Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

TOPOLOGY PROCEEDINGS
Volume 31, No. 1, 2007
Pages 7-18
http://topology.auburn.edu/tp/

MULTI-PARAMETER OSCILLATORY COUPLING FUNCTIONS IN NEURAL NETWORKS

FERNANDA BOTELHO, JAMES JAMISON, AND ANGELA MURDOCK

Abstract. We study the existence and stability of stationary solutions of an integro-differential equation modeling the coarse-grained averaged activity of a single layer of interconnected neurons. The neuronal multi-parameter connections considered are laterally oscillatory with an exponential rate of decay. We identify regions in the parameter space where solutions exhibit areas of excitation with single pulses. We also discuss the stability behavior of these pulses.

1. Introduction

Integro-differential equations of the type

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{\mathbb{R}} w(x-y) f[u(y,t)]dy \tag{1.1}$$

model persistent pattern formations which are observed in brain activity. These models were derived by Hugh R. Wilson and Jack D. Cowan [16]; see also [1] and [15].

The function $u(x,t)$ represents the neuronal activity measured at the location $x \in (-\infty, \infty)$ and time $t \geq 0$. The function $w(x)$ represents the connecting strength between neurons. The neuronal

---

2000 *Mathematics Subject Classification.* Primary 34A36, 37N25; Secondary 45G10.

Key words and phrases. neural networks, oscillatory connections, pulses, single pulses, stationary solutions.

©2007 Topology Proceedings.

response to an incoming activation u is given by the action of the "gain function," $f(u)$. A mathematical model that represents brain activity in a memory process should maintain persistent activity. This motivates the search for stable multi-pulse stationary solutions of (1.1). More precisely, we refer to solutions of

$$u(x) = \int_{\mathbb{R}} w(x-y) f[u(y)] dy, \tag{1.2}$$

where $u(x) > u_T$ over a finite and disjoint union of bounded open intervals.

Many of the previous analytical studies of this problem have used the Heaviside gain function H; see [2] or [10]. In [7], Yixin Guo and Carson C. Chow investigated the existence of pulse solutions for a class of neural networks, modeled by (1.1) for gain functions with no saturation, $f(u) = (\alpha(u - u_T) + \beta)H(u - u_T)$, and connections of the form $\omega(x) = Ae^{-a|x|} - e^{-|x|}$. Moreover, they described a construction of potential single pulse solutions and provided numerical evidence supporting their existence.

Typically, connecting functions are locally excitatory and distally inhibitory; see [6], [3], or [4]. However, Carlo R. Laing and William C. Troy [10] addressed experimental observations by considering the following oscillatory type of connections:

$$w(x) = e^{-a|x|}(\cos x + a \sin |x|).$$

As it is asserted in [11], studies have shown that neural groups from the prefrontal cortex appear to couple in a spatially almost periodic manner (cf. [9]). This justifies a quest for oscillatory coupling functions that best represent experimental observations. In this paper, we propose the four parameter family of connecting weights:

$$w(x) = \gamma\, e^{-a|x|} \cos bx + \eta\, e^{-a|x|} \sin b|x|.$$

This family exhibits oscillatory behavior and rapid decay, with independent period and decay parameters. Since

$$\int_{\mathbb{R}} w(x) dx = \frac{2a\gamma + 2b\eta}{a^2 + b^2},$$

the overall effect on the neurons is excitatory or inhibitory depending on the parameters. We follow Guo and Chow's constructive approach to the existence of single pulse solutions of the steady state equation, (1.2), now for this broader class of oscillatory connecting functions. We show analytically the existence of pulse solutions and study their linear stability for convenient parameter values.

2. Construction of pulses

We determine the existence of pulse stationary solutions for (1.1) or equivalently, time independent symmetric functions satisfying (1.2) and which are greater than u_T only over an interval $(-x_T,\, x_T)$. The value x_T is referred to as the "width" of u. A region in the brain directly affected by a stimulus presents a persistent excitatory response which is typically surrounded by a region of little to no response. This is represented by pulse functions (cf.[7]).

