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A PRECOMPACTNESS TEST FOR TOPOLOGICAL GROUPS IN THE MANNER OF GROTHENDIECK

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Abstract. Let G be an arbitrary topological group. We show that G is precompact if and only if for any two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G, the two subsets $\{s_m t_n \mid m \leq n\}$ and $\{s_m t_n \mid m > n\}$ of G are right proximal. This result and Grothendieck's characterization of weak almost periodicity in terms of the double limit property allow us to obtain, with a simplified proof, the theorem due to M. G. Megrelishvili, V. G. Pestov, and V. V. Uspenskij: G is precompact if and only if every bounded real-valued right uniformly continuous function on G is weakly almost periodic. We also present a class of G for which these results hold when considering "lower proximality" instead of "right proximality" and "lower uniform continuity" instead of "right uniform continuity."

1. Introduction

Let G be an arbitrary topological group and let $\mathcal{C}(G)$ be the space of all bounded real-valued continuous functions on G. Let $\mathcal{RUC}(G)$ (respectively, $\mathcal{UC}(G)$) be the space of all functions in $\mathcal{C}(G)$ which are right (respectively, lower) uniformly continuous, and let $\mathcal{WAP}(G)$ be the space of all functions in $\mathcal{C}(G)$ which are weakly almost periodic. (This and other notions will be defined in section 2.) As is well known, the inclusion $\mathcal{WAP}(G) \subset \mathcal{UC}(G)$ holds [2]

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(and of course, $\mathcal{UC}(G) \subset \mathcal{RUC}(G)$). In [1], R. B. Burckel proved that for G locally compact, $\mathcal{C}(G) = \mathcal{WAP}(G)$ if and only if G is compact. In [9], Edmond E. Granirer improved this result by observing that $\mathcal{C}(G)$ can be replaced by $\mathcal{UC}(G)$ (cf. also [5] and [6]). Much more recently, M. G. Megrelishvili, V. G. Pestov, and V. V. Uspenskij [14] showed that for any G, $\mathcal{RUC}(G) = \mathcal{WAP}(G)$ if and only if G is precompact. Their method of proof consists of a (non-immediate) reduction to the case of a separable metrizable group and an application of the following result of Jan K. Pachl [16]: Every uniquely amenable separable metrizable topological group is precompact.

In the present note, we prove in section 3 that G is precompact if and only if for any two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G, the two subsets $\{s_m t_n \mid m \leq n\}$ and $\{s_m t_n \mid m > n\}$ of G are right proximal. This result and Grothendieck's characterization of weak almost periodicity in terms of the double limit property allow us to obtain the theorem of Megrelishvili, Pestov and Uspenskij by another method.

In section 5, we introduce the class of $\aleph_0$-FASIN topological groups. For every G in this class, the preceding results hold when considering "lower proximality" instead of "right proximality" and "lower uniform continuity" instead of "right uniform continuity." As every locally precompact topological group belongs to this class, Theorem 5.6 below contains as a particular case the above result by Granirer.

Our terminology is the same as in [7] and [18].

2. Preliminaries

If G is a topological group, we always denote by e its identity element and by $\mathcal{V}_G(e)$ the set of all neighborhoods of e in G.

Definition 2.1. Let G be a topological group and $\mathcal{C}(G)$ the Banach space of all bounded real-valued continuous functions on G.

1) For each $V \in \mathcal{V}_G(e)$, let $V_{\mathcal{L}}$ (respectively, $V_{\mathcal{R}}$) be the set of all $(x, y) \in G \times G$ such that $x^{-1}y \in V$ (respectively, $xy^{-1} \in V$); then the set of all $V_{\mathcal{L}}$ (respectively, $V_{\mathcal{R}}$), as V runs through $\mathcal{V}_G(e)$, is a fundamental system of entourages of the *left* (respectively, *right*) uniform structure $\mathcal{L}_G$ (respectively, $\mathcal{R}_G$) on G [18, 2.1]. For each $V \in \mathcal{V}_G(e)$, let us put $\widetilde{V} = \{(x, y) \in G \times G \mid y \in VxV\}$; then the

