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SUBCONTINUA OF FIBONACCI-LIKE INVERSE LIMIT SPACES

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ABSTRACT. We study the subcontinua of inverse limit spaces of Fibonacci-like unimodal maps. Under certain combinatorial constraints no other subcontinua than points, arcs, and $\sin \frac{1}{x}$ -curves are shown to exist. From the way these $\sin \frac{1}{x}$ -curves accumulate onto each other, a method of partially distinguishing the Fibonacci-like inverse limit spaces is proposed.

1. INTRODUCTION

The classification of inverse limit spaces of unimodal map is a tenacious problem. The main conjecture, posed by Ingram, is

If f and \tilde{f} are two non-conjugate unimodal maps, then the corresponding inverse limit spaces (I, f) and (I, \tilde{f}) are non-homeomorphic.

Let us restrict the unimodal map $f : I \rightarrow I$ to its core $I = [c_2, c_1]$, where c is the turning point and $c_k = f^k(c)$. We assume that f is locally eventually onto. Any such map is conjugate to a tent map T_s with slope $\pm s$ (where $\log s = h_{top}(f)$), and the resulting inverse limit space (I, f) is an indecomposable continuum.

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The classification (and an affirmative answer to Ingram conjecture) have been obtained for maps with a finite critical orbit [9], [10], [12], [3], but for the case that $\text{orb}(c)$ is infinite, and especially when c is recurrent, few results are known, see [1], [4], [2], [11].

In this paper, we extend the study of one example in [4], *the Fibonacci map*, and use subcontinua as a tool for partial classification of Fibonacci-like unimodal inverse limit spaces.

Fibonacci-like unimodal maps are defined combinatorially by a condition on their kneading maps Q , or equivalently their *cutting times* $\{S_k\}_{k \geq 0}$. Cutting times are those iterates n of the map such that the central branch of f^n maps onto c ; they satisfy a recursive relation $S_0 = 1$ and $S_k = S_{k-1} + S_{Q(k)}$, where $Q : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ is called the *kneading map*. In the sequel it will be more convenient to use

$$R(k) := Q(k+1), \quad \text{so } S_{k+1} = S_k + S_{R(k)}.$$

If $Q(k) = \max\{k-2, 0\}$, then the Fibonacci map is obtained (and the S_k s are the Fibonacci numbers, hence the name). We will consider unimodal maps are like Fibonacci map in the sense that $Q(k) \rightarrow \infty$. For such maps, it is known that c is recurrent and the critical omega-limit set $\omega(c)$ is a minimal Cantor set [6]. As a result, (I, f) will have uncountably many end-points which densely fill a Cantor set, but away from this Cantor set, (I, f) is locally homeomorphic to a Cantor set cross and arc; see, e.g., [7]. We will make further restrictions, reducing the complexity of the subcontinua in (I, f) even more, but there remains a rich structure and variety in the (arrangement of) subcontinua, so that some coarse classifications is still possible.

It is well-known that if $\text{orb}(c)$ is finite, then the only proper subcontinua of (I, f) are points and arcs. The same is true if f is *long-branched*, see [4]. In this paper we will mainly encounter *arc + ray continuum* which consist of a ray (or half-ray) and two (or one) arcs. The simplest such continuum is the $\sin \frac{1}{x}$ -curve, but more complicated arc + ray subcontinua can be found in many unimodal inverse limit spaces, see [4]. The arc(s) of an arc + ray continua are also called the *bar(s)* of the continuum \mathcal{H} . We will denote this bar by $\text{bar}(\mathcal{H})$.

Theorem 1.1. *If f is a unimodal map such that*

$$(1) \quad Q(k) \rightarrow \infty \text{ and } R(1+k) > 1 + R^2(k)$$

for all k sufficiently large, then the only proper subcontinua of (I, f) are points, arcs and $\sin \frac{1}{x}$ -curves.

Let \hat{f} denote the induced homeomorphism on (I, f) .

Theorem 1.2. *If (1) is satisfied, then there is a one-to-one correspondence between \hat{f} -orbits of $\sin \frac{1}{x}$ -curves in (I, f) and infinite backward R -orbits in \mathbb{N} .*

The following corollaries are immediate.

