{B}\}$.

Let $WG(\mathbf{K}, X)$ be the following positional game with perfect information. There are two players, player one and player two. They choose alternatively consecutive terms of a sequence $(E_n : n \in \omega)$ of subsets of X, so that each player knows $\mathbf{K}, E_0, E_1, \ldots, E_n$ when it is choosing E_{n+1}.

A sequence $(E_n : n \in \omega)$ of subsets of X is a play of $WG(\mathbf{K}, X)$, if $E_0 = X$ and if for each $n \in \omega$:

(1) E_{n+1} is the choice of player one;
(2) E_{2n+2} is the choice of player two;
(3) $E_{2n+1} \in \mathbf{K}$;
(4) $E_n \in 2^X$;
(5) $E_{2n+1} \subset E_{2n}$;
(6) For each $x \in E_{2n+1}$, player two chooses an open set $U(x) \in \mathcal{B}(x)$, and $E_{2n+2} = E_{2n} \setminus \cup\{U(x) : x \in E_{2n+1}\}$;

If $\cap\{E_{2n} : n \in \omega\} = \emptyset$, then the player one wins the play $(E_n : n \in \omega)$.

A finite sequence $(E_m : m \leq n)$ of subsets of X is admissible sequence for $WG(\mathbf{K}, X)$ if the sequence $(E_0, E_1, \ldots, E_n, 0, 0, \ldots, 0, \ldots)$ is a play of $WG(\mathbf{K}, X)$. A function s is a strategy for player one in $WG(\mathbf{K}, X)$ if the domain consists of admissible sequences $(E_0, \ldots, E_n)$ with n even, such that $s(E_0, \ldots, E_n) = E_{n+1} \in 2^X \cap \mathbf{K}$ and $(E_0, \ldots, E_n, E_{n+1})$ is admissible.

A strategy s is said to be winning for player one, if player one wins each play of $WG(\mathbf{K}, X)$ by s. $WI(\mathbf{K}, X)$ denotes the set of all winning strategies of player one in $WG(\mathbf{K}, X)$. If there is a base $\mathcal{B}$ of X, such that $WI(\mathbf{K}, X) \neq \emptyset$, then X is called a weak **K**–like space. We may say that X is a weak **K**–like space for base $\mathcal{B}$.

If $\mathcal{B} = \mathcal{T}$, then X is said to be a **K**–like space(cf. [1]). The definitions of **K**–like spaces and weak **K**–like spaces are very similar. In the following we will discuss the relation between the two concepts.

Theorem 1. *A space X is a* **K**–*like space if and only if X is a weak* **K**–*like space.*

Proof. It is obvious that every **K**–like space is a weak **K**–like space. Next we will show that every weak **K**–like space is a **K**–like space.

Let $\mathcal{B}$ be a base of X such that $WI(\mathbf{K}, X) \neq \emptyset$. We let $s \in WI(\mathbf{K}, X)$. Next we will define $t \in I(\mathbf{K}, X)$. Let $E_0 = A_0 = X$, $E_1 = s(E_0) \in \mathbf{K}$. Let $t(A_0) = A_1 = E_1$. For any $A_2 \in 2^X$ and $A_2 \cap A_1 = \emptyset$, (A_0, A_1, A_2) is an admissible sequence of $G(\mathbf{K}, X)$. For any $x \in E_1$, player two chooses an element $U(x) \in \mathcal{B}(x)$, such that $U(x) \cap A_2 = \emptyset$. Let $E_2 = X \setminus \cup\{U(x) : x \in E_1\}$. Thus (E_0, E_1, E_2) is an admissible sequence of $WG(\mathbf{K}, X)$, and $A_2 \subset E_2$. X is a weak **K**–like space, we may easily see that there is some $a_1 \in N$, such that $(E_0, E_1, \ldots, E_{2a_1})$ is an admissible sequence of $WG(\mathbf{K}, X)$, satisfying $A_2 \subset E_{2n}$, $E_{2(n-1)+1} \cap A_2 = \emptyset$ for all $n \leq a_1$, and $E_{2a_1+1} = s(E_0, \ldots, E_{2a_1}) \cap A_2 \neq \emptyset$.