lower uniform structure $\mathcal{L}_G \wedge \mathcal{R}_G$ on G admits $\{\widetilde{V} \mid V \in \mathcal{V}_G(e)\}$ as a fundamental system of entourages [18, 2.5]. A mapping f of G into a uniform space is said to be *left* (respectively, *right*) *uniformly continuous* if it is uniformly continuous when G is equipped with $\mathcal{L}_G$ (respectively, $\mathcal{R}_G$); if f is both left and right uniformly continuous, or equivalently, if f is uniformly continuous when G is equipped with $\mathcal{L}_G \wedge \mathcal{R}_G$, then f is said to be *lower uniformly continuous.* Recall that $\mathcal{LUC}(G)$ (respectively, $\mathcal{RUC}(G)$) denotes the space of all bounded real-valued functions on G that are left (respectively, right) uniformly continuous, and that $\mathcal{UC}(G) = \mathcal{LUC}(G) \cap \mathcal{RUC}(G)$.

2) For all $f \in \mathcal{C}(G)$ and all $g \in G$, let f_g (respectively, f^g) be the *left translate* (respectively, *right translate*) of f by g defined by $f_g(x) = f(gx)$ (respectively, $f^g(x) = f(xg)$) for all $x \in G$. A function $f \in \mathcal{C}(G)$ is said to be *weakly almost periodic* if $\{f_g \mid g \in G\}$ is a weakly relatively compact subset of $\mathcal{C}(G)$. It is equivalent to suppose that $\{f^g \mid g \in G\}$ is a weakly relatively compact subset of $\mathcal{C}(G)$ [10]. (Cf. 2.2 below.)

Recall that $\mathcal{WAP}(G)$ denotes the space of all functions in $\mathcal{C}(G)$ that are weakly almost periodic, and recall that $\mathcal{WAP}(G) \subset \mathcal{UC}(G)$. (Cf. for instance [2].)

The following important criterion for weak almost periodicity is due to Grothendieck [10]. This is the main tool of this note. It is used to obtain theorems 3.6 and 5.6 below.

Lemma 2.2. *Let f be a bounded real-valued continuous function on a topological group G. Then f is weakly almost periodic if and only if*

$$\lim_{m\to+\infty} \lim_{n\to+\infty} f(s_m t_n) = \lim_{n\to+\infty} \lim_{m\to+\infty} f(s_m t_n)$$

whenever $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences in G such that all the limits exist.

A second key lemma used to obtain theorems 3.6 and 5.6 is Lemma 2.4 below, which is a particular case of the well-known Katětov theorem [11], [12]. A concise proof of this theorem, given by T. E. Gantner (cf. [8] or [7, 8.5.6]), contains the short proof of 2.4 given below. First, let us recall a definition.

Definition 2.3. Let $(X,\mathcal{U})$ be a uniform space. Two subsets A and B of X are said to be *proximal*, or *near*, if $V[A]\cap B$ is non-empty for every entourage V of $\mathcal{U}$.

Lemma 2.4. *Let A and B be two non-empty subsets of a uniform space $(X,\mathcal{U})$. If (and only if) A and B are not proximal, there exists a bounded real-valued uniformly continuous function f on X such that $f(A)=\{1\}$ and $f(B)=\{0\}$.*

Proof: Let $V\in\mathcal{U}$ such that $V[A]\cap B=\emptyset$. Let d be a pseudometric on X which is uniform with respect to $\mathcal{U}$ and satisfies the conditions $d\le 1$ and $\{(x,y)\in X\times X \mid d(x,y)<1\}\subset V$ [7, 8.1.11]. It suffices to define f by $f(x)=\inf\{d(x,y)\mid y\in B\}$ for every $x\in X$. □

3. The equality $\mathcal{RUC}(G)=\mathcal{WAP}(G)$

First, recall a definition.