Corollary 1.1. *If R is eventually surjective, then there are only countably many $\sin \frac{1}{x}$ -curves in (I, f) . In particular, if $Q(k) = \max\{0, k - d\}$, then there are $d - 1$ distinct \hat{f} -orbits of $\sin \frac{1}{x}$ -curves.*

However, we do not know if (I, f) are distinct for different values of d .

Corollary 1.2. *If $\#R^{-1}(k) \geq 2$ for all sufficiently large k , then there are uncountably many \hat{f} -orbits of $\sin \frac{1}{x}$ -curves in (I, f) .*

Although the subcontinua presented here are all $\sin \frac{1}{x}$ -curves, they can be used, to some extent, to distinguish inverse limit spaces by the way they (or rather their bars) accumulate onto each other.

Proposition 1.3. *For each $k \in \mathbb{N}$ that appears in infinitely many backward R -orbits there is a $\sin \frac{1}{x}$ -curve whose bar is the limit of the bars of other $\sin \frac{1}{x}$ -curves.*

The orbit structure of R can be expressed by a tree with vertices labeled by \mathbb{N} and arrows $k \rightarrow l$ if $l = R(k)$; see, e.g., Figure 1. The tree is rooted at 0. Although $R(0) = 0$, we do not write the arrow $0 \rightarrow 0$. A backward R -orbit is an infinite backward path in this tree.

Based on an idea of Brian E. Raines [11], and using Proposition 1.3, we can define the equivalent of Cantor-Bendixson depth for $\sin \frac{1}{x}$ -curves. We say that the *depth* of a $\sin \frac{1}{x}$ -curve is 0 if its bar B is not the accumulation of a sequence of bars of other $\sin \frac{1}{x}$ -curves (even though a sequence of bars may accumulate to a proper subset of B). We continue inductively, saying that a $\sin \frac{1}{x}$ -curve has *depth* d if, after removing all $\sin \frac{1}{x}$ -curves of lower depth have been removed, its bar is not the accumulation of a sequence of bars of other $\sin \frac{1}{x}$ -curves. The $\sin \frac{1}{x}$ -curves that are never removed in this process are said to have *depth* ∞ .

Example 1. Figure 1 gives an example of a backward tree of the map R , and indicates the depths of some of its $\sin \frac{1}{x}$ -curves. The vertices in the lower half of the picture (including the leftmost vertex) all have two preimages, and therefore infinitely many backward R -orbits. Hence, all the subcontinua corresponding to such a backward R -orbit have depth ∞ . The vertices in the upper half, excluding the uppermost path, have only one backward R -orbit; the corresponding subcontinua have depth 0. After removing these subcontinua (and hence the corresponding backward R -orbits), the vertices in the remaining uppermost have only one backward R -orbit left, so their depths are 1.

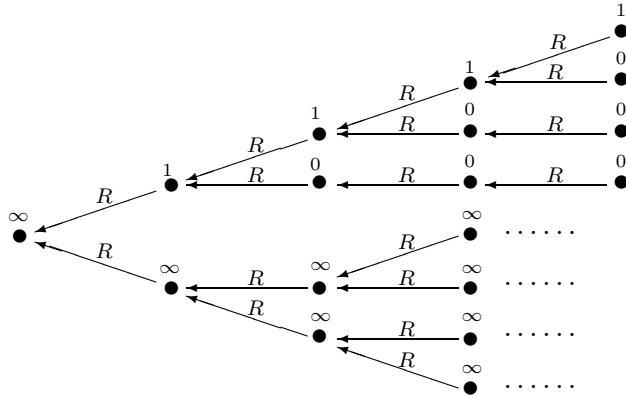


FIGURE 1. Example of a backward tree of R with the depths of each vertex indicated.