For some $n \in N$, we have that $(A_0, A_1, \ldots, A_{2n})$ is an admissible sequence of $G(\mathbf{K}, X)$, and there is an admissible sequence $(E_0, \ldots, E_{2a_1}, \ldots, E_{2a_{(n-1)}}, E_{2a_{(n-1)}+1})$ of $WG(\mathbf{K}, X)$, such that for any $k \leq a_{(n-1)}$, $E_{2k+1} = s(E_0, \ldots, E_{2k})$. For any $m \leq n$, $E_{2a_{(m-1)}+1} \cap A_{2(m-1)} \neq \emptyset$, $A_{2(m-1)+1} = t(A_0, \ldots, A_{2(m-1)}) =$

$E_{2a_{(m-1)}+1} \cap A_{2(m-1)}$. For any $2 \le m \le n$, and $a_{m-2} < d \le a_{m-1}(a_0 = 0), A_{2(m-1)} \subset E_{2d}$, $E_{2(d-1)+1} \cap A_{2(m-1)} = \emptyset$. Thus $A_{2n-1} = E_{2a_{(n-1)}+1} \cap A_{2n-2} \in \mathbf{K}$. For any $x \in E_{2a_{(n-1)}+1}$, player two chooses an element $U(x) \in \mathcal{B}(x)$, such that $U(x) \cap A_{2n} = \emptyset$. Let $E_{2a_{(n-1)}+2} = E_{2a_{(n-1)}} \setminus \cup\{U(x)\colon x \in E_{2a_{(n-1)}+1}\}$. Thus $A_{2n} \subset E_{2a_{(n-1)}+2}$.

Since $s \in WI(\mathbf{K}, X)$, there exists some $a_n \in N, a_n > a_{(n-1)}$, such that $E_{2a_n+1} \cap A_{2n} \neq \emptyset$. $(E_0, \ldots, E_{2a_{(n-1)}}, \ldots, E_{2a_n+1})$ is an admissible sequence of $WG(\mathbf{K}, X)$, and $E_{2l} \supset A_{2n}$, $E_{2l+1} \cap A_{2n} = \emptyset$ for $a_{n-1} < l < a_n$. Thus we let $A_{2n+1} = t(A_0, \ldots, A_{2n}) = E_{2a_n+1} \cap A_{2n} \in \mathbf{K}$.

In this way, we may get a play $(A_n : n \in N)$ of $G(\mathbf{K}, X)$, and a play $(E_0, E_1, \ldots, E_{2a_n}, \ldots)$ of $WG(\mathbf{K}, X)$, such that $A_{2n+1} = t(A_0, \ldots, A_{2n}) = E_{2a_n+1} \cap A_{2n}$. For any $l \in N$, if $a_{n-1} < l \le a_n$, then $A_{2n} \subset E_{2l}$ and $E_{2(l-1)+1} \cap A_{2n} = \emptyset$. For each $n \in \omega$, $E_{2a_n+1} \cap A_{2n} \neq \emptyset$. Since $s \in WI(\mathbf{K}, X)$, we have $\cap\{E_{2m} : m \in \omega\} = \emptyset$. Thus $\cap\{E_{2a_n} : n \in N\} = \emptyset$. So $\cap\{A_{2n} : n \in N\} = \emptyset$. Thus $t \in I(\mathbf{K}, X)$. We have proved that X is a $\mathbf{K}$–like space. □

In the following we will discuss the $\mathbf{W}$–likeness of $(X, \mathcal{T}_\omega)$, if $(X, \mathcal{T})$ is a $\mathbf{W}$–like space.

Theorem 2. *If $(X, \mathcal{T})$ is a $\mathbf{W}$–like space, then $(X, \mathcal{T}_\omega)$ is a $\mathbf{W}$–like space.*

Proof. It is enough to prove that $(X, \mathcal{T}_\omega)$ is a weak $\mathbf{W}$–like space for the base $\mathcal{B}_\omega = \{U \setminus C : U \in \mathcal{T}, C \subset X, |C| \le \omega\}$ by Theorem 1. Let $s \in I(\mathbf{W}, X)$, $E_0 = X$. We will define $t \in WI(\mathbf{W}, X)$. Let $A_0 = X, E_1 = s(E_0) \in \mathbf{W}$. We may assume $E_1 = \{x^1_k : k \in N\}$. Let $A_1 = t(A_0) = E_1$. For each $k \in N$, player two chooses an element $U^1_k \setminus C^1_k$ from $\mathcal{B}_\omega$, such that $x^1_k \in U^1_k \setminus C^1_k$. Let $A_2 = X \setminus \cup\{U^1_k \setminus C^1_k : x^1_k \in A_1\}$. $E_2 = X \setminus \cup\{U^1_k : k \in N\}$. Thus $E_2 \subset A_2 \subset E_0$, and $|A_2 \setminus E_2| \le \omega$. (E_0, E_1, E_2) is an admissible sequence of $G(\mathbf{K}, X)$, (A_0, A_1, A_2) is an admissible sequence of $WG(\mathbf{W}, X)$.