Definition 3.1. Let $(X,\mathcal{U})$ be a uniform space. A family $(x_i)_{i\in I}$ of points of X is said to be *uniformly discrete in* X if there exists $V\in\mathcal{U}$ such that $V[x_i]$ and $V[x_j]$ are disjoint whenever $i,j\in I$ and $i\neq j$. Recall that $(X,\mathcal{U})$ is said to be *precompact* if for every $V\in\mathcal{U}$ there is a finite subset F of X such that $X=V[F]$. It is easy to verify that $(X,\mathcal{U})$ is precompact if and only if every uniformly discrete family in X is finite.

Let G be a topological group. G is said to be precompact if the uniform space $(G,\mathcal{R}_G)$ is precompact. G is said to be locally precompact if there is a precompact uniform subspace of $(G,\mathcal{R}_G)$ which belongs to $\mathcal{V}_G(e)$. Equivalent notions are obtained when replacing $\mathcal{R}_G$ by $\mathcal{L}_G$ or by $\mathcal{L}_G\vee\mathcal{R}_G$ (but not by $\mathcal{L}_G\wedge\mathcal{R}_G$: G may be precompact with respect to $\mathcal{L}_G\wedge\mathcal{R}_G$ without being a locally precompact topological group) [18]. As is well known, Hausdorff (locally) precompact topological groups are exactly subgroups of (locally) compact topological groups.

The proof of Theorem 3.4 rests on the following two lemmas.

Lemma 3.2. *Let G be a topological group. Let us suppose that G is not locally precompact. Then there exist two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G such that the double sequence $(s_mt_n)_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ is right uniformly discrete in G.*

Proof: As G is not right precompact, there exist a sequence $(t_n)_{n\in\mathbb{N}}$ in G and $W \in \mathcal{V}_G(e)$ such that Wt_i and Wt_j are disjoint whenever $i, j \in \mathbb{N}$ and $i \neq j$. Let $V \in \mathcal{V}_G(e)$ such that $V^2 \subset W$. As V is not right precompact, there exist a sequence $(s_m)_{m\in\mathbb{N}}$ in V and $U \in \mathcal{V}_G(e)$ such that Us_i and Us_j are disjoint whenever $i, j \in \mathbb{N}$ and $i \neq j$. Obviously, U can be chosen such that $U \subset V$. Let us consider two distinct points (k, l) and (m, n) of $\mathbb{N} \times \mathbb{N}$, and let us verify that $Us_kt_l \cap Us_mt_n = \emptyset$, which will prove the lemma. If $l \neq n$, this follows from $Us_k \subset V^2 \subset W$, $Us_m \subset V^2 \subset W$, and $Wt_l \cap Wt_n = \emptyset$; if $l = n$, then $k \neq m$ and consequently, $Us_k \cap Us_m = \emptyset$, which implies $Us_kt_l \cap Us_mt_n = \emptyset$ since $t_l = t_n$. □

Lemma 3.3. *Let G be a topological group. Let us suppose that for some neighborhood V of e in G, $G \not\subset VFVF$ for any finite subset F of G. Then there exist two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G such that the two subsets $\{s_mt_n \mid m \leq n\}$ and $\{s_mt_n \mid m > n\}$ of G are not lower proximal.*

Proof: Obviously, we may assume V is symmetric, so that $G \not\subset FVFV$ for any finite subset F of G. By induction, we build sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G in the following way. Put $s_0 = e$, $t_0 = e$ and let us assume that $s_0, \ldots, s_n$ and $t_0, \ldots, t_n$ have been defined. First, we choose s_{n+1} in G which does not belong to $Vs_pt_qVt_r^{-1}$ for all p, q, r such that $0 \leq p \leq q \leq n$ and $0 \leq r \leq n$; next we choose t_{n+1} in G which does not belong to $s_p^{-1}Vs_qt_rV$ for all p, q, r such that $0 \leq p \leq n+1$ and $0 \leq r < q \leq n+1$. Let us verify that