Definition 2.1. A function u is said to be a pulse function of width x_T ($x_T > 0$) if and only if it satisfies the following conditions:

(1) u is symmetric and differentiable almost everywhere up to the fourth order,

(2) $\left(u, u^{'}, u^{''}, u^{'''}\right) \to (0,0,0,0)$ exponentially fast as $|x| \to \infty$, and

(3) $\begin{cases} u(x) > u_T, & \text{if } \ x \in (-x_T, x_T) \\ u(x) = u_T, & \text{if } \ x = -x_T \text{ or } x = x_T \\ u(x) < u_T, & \text{otherwise.} \end{cases}$

If, in addition, $u^{''}(0) < 0$, $u(0)$ is the absolute maximum value of u and $x = 0$ is the only critical point on $[-x_T, x_T]$, then u is said to be a single pulse function.

We now seek pulse and single pulse solutions of the equation

$$u(x) = \int_{\mathbb{R}} w(x-y) f[u(y)] dy = [w * f(u)](x),$$

with

$$w(x) = \gamma\, e^{-a|x|} \cos bx + \eta\, e^{-a|x|} \sin b|x|,$$

and $f(u) = (\alpha(u - u_T) + \beta) H(u - u_T)$. We first construct the general form for such pulse solutions. We apply the Fourier Transform,

$$\mathcal{F}(g(x)) = \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx,$$

to the convolution equation $u = w * f(u)$:

$$\mathcal{F}(u) = \mathcal{F}(w)\mathcal{F}[f(u)], \text{ where} \tag{2.1}$$

$$\mathcal{F}(w) = \frac{2a\,\gamma\,(a^2+b^2+\xi^2) + 2b\,\eta\,(a^2+b^2-\xi^2)}{(a^2+(b+\xi)^2)(a^2+(b-\xi)^2)}.$$

The coupling function w is not a traditional "Mexican hat" function (see [1], [7], or [11]); hence, previous numerical studies supporting the existence of one pulse solutions do not apply. The Inverse Fourier Transform applied to (2.1) yields

(2.2)

$$u^{(4)} - 2(a^2-b^2)u'' + (a^2+b^2)^2 u = (2a\gamma + 2b\eta)(a^2+b^2)f(u) +$$
$$2(a\gamma - b\eta)\left\{\beta\,(\delta'_{x_T} + \delta'_{-x_T}) + \alpha\, u'(x_T)(\delta_{x_T} + \delta_{-x_T}) - \alpha u''\, H(u-u_T)\right\}$$

(δ_a is the Dirac distribution at a), since

$$\mathcal{F}[\xi^2 f[u(x)]] = \beta\,(-i\xi e^{i\xi x_T} + i\xi e^{-i\xi x_T}) \;+\alpha\, u'(x_T)(e^{i\xi x_T} + e^{-i\xi x_T}) - \int_{-\infty}^{\infty} e^{i\xi x}\alpha u''\, H(u-u_T)dx,$$

(cf. [7]). We observe that

$$\beta\,(\delta'_{x_T} + \delta'_{-x_T}) + \alpha\, u'(x_T)(\delta_{x_T} + \delta_{-x_T})$$

is the zero distribution over the set of infinitely differentiable functions with compact support contained in $R \setminus \{\pm x_T\}$. We consider region I to be the open interval $(-x_T, x_T)$ and regions II and III to be the open interval (x_T, ∞) and $(-\infty, -x_T)$, respectively. Equation (2.2) reduces to

$$u^{(4)} - 2\,\left[(a^2-b^2) - (a\gamma - b\eta)\alpha\right] u''$$
$$+ \left[(a^2+b^2)^2 - 2\alpha(a\gamma + b\eta)(a^2+b^2)\right] u =$$
$$(\beta - \alpha u_T)[(2\gamma a + 2b\eta)(a^2+b^2)],$$

over region I, and to

$$u^{(4)} - 2\,\left[(a^2-b^2) - (a\gamma - b\eta)\alpha\right] u''$$
$$+ \left[(a^2+b^2)^2 - 2\alpha(a\gamma + b\eta)(a^2+b^2)\right] u = 0,$$

over regions II and III.