For some $n \in N$, we have an admissible sequence $(A_0, A_1, \ldots, A_{2n})$ of $WG(\mathbf{K}, X)$, satisfying that $E_{2m} \subset A_{2m} \subset E_{2(m-1)}$ and $|A_{2m} \setminus E_{2m}| \le \omega$, $m \le n$. If $m < n$, then $E_{2m+1} = s(E_0, \ldots, E_{2m})$, $A_{2m+1} = (A_{2m} \setminus E_{2m}) \cup E_{2m+1} = \{x^{2m+1}_k : k \in N\}, t(A_0, \ldots, A_{2m}) = A_{2m+1}$. For each $k \in N$, player two chooses a member $U^{2m+1}_k \setminus C^{2m+1}_k$ from $\mathcal{B}_\omega$, such that $x^{2m+1}_k \in U^{2m+1}_k \setminus C^{2m+1}_k$. Let

$A_{2m+2} = E_{2m} \cap (X \setminus \cup \{U_k^{2m+1} \setminus C_k^{2m+1} : x_k^{2m+1} \in A_{2m+1}\})$, $E_{2m+2} = E_{2m} \cap (X \setminus \cup \{U_k^{2m+1} : k \in N\})$. Now we define $t(A_0, A_1, \ldots, A_{2n}) = A_{2n+1} = E_{2n+1} \cup (A_{2n} \setminus E_{2n}) = \{x_k^{2n+1} : k \in N\}$. Player two chooses a member $U_k^{2n+1} \setminus C_k^{2n+1}$ from $\mathcal{B}_\omega$, such that $x_k^{2n+1} \in U_k^{2n+1} \setminus C_k^{2n+1}$. Let $A_{2n+2} = E_{2n} \cap (X \setminus \cup \{U_k^{2n+1} \setminus C_k^{2n+1} : x_k^{2n+1} \in A_{2n+1}\})$, $E_{2n+2} = E_{2n} \cap (X \setminus \cup \{U_k^{2n+1} : k \in N\})$.

Thus $E_{2n+2} \subset A_{2n+2} \subset E_{2n}$, and $|A_{2n+2} \setminus E_{2n+2}| \leq \omega$. In this way, we have a play $(A_n : n \in \omega)$ of $WG(\mathbf{W}, X)$, and a play $(E_n : n \in \omega)$ of $G(\mathbf{K}, X)$, such that $A_{2n+2} \subset E_{2n}$, and $\cap \{E_{2n} : n \in \omega\} = \emptyset$. From the proof we know that $\cap \{A_{2n} : n \in \omega\} \subset \cap \{E_{2n-2} : n \in N\}$. Thus $\cap \{A_{2n} : n \in \omega\} = \emptyset$. Then $t \in WI(\mathbf{W}, X)$. Thus $(X, \mathcal{T}_\omega)$ is a weak $\mathbf{W}$–like space. So X is a $\mathbf{W}$–like space by Theorem 1. □

Problem ([1])**:** Is every regular Lindelöf **DK**–like space a **K**–like space?

The positive answer to the above problem appeared in [2] and [3]. Next we will discuss non–regular case of this problem. The following Explanation shows that the answer to the non–regular case of the Telgársky's problem is negative.

Explanation. In [4], it was noted that there is a Tychonoff space X which is a **W**–like space but it is not a **1**–like space (under CH). We denote this space $(X^1, \mathcal{T}^1)$. We know that $(X^1, \mathcal{T}^1_\omega)$ is a **W**–like space by Theorem 2. Since any countable set of $(X^1, \mathcal{T}^1_\omega)$ is a discrete closed set, we know that $(X^1, \mathcal{T}^1_\omega)$ is a Lindelöf **D1**–like space. Let $f : (X^1, \mathcal{T}^1_\omega) \to (X^1, \mathcal{T}^1)$, such that $f(x) = x$ for any $x \in X$. We may easily see that f is a continuous map. The continuous image of a **1**–like space is also a **1**–like space (cf. [1]). Thus $(X^1, \mathcal{T}_1)$ is a **1**–like space, if $(X_1, \mathcal{T}^1_\omega)$ is a **1**–like space. We have known that $(X^1, \mathcal{T}^1)$ is not a **1**–like space. So $(X^1, \mathcal{T}^1_\omega)$ is not a **1**–like space. A regular Lindelöf **D1**–like space is a **1**–like space (cf.[2-3]), so $(X^1, \mathcal{T}^1_\omega)$ is a non-regular space. Thus $(X^1, \mathcal{T}^1_\omega)$ is a non-regular Lindelöf **D1**–like space, but it is not a **1**–like space.