$$(V\{s_mt_n \mid m \leq n\}V) \cap \{s_mt_n \mid m > n\} = \emptyset.$$

Suppose, on the contrary, that there are $m, n, p, q \in \mathbb{N}$ such that $m \leq n$, $p > q$, and $s_pt_q \in Vs_mt_nV$. We have two cases to consider. If $n \geq p$, then $t_n \in s_m^{-1}Vs_pt_qV$ with $0 \leq m \leq n$ and $0 \leq q < p \leq n$; this contradicts the definition of t_n. If $p > n$, then $s_p \in Vs_mt_nVt_q^{-1}$ with $0 \leq m \leq n < p$ and $0 \leq q < p$; this contradicts the definition of s_p. □

Theorem 3.4. *Let G be a topological group. Then the following statements are equivalent:*

(1) *G is precompact.*

(2) *For any two sequences* $(s_m)_{m\in\mathbb{N}}$ *and* $(t_n)_{n\in\mathbb{N}}$ *in* G*, the two subsets* $\{s_m t_n \mid m \leq n\}$ *and* $\{s_m t_n \mid m > n\}$ *of* G *are right proximal.*

(3) G *is locally precompact and for any two sequences* $(s_m)_{m\in\mathbb{N}}$ *and* $(t_n)_{n\in\mathbb{N}}$ *in* G*, the two subsets* $\{s_m t_n \mid m \leq n\}$ *and* $\{s_m t_n \mid m > n\}$ *of* G *are lower proximal.*

Proof: (1) $\Rightarrow$ (2): Let us suppose condition (2) is not satisfied, and let us choose $V \in \mathcal{V}_G(e)$ such that the sets $V\{s_m t_n \mid m \leq n\}$ and $\{s_m t_n \mid m > n\}$ are disjoint. Let W be a symmetric neighborhood of e in G such that $W^2 \subset V$. If $i, j \in \mathbb{N}$ and if $j > i$, then $W s_i t_i \cap W s_j t_i = \emptyset$; consequently, $W s_i \cap W s_j = \emptyset$, so that $(s_m)_{m\in\mathbb{N}}$ is right uniformly discrete in G. Since there is in G a right uniformly discrete sequence, G is not precompact. (Cf. 3.1.)

(2) $\Rightarrow$ (3): The local precompacity of G follows from Lemma 3.2. Moreover, if two subsets of G are right proximal, then they are obviously lower proximal.

(3) $\Rightarrow$ (1): Let V be a right precompact neighborhood of e in G. By Lemma 3.3, there is a finite subset F of G such that $G = VFVF$; consequently, since a finite product of right precompact subsets of G is right precompact [18, 9.15], G is precompact. □

Remark 3.5. It goes without saying that in conditions (2) and (3) of the above Theorem 3.4, "$\leq$" may be replaced by "$<$."

The proof of Theorem 3.6 below rests on Theorem 3.4 (and on lemmas 2.2 and 2.4). Recall that (1) $\Leftrightarrow$ (2) is a result by Megrelishvili, Pestov and Uspenskij [14], and that (1) $\Leftrightarrow$ (3) is a result by Granirer [9].

Theorem 3.6. *Let* G *be a topological group. Then the following statements are equivalent.*

(1) G *is precompact.*
(2) $\mathcal{RUC}(G) = \mathcal{WAP}(G)$*.*
(3) G *is locally precompact and* $\mathcal{UC}(G) = \mathcal{WAP}(G)$*.*

Proof: If the topological group G is precompact, then every bounded real-valued right uniformly continuous function f on G is almost periodic, that is to say $\{f_g \mid g \in G\}$ (or equivalently $\{f^g \mid g \in G\}$) is norm relatively compact in $\mathcal{C}(G)$. (This well-known property is proved, for instance, in [2].) Consequently, (1) $\Rightarrow$ (2).