Example 2. Let $k_1 = 1$, $k_2 = 3$, and $k_i = k_{i-1} + i = \frac{i(i+1)}{2}$ be the i -th triangular number. Define two unimodal maps f and \tilde{f} by means of their functions R and \tilde{R} , respectively:

$$\begin{cases} R(1) = R(2) = 0; \\ R(k_i) = k_{i-1} \text{ for } i \geq 2; \\ R(k_i + l) = k_{i-1} + l \text{ for } 1 \leq l < i; \\ R(k_i + i) = k_{i-1} + i - 1, \end{cases}$$

and

$$\begin{cases} \tilde{R}(1) = \tilde{R}(2) = 0; \\ \tilde{R}(k_i) = k_{i-1} \text{ for } i \geq 2; \\ \tilde{R}(k_i + l) = k_{i-1} \text{ for } 1 \leq l < i; \\ \tilde{R}(k_i + i) = k_{i-1} + i - 1. \end{cases}$$

Then (I, f) and (I, \tilde{f}) are non-homeomorphic. Indeed, (I, f) has $\sin \frac{1}{x}$ -curves of Cantor-Bendixson depth 0 and 1, whereas (I, \tilde{f}) has only $\sin \frac{1}{x}$ -curves of Cantor-Bendixson depth 0. This can be seen from the trees of the backward orbits of R and \tilde{R} , respectively; see Figure 2.

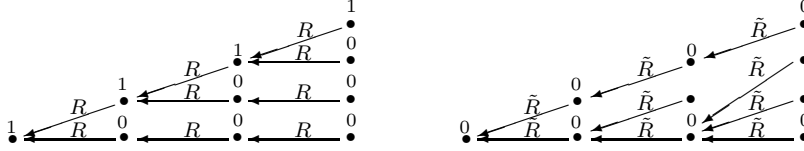


FIGURE 2. The backward trees of R and \tilde{R} with the depths of each vertex indicated.

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2. PRELIMINARIES

The tent map $T_s : [0, 1] \rightarrow [0, 1]$ with slope $\pm s$ is defined as

$$T_s(x) = \begin{cases} sx & \text{if } x \leq \frac{1}{2}, \\ s(1-x) & \text{if } x > \frac{1}{2}. \end{cases}$$

We fix $s \in (\sqrt{2}, 2]$ and call $f = T_s$. Write $c = \frac{1}{2}$ and $c_k = f^k(c)$. Let $I = [c_2, c_1]$ be the *core* of the map. It is well-known that f is *locally eventually onto*; i.e., for every non-degenerate interval J , there is n such that $f^n(J) = I$.

The inverse limit space (I, f) is

$$(I, f) = \{x = (x_0, x_1, x_2, \dots) : I \ni x_i = f(x_{i+1}) \text{ for all } i \geq 0\},$$

equipped with metric $d(x, y) = \sum_{n \geq 1} |x_n - y_n| 2^{-n}$ and induced (or shift) homeomorphism

$$\hat{f}(x_0, x_1, x_2, \dots) = (f(x_0), x_0, x_1, x_2, \dots).$$

Let $\pi_k : (I, f) \rightarrow I$, $\pi_k(x) = x_k$ be the k -th projection map.

To describe the combinatorial structure of f , we recall the definition of cutting times and kneading map from [8], [6]. If J is a maximal (closed) interval on which f^n is monotone, then $f^n : J \rightarrow f^n(J)$ is called a *branch*. If $c \in \partial J$, then $f^n : J \rightarrow f^n(J)$ is a *central branch*. Obviously, f^n has two central branches, and they have the same image if n is sufficiently large. Denote this image (or the largest of the two) by D_n . If $D_n \ni c$, then n is called a *cutting time*. Denote the cutting times by $\{S_i\}_{i \geq 0}$, $S_0 < S_1 < S_2 < \dots$. If the slope $s > 1$, then $S_0 = 1$ and $S_1 = 2$. It can be shown that there is a map $Q : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$, called a *kneading map*, such that $S_k - S_{k-1} = S_{Q(k)}$ for all $k \geq 1$. This map satisfies $Q(k) < k$ plus other conditions that will not be of our concern in this paper. Recall that $R(k) = Q(k+1)$.