Hence, (2.2) determines the following initial value problem:
(2.3)

$$\begin{cases} u^{(4)} - 2\,Cu'' + [(a^2+b^2)^2 - D\alpha]u = (\beta - \alpha u_T)D, \text{ if } u > u_T \\ u^{(4)} - 2\,Cu'' + [(a^2+b^2)^2 - D\alpha]u = 0, \text{ if } u < u_T\ , \end{cases}$$
$$u_I(x_T) = u_{II}(x_T);\ u_I'(x_T) = u_{II}'(x_T);$$
$$u_I''(x_T) = u_{II}''(x_T) - 2\,\beta\,(a\gamma - b\eta);$$
$$u_I'''(x_T) = u_{II}'''(x_T) - 2\,\alpha\,(a\gamma - b\eta)\,u'(x_T),$$

(where $C = \left[(a^2-b^2) - (a\gamma - b\eta)\alpha\right]$ and $D = 2(a\gamma + b\eta)(a^2+b^2)$).

A function u, so defined, is a solution of (2.2) in the distribution sense or a weak solution relative to the space of infinitely differentiable functions with compact support (test functions).

We describe the construction under the assumptions $\gamma = b = 1$, and $\eta = a$, and $\alpha = 0$. This leads to $C = a^2 - 1$ and $D = 4a(a^2+1)$. Therefore, (2.3) reduces to

$$\begin{cases} u^{(4)} - 2(a^2-1)u'' + (a^2+1)^2 u = 4a(a^2+1)\beta \text{ if } u > u_T \\ u^{(4)} - 2(a^2-1)u'' + (a^2+1)^2 u = 0, \text{ if } u < u_T\ , \end{cases}$$
$$u_I(x_T) = u_{II}(x_T), u_I'(x_T) = u_{II}'(x_T), u_I''(x_T) = u_{II}''(x_T),$$
$$u_I'''(x_T) = u_{II}'''(x_T). \quad \text{(cf. [7])}$$

The solutions are of the form

$$u_I(x) = c_1(e^{-ax} + e^{ax})\cos x + c_2(e^{-ax} - e^{ax})\sin x + \tfrac{4a\beta}{a^2+1} \quad \text{if } |x| < x_T$$
$$u_{II}(x) = (c_3 cos(x) + c_4 sin(x))e^{-ax}, \quad \text{if } x > x_T,$$
$$u_{III}(x) = (c_3 cos(x) - c_4 sin(x))e^{ax}, \quad \text{if } x < -x_T.$$

We use Maple to solve the system determined by the initial conditions, and we obtain the following coefficients:

$$c_1 = \frac{(1-a^2)\ \sin(x_T) - 2a\ \cos(x_T)}{e^{ax_T}(a^2+1)}\,\beta$$

$$c_2 = \frac{(1-a^2)\ \cos(x_T) + 2a\ \sin(x_T)}{e^{ax_T}(a^2+1)}\,\beta$$

$$c_3 = -\frac{(a^2-1)\sin x_T(e^{ax_T} + e^{-ax_T}) + 2a\cos x_T(e^{-ax_T} - e^{ax_T})}{a^2+1}\,\beta$$

$$c_4 = \frac{(a^2-1)\cos x_T(e^{ax_T} - e^{-ax_T}) + 2a\sin x_T(e^{ax_T} + e^{-ax_T})}{a^2+1}\,\beta.$$

Given a threshold value u_T, a pulse function must attain the value u_T at both $\pm x_T$. Therefore, $\Phi(x_T) = u_I(x_T) = u_T$, a value in the range of the so-called existence function (see [7]),

$$\Phi(x) = [(1-a^2)\sin(2x) - 2a\cos(2x)]\frac{\beta\, e^{-2ax}}{a^2+1} + \frac{2a\beta}{a^2+1},$$

$(x > 0)$. This function provides information on the existence of potential pulse solutions. For simplicity of exposition, we assume $\beta \geq 0$.