Let us suppose that G is not locally precompact (respectively, locally precompact but not precompact). Then by Theorem 3.4, there exist sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G such that the two subsets $A = \{s_m t_n \mid m \leq n\}$ and $B = \{s_m t_n \mid m > n\}$ of G are not right (respectively, lower) proximal. By Lemma 2.4, there exists a bounded real-valued right (respectively, lower) uniformly continuous function f on G such that $f(A) = \{1\}$ and $f(B) = \{0\}$. Since

$$\lim_{m\to+\infty} \lim_{n\to+\infty} f(s_m t_n) = 1 \text{ and } \lim_{n\to+\infty} \lim_{m\to+\infty} f(s_m t_n) = 0,$$

it follows from Lemma 2.2 that f is not weakly almost periodic. Since the inclusions $\mathcal{WAP}(G) \subset \mathcal{UC}(G) \subset \mathcal{RUC}(G)$ always hold [2], this proves that (2) implies (3) and that (3) implies (1). $\square$

4. A remark about Theorem 3.4

Let G be a topological group. If G is not precompact, then by Theorem 3.4, there exist two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G such that the subsets $\{s_m t_n \mid m \leq n\}$ and $\{s_m t_n \mid m > n\}$ of G are not right proximal. This follows from Lemma 3.2 if G is not locally precompact and from Lemma 3.3 if G is locally precompact. One can also obtain such sequences by a straightforward adaptation of the proof of Lemma 3.3, by using the following fact pointed out (without proof) by Vladimir Uspenskij in [19]: If G is not precompact, then for at least one neighborhood V of e in G, $G \not\subset FVF$ for any finite subset F of G. This method has the advantage of not distinguishing the locally precompact case from the others; however, it does not bring out the extra information about the double sequence $(s_m t_n)_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ given by Lemma 3.2 (in the non locally precompact case) and by Lemma 3.3 (in the locally precompact case).

For the sake of completeness, we include a proof for Uspenskij's observation. We begin with an algebraic lemma.

Lemma 4.1. *Let $(A_1, \ldots, A_n)$ be a finite sequence of subsets of a group G. Let us suppose that there is a finite subset F of G such that $G = \bigcup_{i=1}^{n} A_i F$. Then there exist $j \in \{1, \ldots, n\}$ and a finite subset L of G such that $G = A_j^{-1} A_j L$.*

Proof: We proceed by induction on $n \geq 1$. If $n = 1$, then $G = A_1 F$, and by choosing $a \in A_1$, we obtain $G = a^{-1} A_1 F \subset A_1^{-1} A_1 F$.

Assume that the property is satisfied for n, and let us consider a finite sequence $(A_1, \dots, A_{n+1})$ of subsets of G such that the equality $G = \bigcup_{i=1}^{n+1} A_i F$ holds for some finite subset F of G; we must show that there exist $j \in \{1, \dots, n+1\}$ and a finite subset L of G such that $G = A_j^{-1} A_j L$. If $G = A_{n+1}^{-1} A_{n+1} F$, then $j = n+1$ and $L = F$ are appropriate. If not, consider any g in $G \setminus A_{n+1}^{-1} A_{n+1} F$. Then $A_{n+1} g \cap A_{n+1} F = \emptyset$, so that $A_{n+1} g \subset \bigcup_{i=1}^{n} A_i F$ which implies $A_{n+1} F \subset \bigcup_{i=1}^{n+1} A_i F g^{-1} F$; consequently,

$$G = \bigcup_{i=1}^{n} A_i (F \cup (F g^{-1} F)),$$

and as $F \cup (F g^{-1} F)$ is a finite subset of G, the induction hypothesis allows us to conclude. □

Remark 4.2. In a well-known paper by B. H. Neumann [15], it is proved that if a finite union of cosets of subgroups of a group G is equal to G, then at least one of these subgroups has a finite index in G. The above Lemma 4.1 is adapted from that of Neumann.