In this paper, we are interested in maps with kneading map $Q(k) \rightarrow \infty$. In this case, the critical omega-limit set $\omega(c)$ is a minimal Cantor set [6]. We call z a *closest precritical point* if $f^n(z) = c$ for some $n \geq 1$ and $f^m((c, z)) \not\ni c$ for $m < n$. It is not hard to show (see [6]), that the integers n for which this happens are exactly the cutting times, so we write z_k when $n = S_k$. Moreover, $z_{R(k)}$ is the closest precritical point with smallest index in $[c, c_{S_k}]$. (There is a closest precritical point of the same index at either side of c ; we will denote the one in $[c, c_{S_k}]$ by $z_{R(k)}$ and the other by $\hat{z}_{R(k)}$.) Since $|c - c_{S_k}| \leq |c - z_{R(k)-1}| \rightarrow 0$ as $R(k) \rightarrow \infty$, the condition $\lim_{k \rightarrow \infty} R(k) = \infty$ obviously implies that $|c - c_{S_k}| \rightarrow 0$ as $k \rightarrow \infty$.

We will use the following terminology. A *ray* is a continuous copy of $(0, 1)$; a *half-ray* is a continuous copy of $[0, 1)$. A *continuum* is a compact connected metric space and a *subcontinuum* \mathcal{H} is a subset of a continuum which is closed and connected itself. The $\sin \frac{1}{x}$ -curve is the homeomorphic image of the graph $\{(t, \sin \frac{1}{t}) : t \in (0, 1]\}$ together with the arc (called *bar*) $\{0\} \times [-1, 1]$. Given a subcontinuum \mathcal{H} of (I, f) , its *critical projections times* are the integers n such that $\mathcal{H}_n \ni c$.

Lemma 2.1. *If $\mathcal{H} \subset (I, f)$ is a subcontinuum that is not an arc or point, there is an infinite sequence of critical projections times $\{n_i\}_{i \geq 0}$ such that*

- (1) *if \mathcal{H} is a proper subcontinuum, then $|\mathcal{H}_n| \rightarrow 0$ as $n \rightarrow \infty$;*
- (2) *$n_i - n_{i-1} = S_{k_i}$ for some k_i and $k_i \rightarrow \infty$ as $i \rightarrow \infty$;*

- (3) $\pi_{n_{i-1}}(\mathcal{H}) \supset [c, c_{S_{k_i}}]$;
 (4) $k_{i-1} \leq R(k_i) \leq k_i$ for all i , so $\{k_i\}_{i \geq 1}$ is non-decreasing.

Proof: If there are only finitely many critical projections times, say n is the largest, then \mathcal{H} can be parametrized by $t \in \mathcal{H}_n$, so that \mathcal{H} is a point or an arc. Therefore, subcontinua other than arcs or points have infinitely many critical projections times.

Let us prove the other statements:

- (1) The first statement follows because f is locally eventually onto. If $\liminf_n |\mathcal{H}_n| := \varepsilon > 0$, then there is an interval J that belongs to infinitely many \mathcal{H}_n . Take N such that $f^N(J) = I$. Then $I \in \mathcal{H}_{n-N} = f^N(\mathcal{H})$ infinitely often, so $\mathcal{H} = (I, f)$.
 (2) Let k_i be the lowest index of closest precritical points in \mathcal{H}_{n_i} . Then the next iterate such that $\mathcal{H}_n \ni c$ is $n_{i-1} = n_i - S_{k_i}$ iterates.
 (3) Follows immediately from 2.
 (4) As $\mathcal{H}_{n_{i-1}} \supset [c, c_{S_{k_i}}] \ni z_{R(k_i)}$, $S_{k_{i-1}} = n_i - n_{i-1} \leq S_{R(k_i)}$, and (4) follows. \square

For each i there are closed intervals M_{n_i} and L_{n_i} such that

$$\mathcal{H}_{n_i} = M_{n_i} \cup L_{n_i}, \quad M_{n_i} \cap L_{n_i} = \{c\} \text{ and } f^{n_i - n_{i-1}}(M_{n_i}) = \mathcal{H}_{n_{i-1}}$$

Lemma 2.2. *Every subcontinuum \mathcal{H} contains a dense ray*

$$\text{ray}(\mathcal{H}) := \{x \in \mathcal{H} : x_{n_i} \in M_{n_i} \text{ for all } i \text{ sufficiently large}\}.$$

Proof: This proof is given in [4], but we include it for completeness. We will define a parametrization $\varphi : \mathbb{R} \rightarrow \mathcal{H}$ of a subset of \mathcal{H} and show that $\varphi(\mathbb{R})$ lies dense in \mathcal{H} . For the construction of φ , it suffices to construct

$$\varphi_i : \mathbb{R} \rightarrow \mathcal{H}_{n_i} \text{ such that } \varphi_j = f^{n_i - n_j} \circ \varphi_i$$

for all $1 \leq j \leq i$. We do this inductively. First, let $\varphi_0 : [-1, 1] \rightarrow \mathcal{H}_{n_0}$ be any homeomorphism, and set $a_0 = -1$ and $b_0 = 1$.