2. Some applications of weak K–like spaces

If the intersection of countable open sets of a space X is also an open set of X, then the space X is called a P–space (cf.[5]).

It is obvious that $b(X, \mathcal{T})$ is a P–space for any space $(X, \mathcal{T})$.

Theorem 3. *If $(X, \mathcal{T})$ is a Hausdroff* $\mathbf{1}$*–like space, then $(X, \mathcal{T}_b)$ is a* $\mathbf{1}$*–like space.*

Proof. We firstly prove that $(X, \mathcal{T}_b)$ is a $\mathbf{W}$–like space, if $(X, \mathcal{T})$ is a $\mathbf{1}$–like space. We let β_δ be the base of $(X, \mathcal{T}_b)$, that is, every member of β_δ is a G_δ–set of $(X, \mathcal{T})$.

Let $s \in I(\mathbf{1}, X)$, $E_0 = A_0 = X$. Next we will define $t \in WI(\mathbf{W}, X)$. $E_1 = s(E_0) \in \mathbf{1}$. We define $A_1 = E_1 = t(A_0)$. For any $B_1 \in \beta_\delta$, if $s(E_0) \subset B_1$, then $X \setminus B_1 = A_2 = \cup\{E(2, l_1) : l_1 \in N\}$, where $E(2, l_1) \in 2^X$ for any $l_1 \in N$. So $(E_0, E_1, E(2, l_1))$ is an admissible sequence of $G(\mathbf{1}, X)$, and (A_0, A_1, A_2) is an admissible sequence of $WG(\mathbf{1}, X)$. Let $E(3, l_1) = s(E_0, E_1, E(2, l_1)) \in \mathbf{1}$. We let $A_3 = \cup\{E(3, l_1) : l_1 \in N\}$, thus $A_3 \in \mathbf{W}$. We define $t(A_0, A_1, A_2) = A_3$, $D_2 = \{E(2, l_1) : l_1 \in N\}$. Let $B(3, l_1)$ be any set of β_δ such that $E(3, l_1) \subset B(3, l_1)$ for any $l_1 \in N$. Let $A_4 = A_2 \setminus \cup\{B(3, l_1) : l_1 \in N\}$, so $(A_0, A_1, A_2, A_3, A_4)$ is an admissible sequence of $WG(\mathbf{W}, X)$.

For some $n \in N, n \geq 2$, we have $(A_0, \ldots, A_{2n})$ which is an admissible sequence of $WG(\mathbf{W}, X)$, and satisfies the following conditions: For any $2 \leq m < n$,

$D_{2m} = \{E(2k, l_1 l_2 \ldots l_k) : k \in N, k \geq m, l_1, \ldots, l_k \in N\}$, where $(E_0, \ldots, E(2k, l_1 l_2 \ldots l_k))$ is an admissible sequence of $G(\mathbf{1}, X)$, and $E(2k+1, l_1 l_2 \ldots l_k) = s(E_0, \ldots, E(2k, l_1 l_2 \ldots l_k)) \subset A_{2m}$, $A_{2m} \subset \cup D_{2m}$, $A_{2m+1} = \cup\{E(2k+1, l_1 l_2 \ldots l_k) : E(2k+1, l_1 l_2 \ldots l_k) = s(E_0, E_1, \ldots, E(2k, l_1 l_2 \ldots l_k)), E(2k, l_1 \ldots l_k) \in D_{2m}\}$.

For any $x \in A_{2m+1}$, there is a $E(2k, l_1 l_2 \ldots l_k) \in D_{2m}$, such that $E(2k+1, l_1 l_2 \ldots l_k) = \{x\}$. Let B_x be any set of β_δ, such that $x \in B_x$. We let $B(2k+1, l_1 l_2 \ldots l_k) = B_x$, and let $A_{2m+2} = A_{2m} \setminus \cup\{B(2k+1, l_1 l_2 \ldots l_k)$: $E(2k, l_1 l_2 \ldots l_k) \in D_{2m}\}$.