Proposition 4.3. *Let G be a topological group. Let us suppose that for every neighborhood V of e in G there exists a finite subset F of G such that $G = FVF$. Then G is precompact.*

Proof: Let $V \in \mathcal{V}_G(e)$. Let W be a symmetric neighborhood of e in G such that $W^2 \subset V$, and let F be a finite subset of G such that $G = FWF$. By Lemma 4.1, there is $x \in F$ and a finite subset L of G such that $G = (xW)^{-1}(xW)L$; obviously $G = VL$, and consequently G is precompact. □

5. The equality $\mathcal{UC}(G) = \mathcal{WAP}(G)$

In this section, our goal is to extend Granirer's result to the class of $\aleph_0$-FASIN topological groups by using the same methods as in section 3. (It cannot be extended to all topological groups [19].) Before defining this class, let us recall some basic concepts and properties [18].

Let G be a topological group. If there is at least one $W \in \mathcal{V}_G(e)$ such that $\cap_{x \in W} x^{-1} V x \in \mathcal{V}_G(e)$ for each $V \in \mathcal{V}_G(e)$, then G is said to be *almost SIN* (*ASIN*). If G is locally precompact, then G is ASIN. (Any right precompact $W \in \mathcal{V}_G(e)$ works.) In another direction, if G is extremally disconnected, then G contains an open abelian subgroup [13], and consequently G is ASIN. Let $A \subset G$; if for every $V \in \mathcal{V}_G(e)$, there is $U \in \mathcal{V}_G(e)$ such that $AU \subset VA$, then A is said to be *right neutral in* G. If every subset of G is right neutral in G, or equivalently [17], if $\mathcal{LUC}(G) = \mathcal{RUC}(G)$, then G is said to be *functionally SIN* (*FSIN*). If every countable subset of G is right neutral in G, then G is said to be $\aleph_0$-*FSIN* [3].

Definition 5.1. Let us complete the previous concepts as follows. Let $A \subset G$. If for every $f \in \mathcal{RUC}(G)$ the set $\{f_g \mid g \in A\}$ is equicontinuous, or equivalently, if every subset of A is right neutral in G, then A is said to be *left functionally thin (Fthin) in* G [4]. If every countable subset of A is right neutral in G, then A is said to be *left* $\aleph_0$*-Fthin in* G. If there is at least one $V \in \mathcal{V}_G(e)$ that is left Fthin in G, then G is said to be *FASIN*. If there is at least one $V \in \mathcal{V}_G(e)$ that is left $\aleph_0$-Fthin in G, then G is said to be $\aleph_0$-*FASIN*. For instance, if G is a P-space (countable intersections of open sets in G are open), then G is $\aleph_0$-FASIN. Obviously, for any topological group, ASIN $\Rightarrow$ FASIN $\Rightarrow$ $\aleph_0$-FASIN (and as pointed out above, every locally precompact topological group is ASIN).

The following lemmas 5.2, 5.3, and 5.4 are used in the proof of Theorem 5.5; we prove them only for the $\aleph_0$-Fthin case, the other case being entirely analogous.

Lemma 5.2. *Let A be a left $\aleph_0$-Fthin (respectively, Fthin) subset of a topological group G. Then for any $x, y \in G$, xAy is a left $\aleph_0$-Fthin (respectively, Fthin) subset of G.*

Proof: Let B be a countable subset of xAy. Let $V \in \mathcal{V}_G(e)$; since $x^{-1}Vx \in \mathcal{V}_G(e)$ and since $x^{-1}By^{-1}$ is a countable subset of A, there exists $U \in \mathcal{V}_G(e)$ such that $(x^{-1}By^{-1})U \subset (x^{-1}Vx)(x^{-1}By^{-1})$; this implies $B(y^{-1}Uy) \subset VB$ and since $y^{-1}Uy \in \mathcal{V}_G(e)$, we can see that B is right neutral. □

Lemma 5.3. *Let G be a topological group and let $(A_i)_{i\in I}$ be a finite family of left $\aleph_0$-Fthin (respectively, Fthin) subsets of G. Then $\cup_{i\in I} A_i$ is a left $\aleph_0$-Fthin (respectively, Fthin) subset of G.*