We first notice that $\Phi'(x) = 2\beta[\cos(2x) + a\sin(2x)]e^{-2ax}$ and $\Phi''(x) = -4\,\beta\, e^{-2ax}(a^2+1)\sin(2x)$.

The critical points of $\Phi(x)$ are at $x_j = \frac{1}{2}\tan^{-1}\left(-\frac{1}{a}\right) + \frac{\pi}{2}j$, with $j \in \mathbb{Z}^+$. We choose $\xi = \frac{1}{2}\tan^{-1}(-\frac{1}{a})$, a value in the interval $(\frac{\pi}{4}, \frac{\pi}{2})$. The function $\Phi(x)$ attains a local maximum, M_j, at $x_j = \xi + \frac{\pi}{2}j$, if j is even and a local minimum, m_j, if j is odd. The sequence $\{\Phi(x_{2k-2})\}_{k\in\mathbb{N}}$ is strictly decreasing and $\{\Phi(x_{2k-1})\}_{k\in\mathbb{N}}$ is strictly increasing. The absolute maximum value of Φ is

$$M_0 = \sin(2\xi)\beta e^{-2a\xi} + \frac{2a\beta}{a^2+1}.$$

Definition 2.2. We say that a pair (x_T, u_T) determines a pulse candidate provided that $\Phi(x_T) = u_T$. A pulse candidate u, such that $u(x) > u_T$ for $|x| < x_T$ and $u(x) \leq u_T$, otherwise, is designated a pulse solution. If, in addition, u has a single maximum on the interval $(-x_T, x_T)$, attained at $x = 0$, we simply say that u is a single pulse.

We observe that $\phi(x_T) = u_T$ is a necessary condition for the function u, defined as a concatenation of the three pieces, u_I, u_{II}, and u_{III}, to be a pulse solution of (1.1). The next proposition addresses the existence of pulse candidates for different values of u_T. The proof is straightforward.

Proposition 2.3. (1) *If $u_T > M_0$, then there exist no pulse solutions.*

(2) *If min$\{0, m_1\} < u_T \leq M_0$, there exists at least one pulse candidate.*

(3) *If $j = 1, 2, \ldots$ and*

$$M_{2j-2} > u_T > M_{2j},$$

then there exist $2j$ pulse candidates.

(4) *If $j = 1, 2, \ldots$ and*

$$m_{2j+1} > u_T > m_{2j-1},$$

then there exist at most $2j+1$ pulse candidates.

(5) *If* $u_T = 2a\beta/(a^2+1)$, *then there are infinitely many pulse candidates.*

3. Existence of single pulse solutions

The next theorem asserts the existence of one pulse stationary solutions for the system considered in a region of the parameter space (a, β, x_T), under the additional parameter constraints $\gamma = b = 1$, and $\eta = a$, and $\alpha = 0$.

Theorem 3.1. *If* $a \geq 1$, $\beta > 0$, *and* $0 < x_T < \frac{\pi}{4}$,

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{\mathbb{R}} w(x-y) f[u(y,t)]dy$$

has single pulse stationary solutions.