For $m = n - 1$, we have
$A_{2n} = A_{2(m-1)} \setminus \cup\{B(2k+1, l_1 \ldots l_k)$: $E(2k, l_1 \ldots l_k) \in D_{2(n-1)}\}$.

For any $E(2k, l_1 \ldots l_k) \in D_{2(n-1)}$,
$E(2k, l_1 l_2 \ldots l_k) \setminus B(2k+1, l_1 \ldots l_k) = \cup\{E(2k+2, l_1 \ldots l_{k+1}) : l_{k+1} \in N\}$.

For any $l_{k+1} \in N$,
$E(2k+3, l_1 \ldots l_{k+1}) = s(E_0, \ldots, E(2k+2, l_1 \ldots l_{k+1}))$. If $E(2k+3, l_1 \ldots l_{k+1}) \not\subset A_{2n}$, then we let $B(2k+3, l_1 \ldots l_{k+1})$ be any set of β_δ such that $E(2k+3, l_1 \ldots l_{k+1}) \subset B(2k+3, l_1 \ldots l_{k+1})$, and $B(2k+3, l_1 \ldots l_{k+1}) \cap A_{2n} = \emptyset$. Thus $E(2k+2, l_1 \ldots l_{k+1}) \setminus B(2k+3, l_1 \ldots l_{k+1}) = \cup\{E(2k+4, l_1 \ldots l_{k+1} l_{k+2}) : l_{k+2} \in N\}$.

If $s(E_0, \ldots, E(2k+4, l_1 \ldots l_{k+2})) = E(2k+5, l_1 \ldots l_{k+2}) \not\subset A_{2n}$, then we repeat what have done in the previous paragraph with $E(2k+3, l_1 \ldots l_{k+1})$. So we can get a family

$$D_{2n} = \{E(2k, l_1 \ldots l_k) : k \in N, k \geq n, l_1, \ldots, l_k \in N, \\ s(E_0, \ldots, E(2k, l_1 \ldots l_k)) = E(2k+1, l_1 \ldots l_k) \subset A_{2n}\}.$$

Since $s \in I(\mathbf{1}, X)$, we may know that $A_{2n} \subset \cup D_{2n}$.

Let $A_{2n+1} = \cup\{E(2k+1, l_1 \ldots l_k) : E(2k+1, l_1 \ldots l_k) = s(E_0, \ldots, E(2k, l_1 \ldots l_k))$ and $E(2k, l_1 \ldots l_k) \in D_{2n}\}$. So $|A_{n+1}| \leq \omega$. We define $t(A_0, \ldots, A_{2n}) = A_{2n+1}$. For any $x \in A_{2n+1}$, there is some $E(2k, l_1 \ldots l_k) \in D_{2n}$, such that $E(2k+1, l_1 \ldots l_k) = \{x\}$. Let B_x be any element of β_δ, such that $x \in B_x$, we let $B_x = B(2k+1, l_1 \ldots l_k)$. We let $A_{2n+2} = A_{2n} \setminus \cup\{B(2k+1, l_1 \ldots l_k) : E(2k, l_1 \ldots l_k) \in D_{2n}\}$.

In this way we get a play $(A_0, \ldots, A_{2n}, A_{2n+1}, \ldots)$ of $WG(\mathbf{W}, X)$, and satisfies that $A_{2n+1} =$
$t(A_0, \ldots, A_{2n}) = \cup\{E(2k+1, l_1 \ldots l_k) : E(2k+1, l_1 \ldots l_k) \in D_{2n}\}$, where

$D_{2n} = \{E(2k, l_1 \ldots l_k)\colon k \in N, k \geq n, l_1, \ldots, l_k \in N\}$ and
$E(2k+1, l_1 \ldots l_k) = s(E_0, \ldots, E(2k, l_1 \ldots l_k)) \subset A_{2n}$, $A_{2n} \subset \cup D_{2n}$.

In the following we will show that $\cap\{A_{2n} : n \in N\} = \emptyset$.