Proof: Let B be a countable subset of $\cup_{i\in I} A_i$. Let $V \in \mathcal{V}_G(e)$. Since $B \cap A_i$ is a countable subset of A_i, there exists $U_i \in \mathcal{V}_G(e)$ such that $(B \cap A_i)U_i \subset V(B \cap A_i)$; let $U = \cap_{i\in I} U_i$; then $U \in \mathcal{V}_G(e)$ and $BU \subset VB$. Consequently, B is right neutral in G. □

Lemma 5.4. *Let G be a topological group. Let us suppose that for every neighborhood V of e in G there is a left $\aleph_0$-Fthin (respectively, Fthin) subset A of G such that $G = VA$. Then G is $\aleph_0$-FSIN (respectively, FSIN).*

Proof: Let B be a countable subset of G and let us verify that B is right neutral. Let $V \in \mathcal{V}_G(e)$. Let W be a symmetric neighborhood of e in G such that $W^3 \subset V$, and A be an $\aleph_0$-Fthin subset of G such that $G = WA$. For any $x \in G$, choose $w_x \in W$ and $a_x \in A$ such that $x = w_x a_x$, and put $D = \{a_x \mid x \in B\}$. Since D is a countable subset of A, there is $U \in \mathcal{V}_G(e)$ such that $DU \subset WD$. Let us verify that $BU \subset W^3 B$, which will prove the lemma. Let $y \in BU$. Let $x \in B$ and $u \in U$ such that $y = xu$; then $y = xu = w_x a_x u \in WDU$, and as $DU \subset WD$, we obtain that $y \in W^2 D$, i.e., $BU \subset W^2 D$. To conclude, it now suffices to remark that since for every $x \in B$, $a_x = w_x^{-1} x$, the inclusion $D \subset WB$ holds. □

Theorem 5.5. *Let G be an $\aleph_0$-FASIN topological group. Then the following statements are equivalent.*

1. *G is precompact.*
2. *For any two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G, the two subsets $\{s_m t_n \mid m \leq n\}$ and $\{s_m t_n \mid m > n\}$ of G are lower proximal.*

Proof: (1) ⇒ (2) has already been pointed out in Theorem 3.4, so suppose that condition (2) is satisfied and let us show that G is precompact.

First, let us verify that G is $\aleph_0$-FSIN by applying Lemma 5.4. Let $V \in \mathcal{V}_G(e)$. Since G is $\aleph_0$-FASIN, there is a left $\aleph_0$-Fthin neighborhood W of e in G such that $W \subset V$. By Lemma 3.3, there is a finite subset F of G such that $G = WFWF$; let us put $A = FWF$; then $G = VA$ and since $A = \cup_{(x,y)\in F\times F} xWy$ is $\aleph_0$-Fthin by 5.2 and 5.3, it follows from 5.4 that G is $\aleph_0$-FSIN.

Next, consider two sequences $(s_m)_{m\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in G, and $V \in \mathcal{V}_G(e)$; to prove that G is precompact, it suffices by virtue of Theorem 3.4 to show that $V\{s_m t_n \mid m \leq n\}$ and $\{s_m t_n \mid m > n\}$ intersect. Let W be a symmetric neighborhood of e in G such that $W^2 \subset V$; according to the first part of the proof, there is $U \in \mathcal{V}_G(e)$ such that $\{s_m t_n \mid m \leq n\}U \subset W\{s_m t_n \mid m \leq n\}$. Choose this U contained in W; then $U\{s_m t_n \mid m \leq n\}U \subset V\{s_m t_n \mid m \leq n\}$, and since by hypothesis $U\{s_m t_n \mid m \leq n\}U$ meets $\{s_m t_n \mid m > n\}$, the same is true for $V\{s_m t_n \mid m \leq n\}$. □

The method used in section 3 to obtain 3.6 from 3.4 allows us to obtain the following result from 5.5.

Theorem 5.6. *An $\aleph_0$-FASIN topological group G is precompact if and only if the equality $\mathcal{UC}(G) = \mathcal{WAP}(G)$ holds.*

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