Proof: Under the assumptions of section 2, ($\gamma = b = 1$, and $\eta = a$, and $\alpha = 0$), functions satisfying $u(x) = \int_{\mathbb{R}} w(x-y)f[u(y)]dy$ are of the form

$$u = \begin{cases} u_I(x) = \frac{\beta}{a^2+1}\left[-g(x+x_T) - g(x_T - x) + 4a\right] & \text{if } |x| < x_T \\ u_{II}(x) = \frac{\beta}{a^2+1}\left[g(x - x_T) - g(x+x_T)\right] & \text{if } x \geq x_T \\ u_{III}(x) = u_{II}(-x) & \text{if } x \leq x_T \end{cases}$$

given $g(\xi) = [(a^2-1)\sin(\xi) + 2a\cos(\xi)]e^{-a\xi}$. We start by showing that $u_I > u_T$ over region I and $u_{II} \leq u_T$ over region II. We consider only positive values for x, since u is symmetric. We observe that

$$u_I'(x) = \beta\left\{\left[\cos(x+x_T) + a\sin(x+x_T)\right]e^{-a(x+x_T)} - \right. \\ \left. \left[\cos(x_T - x) + a\sin(x_T - x)\right]e^{-a(x_T-x)}\right\}$$

and

$$u_I''(x) = -\beta(a^2+1)\left[\sin(x+x_T)e^{-a(x+x_T)} + \sin(x_T - x)e^{-a(x_T-x)}\right] < 0,$$

since $0 \leq x + x_T < \frac{\pi}{2}$ and $0 < x_T - x < \frac{\pi}{4}$. Furthermore, there exists η so that $x_T - x < \eta < x_T + x$ and $u_I'(x) = -2x\beta(a^2+1)\sin(\eta)e^{-a\eta}$. Hence, $u_I'(x) < 0$ for $0 < x \leq x_T$.

Now, we show that $u_{II}(x) \leq u_T$ for $x > x_T$. It is sufficient to prove the inequality for the absolute maximum value of u_{II}. This is attained at x_0, a critical point of u_{II}. The Mean Value Theorem (MVT) assures the existence of ζ, $x_0 - x_T < \zeta < x_0 + x_T$, so that

$$\begin{aligned} u'_{II}(x_0) &= -\beta \left\{ [\cos(x_0 - x_T) + a\sin(x_0 - x_T)]\, e^{-a(x_0 - x_T)} \right. \\ &\quad \left. + [\cos(x_T + x_0) + a\sin(x_T + x_0)]\, e^{-a(x_T + x_0)} \right\} \\ &= -2\beta \sin(\zeta) x_T (a^2+1) e^{-a\zeta} = 0. \end{aligned}$$

This implies that $\zeta = 2\pi$. Therefore, $x_0 - x_T < 2\pi < x_0 + x_T$ and $\frac{7\pi}{4} < x_0 < \frac{9\pi}{4}$. In addition, we also have that $2\pi < x_0 + x_T < \frac{5\pi}{2}$. This implies that $\cos(x_0 + x_T) + a\sin(x_0 + x_T) > 0$, and $\cos(x_0 - x_T) + a\sin(x_0 - x_T) > 0$, since

$$\frac{\cos(x_0 + x_T) + a\sin(x_0 + x_T)}{\cos(x_0 - x_T) + a\sin(x_0 - x_T)} = e^{2ax_T}. \tag{3.1}$$

Therefore, $\frac{7\pi}{4} < x_0 - x_T < 2\pi < x_0 + x_T < \frac{9\pi}{4}$.

We set $h(y) = (\sin(y) - a\cos(y))e^{-ay}$, with y in the interval $(\frac{7\pi}{4}, 2\pi)$, then $u_{II}(x_0) = \frac{\beta}{a^2+1}[h(x_0 + x_T) - h(x_0 - x_T)]$, by using (3.1). Again, the MVT assures the existence of μ so that $\frac{7\pi}{4} < x_0 - x_T < \mu < x_0 + x_T < \frac{9\pi}{4}$ and $u_{II}(x_0) = \frac{\beta}{a^2+1} h'(\mu)\,(2x_T)$. Since $h'(\mu) = (a^2+1)\cos(\mu)e^{-a\mu} \le (a^2+1)e^{-\frac{7\pi}{4}a}$, we have $u_{II}(x_0) \le \beta e^{-\frac{7\pi}{4}a}\,(2x_T)$.