Suppose there is a point $x \in X$, such that $x \in \cap\{A_{2n} : n \in N\}$. Thus $x \in A_2$. So there is some $l_1 \in N$, such that $x \in E(2, l_1)$. Let $a_1 = 1$. For any $n \in N$, there is some $a_n \in N$, such that $x \in E(2a_n, l_1 l_2 \ldots l_{a_n})$, $E(2a_n, l_1 l_2 \ldots l_{a_n}) \in D_{2n}$. So $E(2a_n+1, l_1 l_2 \ldots l_{a_n}) = s(E_0, \ldots, E(2a_n, l_1 l_2 \ldots l_{a_n})) \subset A_{2n+1}$. $E(2a_n+1, l_1 l_2 \ldots l_{a_n}) \subset B(2a_n+1, l_1 l_2 \ldots l_{a_n})$, and $x \in A_{2n+2}$. Thus $x \in E(2a_n, l_1 l_2 \ldots l_{a_n}) \setminus B(2a_n+1, l_1 l_2 \ldots l_{a_n}) = \cup\{E(2a_n+2, l_1 l_2 \ldots l_{a_n+1}) : l_{a_n+1} \in N\}$. So there exists $l_{a_n+1} \in N$, such that $x \in E(2a_n+2, l_1 l_2 \ldots l_{a_n+1})$. If $E(2a_n+3, l_1 l_2 \ldots l_{a_n+1}) = s(E_0, \ldots, E(2a_n+2, l_1 l_2 \ldots l_{a_n+1})) \subset A_{2n+2}$, then we let $l_{a_n+1} = l_{a_{n+1}}, a_{n+1} = a_n + 1$. Thus $x \in E(2a_{n+1}, l_1 l_2 \ldots l_{a_{n+1}}) \in D_{2(n+1)}$. If $E(2a_n+3, l_1 l_2 \ldots l_{a_n+1}) \not\subset A_{2n+2}$, then $B(2a_n+3, l_1 l_2 \ldots l_{a_n+1}) \cap A_{2n+2} = \emptyset$. Thus there exists $l_{a_n+2} \in N$, such that $x \in E(2a_n+4, l_1 l_2 \ldots l_{a_n+1} l_{a_n+2})$. The next step is to see whether $E(2a_n+4, l_1 l_2 \ldots l_{a_n+2}) \in D_{2(n+1)}$. Repeat what we have done with $E(2a_n+2, l_1 l_2 \ldots l_{a_n+1})$. Since $s \in I(\mathbf{W}, X)$, we may know that there exists some $l_{a_n+1}$and $a_{n+1} \geq a_n + 1$, such that $x \in E(2a_n+1, l_1 l_2 \ldots l_{a_{n+1}})$, $E(2a_n+1, l_1 l_2 \ldots l_{a_{n+1}}) \in D_{2(n+1)}$.

In this way, we get a play $(E_0, \ldots, E(2a_n, l_1l_2 \ldots l_{a_n}), E(2a_n + 1, l_1l_2 \ldots l_{a_n}), \ldots)$ of $G(\mathbf{1}, X)$, satisfying that $E(2a_n + 1, l_1l_2 \ldots l_{a_n}) = s(E_0, \ldots, E(2a_n, l_1l_2 \ldots l_{a_n}))$ and $x \in \cap\{E(2a_n, l_1l_2 \ldots l_{a_n}) : n \in N\}$. This contradicts with $s \in I(\mathbf{1}, X)$.

Thus $t \in WI(\mathbf{W}, X)$. So $(X, \mathcal{T}_b)$ is a **W**–like space. Hence $(X, \mathcal{T}_b)$ is a Lindelöf space. Since $(X, \mathcal{T})$ is a Lindelöf T_2 space, we may easily prove that $(X, \mathcal{T}_b)$ is a regular space. Thus $(X, \mathcal{T}_b)$ is a regular Lindelöf **D1**–like space. So $(X, \mathcal{T}_b)$ is a **1**–like space (cf. [2-3]). $\square$

Next we will explain that $b(X, \mathcal{T})$ may not be a **W**–like space, if $(X, \mathcal{T})$ is a **W**–like space (under CH).

In [4], it was pointed out that there is a Tychonoff space X which is a **W**–like space but it is not a **1**–like space (under CH). We denote this space $(X^1, \mathcal{T}^1)$. Suppose $(X^1, \mathcal{T}^1_b)$ is a **W**–like space, thus $(X^1, \mathcal{T}^1_b)$ is a regular, Lindelöf **D1**–like space. So $(X^1, \mathcal{T}^1_b)$ is a **1**–like space (cf. [2-3]). Let $f : (X^1, \mathcal{T}^1_b) \to (X^1, \mathcal{T}^1)$, such that $f(x) = x$ for any $x \in X$. Thus f is continuous. So we have that $(X^1, \mathcal{T}^1)$ is a **1**–like space. A contradiction. Thus $(X^1, \mathcal{T}^1_b)$ is not a **W**–like space.

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