If

$$\psi(x) = 2a - e^{-2ax}[(a^2-1)\sin(2x) + 2a\cos(2x)] - 2x(a^2+1)e^{-\frac{7\pi a}{4}},$$

we have $u_T = \frac{\beta}{a^2+1}\,[\psi(x_T) + 2x_T(a^2+1)e^{-\frac{7\pi a}{4}}]$. For fixed a,

$$\psi'(x) = 2(a^2+1)[e^{-2ax}(a\sin(2x) + \cos(2x)) - e^{-\frac{7\pi a}{4}}],$$

$$\psi''(x_T) = -4(a^2+1)^2 \sin(2x_T) e^{-2ax_T} < 0,$$

and

$$\psi'(x_T) > \psi'(\frac{\pi}{4}) = 2(a^2+1)[a\, e^{-\frac{\pi a}{2}} - e^{-\frac{7\pi a}{4}}] > 0.$$

Therefore, ψ is increasing, implying that $\psi(x_T) > \psi(0) = 0$. In particular, this shows that

$$u_T \ge \beta(2x_T)e^{-\frac{7\pi a}{4}} \ge u_{II}(x_T).$$

The remainder of the proof follows from the symmetry of u. $\square$

4. Linear stability of single pulses

In this section, we study linear stability of stationary solutions. Given a stationary solution u, we consider perturbations of the form

$$u_1(x,t) = u(x) + \epsilon v(x)e^{\mu t},$$

where $v(x)e^{\mu t}$ must satisfy the linear approximation to (1.1). The problem is then reduced to the spectral properties of a compact operator. This approach follows a pioneer work by Shun-ichi Amari [1] and later generalized in [13], [14], and [7], among others.

We recall the basic equation:

$$\frac{\partial u}{\partial t} + u(x,t) = \int_{-\infty}^{\infty} w(x-y) f(u(y,t)) dy$$

with

$$f(u) = H(u - u_T),$$

where $H(u-u_T)$ is the Heaviside function and $w(x) = e^{-a|x|}\cos bx + a\, e^{-a|x|}\sin b|x|$. We consider a perturbation of the form $u_1(x,t) = u(x) + \epsilon v(x)e^{\mu t}$, with v in $\mathcal{BC}(\mathbb{R})$. The set $\mathcal{BC}(\mathbb{R})$ consists of all continuous and bounded real-valued functions. Therefore, we have

$$\lambda v(x) + v(x) = \int_{-\infty}^{\infty} \omega(x-y) f[u(y) + \epsilon v(y)e^{\lambda t}]\, dy.$$

The linearization around $\epsilon = 0$ yields the equation

$$(\mu+1)v(x) = -\frac{w(x+x_T)v(-x_T)}{u'(x_T)} - \frac{w(x-x_T)v(x_T)}{u'(x_T)}. \tag{4.1}$$

We define the operator

$$L(v(x)) = -\frac{w(x+x_T)v(-x_T)}{u'(x_T)} - \frac{w(x-x_T)v(x_T)}{u'(x_T)},$$

where $v(x)$, representing a bounded and continuous function on $\mathbb{R}$, is an eigenvector of L associated with an eigenvalue, $\mu + 1$. The following definition states a notion of stability for a stationary solution along an eigenspace of L.

Definition 4.1. The stationary solution $u(x)$ is said to be linearly stable along a vector space V if V is an invariant space of L, and the point spectra of L restricted to V, $\sigma_p(L|_V)$ has real part in the open interval $(-\infty, 1)$.

We observe that L is compact. In fact, given a sequence in $\mathcal{BC}(\mathbb{R})$, we can select a subsequence that converges at x_T and $-x_T$, to y_0 and y_1, respectively. Therefore, the image of such a sequence under L converges to $-\frac{w(x+x_T)y_1}{u'(x_T)} - \frac{w(x-x_T)y_0}{u'(x_T)}$.

Proposition 4.2. *The operator L has three eigenvalues: $\lambda_0 = 0$ and $\lambda_\pm = \frac{1\pm\omega(2x_T)}{1-\omega(2x_T)}$ with associated eigenspaces*

$$V_0 = \{v \in \mathcal{BC}(\mathbb{R}) : \ v(-x_T) = v(x_T) = 0\}, \ \textit{and}$$

$$V_\pm = \{v \in \mathcal{BC}(\mathbb{R}) : v(x) = \frac{c\ (\omega(x+x_T) \pm \omega(x-x_T))}{1+\omega(2x_T)}, \, c \in \mathbb{R}\},$$

respectively.

Proof: Every function $v \in \mathcal{BC}(\mathbb{R})$ has a unique representation as the sum of an odd and an even function:

$$v(x) = v_e(x) + v_o(x),$$

where $v_e = \frac{v(x)+v(-x)}{2}$ and $v_o = \frac{v(x)-v(-x)}{2}$. Therefore,

$$L(v) = -\frac{v_e(x_T)}{u'(x_T)}[\omega(x+x_T)+\omega(x-x_T)] - \frac{v_o(x_T)}{u'(x_T)}[-\omega(x+x_T)+\omega(x-x_T)].$$

In particular, at $x = \pm x_T$, we have

$$L(v(\pm x_T)) = -\frac{1}{u'(x_T)}v_e(x_T)[\omega(2x_T)+1] \pm \frac{1}{u'(x_T)}v_o(x_T)[\omega(2x_T)-1].$$

If $L(v) = 0$, it follows that $v_e(x_T) = v_o(x_T) = 0$; thus, $v(x_T) = 0$. Similarly, we conclude that $v(-x_T) = 0$, by just considering the representation

$$L(v) = -\frac{v_e(-x_T)}{u'(x_T)}[\omega(x+x_T)+\omega(x-x_T)] + \frac{v_o(-x_T)}{u'(x_T)}[-\omega(x+x_T)+\omega(x-x_T)].$$

If L has an eigenvalue $\lambda \neq 0$, then the following system must have a nontrivial solution:

$$\begin{cases} \omega(2x_T)v(-x_T) + (1+\lambda u'(x_T))v(x_T) = 0 \\ (1+\lambda u'(x_T))v(-x_T) + \omega(2x_T)v(x_T) = 0\,, \end{cases}$$

or equivalently, $\lambda^2 u'(x_T)^2 + 2\lambda u'(x_T) - \omega(2x_T)^2 + 1 = 0$. Therefore, $\lambda = \frac{-1\pm\omega(2x_T)}{u'(x_T)}$. We denote the eigenvalues as $\lambda_\pm = \frac{-1\pm\omega(2x_T)}{u'(x_T)}$, with corresponding eigenspaces consisting of functions $v \in \mathcal{BC}(\mathbb{R})$ so that $v(-x_T) = \pm v(x_T)$. This completes the proof. □

Remark 4.3. The previous proof determines the algebraic decomposition $\mathcal{BC}(\mathbb{R}) = ker(L) \oplus V_- \oplus V_+$. Given a function $h \in \mathcal{BC}(\mathbb{R})$, we set $c_1 = \frac{h(x_T)+h(-x_T)}{2}$ and $c_2 = \frac{h(x_T)-h(-x_T)}{2}$. We consider a function $v_+ \in V_+$ so that $v_+(-x_T) = v_+(x_T) = 1$ and $v_- \in V_-$ so that $v_-(-x_T) = -v_-(x_T) = -1$, then $h - (c_1\, v_+ + c_2\, v_-) \in ker(L)$.

Proposition 4.4. *If $a \geq 1$, $\beta > a^2 + 1$, and $0 < x_T < \frac{\pi}{4}$, then u is linearly stable along $V_0 \oplus V_+$.*

Proof: This follows from the Theorem 3.1 and the spectral properties of L. □

References

1. Shun-ichi Amari, *Dynamics of pattern formation in lateral-inhibition type neural fields*, Biol. Cybernet. **27** (1977), no. 2, 77–87.
2. ______, *Mathematical Theory of Neural Nets.* (Japanese). Tokyo: Sangyo-Tosho Publishers, 1978.
3. Marcelo Camperi and Xiao-Jing Wang, *A model of visuospatial working memory in prefrontal cortex: Recurrent network and cellular bistability*, J. Comput. Neurosci. **5** (1998), no. 4, 383–405.
4. Christos Constantinidis and Xiao-Jing Wang, *A neural circuit basis for spatial working memory*, The Neuroscientist **12** (2004), no. 10, 553–565.
5. S. Coombes, G. J. Lord, and M. R. Owen, *Waves and bumps in neuronal networks with axo-dendritic synaptic interactions*, Phys. D **178** (2003), no. 3-4, 219–241.
6. Stefanos E. Folias and Paul C. Bressloff, *Breathing pulses in an excitatory neural network*, SIAM J. Appl. Dyn. Syst. **3** (2004), no. 3, 378–407 (electronic).
7. Yixin Guo and Carson C. Chow, *Existence and stability of standing pulses in neural networks: I. Existence*, SIAM J. Appl. Dyn. Syst. **4** (2005), no. 2, 217–248 (electronic).
8. ______, *Existence and stability of standing pulses in neural networks: II. Stability*, SIAM Journal on Applied Dynamical Systems, **4** (2005), 249–281 (electronic).
9. Boris S. Gutkin, G. Bard Ermentrout, and Joseph O'Sullivan, *Layer 3 patchy recurrent excitatory connections may determine the spatial organization of sustained activity in the primate prefrontal cortex*, Neurocomputing **32-33** (2000), 391–400.

10. Carlo R. Laing, William C. Troy, Boris Gutkin, and G. Bard Ermentrout, *Multiple bumps in a neuronal model of working memory*, SIAM J. Appl. Math. **63** (2002), no. 1, 62–97 (electronic).
11. ______, *PDE methods for nonlocal models*, SIAM J. Appl. Dyn. Syst. **2** (2003), no. 3, 487–516 (electronic).
12. L. A. Peletier and W. C. Troy, *Spatial Patterns. Higher Order Models in Physics and Mechanics*. Progress in Nonlinear Differential Equations and Their Applications, 45. Boston, MA: Birkhuser Boston, Inc., 2001.
13. David J. Pinto and G. Bard Ermentrout, *Spatially structured activity in synaptically coupled neuronal networks. I. Traveling fronts and pulses*, SIAM J. Appl. Math. **62** (2001), no. 1, 206–225 (electronic).
14. ______, *Spatially structured activity in synaptically coupled neuronal networks. II. Lateral inhibition and standing pulses*, SIAM J. Appl. Math. **62** (2001), no. 1, 226–243 (electronic).
15. Hugh R. Wilson and Jack D. Cowan, *Excitatory and inhibitory interactions in localized populations of model neurons*, Biophys. J. **12** (1972), no. 1, 1–24.
16. ______, *A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue*, Kybernetik **13** (1973), 55–80.

(Botelho) Department of Mathematical Sciences; University of Memphis; Memphis, TN 38152

E-mail address: `mbotelho@memphis.edu`

(Jamison) Department of Mathematical Sciences; University of Memphis; Memphis, TN 38152

E-mail address: `jjamison@memphis.edu`

(Murdock) Department of Mathematics; Rhodes College; Memphis, TN 38104

E-mail address: `murdocka@rhodes.edu`