If $\varphi_{i-1} : [a_{i-1}, b_{i-1}] \rightarrow \mathcal{H}_{n_{i-1}}$ is defined, let $\varphi_i[a_{i-1}, b_{i-1}] \rightarrow M_{n_i}$ be such that $\varphi_{i-1} = f^{n_i - n_{i-1}} \circ \varphi_i$. Then either $\varphi_i(a_{i-1}) = c$ or $\varphi_i(b_{i-1}) = c$.

- If $\varphi_i(a_{i-1}) = c$, then set $a_i = a_{i-1} - 1$ and $b_i = b_{i-1}$. Extend φ to $[a_i, a_{i-1}]$ such that it maps homeomorphically on L_{n_i} . Then going inductively downwards from $j = i - 1$ to 1, define $\varphi_j : [a_i, a_{i-1}] \rightarrow \mathcal{H}_{n_j}$ such that $\varphi_j = f^{n_{j+1} - n_j} \circ \varphi_j$.

- If $\varphi_i(b_{i-1}) = c$, then set $b_i = b_{i-1} + 1$ and $a_i = a_{i-1}$. Extend φ to $[b_{i-1}, b_i]$ such that it maps homeomorphically on L_{n_i} . Then going inductively downwards from $j = i - 1$ to 1, define $\varphi_j : [b_{i-1}, b_i] \rightarrow \mathcal{H}_{n_j}$ such that $\varphi_j = f^{n_{j+1}-n_j} \circ \varphi_{j+1}$.

If $a_i \rightarrow -\infty$ and $b_i \rightarrow \infty$, then this defines the ray. If $\inf a_i > -\infty$ and/or $\sup b_i < \infty$, then φ parametrizes a half-ray or arc. In this case, we can restrict φ to $(\inf a_i, \sup b_i)$.

To show that $\varphi(\mathbb{R})$ is dense in \mathcal{H} , take $\varepsilon > 0$ and i so large that $2^{-n_i} < \varepsilon$. Now, for any $x \in \mathcal{H}$, take $t \in \mathbb{R}$ such that $\varphi_i(t) = x_{n_i}$. Then $\varphi(t)_n = x_n$ for $n \leq n_i$ and $d(x, \varphi(t)) \leq \sum_{m > n_i} 2^{-m} = 2^{-n_i} < \varepsilon$. \square

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1: Let \mathcal{H} be a proper subcontinuum of (I, f) which is more complicated than an arc, so it has infinitely many critical projections times $\{n_i\}_{i \geq 1}$ and $S_{k_i} = n_i - n_{i-1}$. By applying \hat{f}^{-1} repeatedly (which has the effect of subtracting a large fixed number from the critical projection times n_i), we can assume that (1) holds for all $k \geq k_1$.

Let i be arbitrary; we know that $(c, c_{S_{k_i}}]$ is one component of $\mathcal{H}_{n_{i-1}} \setminus \{c\}$. Suppose n_{i-1} is such that $M_{n_{i-1}} = [c, c_{S_{k_i}}]$. Then $n_{i-1} - n_{i-2} = S_{k_{i-1}} = S_{R(k_i)}$, and

$$(2) \quad \mathcal{H}_{n_{i-2}} = [c_{S_{k_{i-1}}}, c_{S_{1+k_i}}] = [c_{S_{R(k_i)}}, c_{S_{1+k_i}}] \supset \{z_{R^2(k_i)}, z_{R(1+k_i)}\}.$$

Therefore, $k_{i-2} = \min\{R^2(k_i), R(1+k_i)\} = R^2(k_i)$ by (1). It follows that $M_{n_{i-2}} = [c, c_{S_{R(k_i)}}]$ and $L_{n_{i-2}} = [c, c_{S_{1+k_i}}]$. We obtain by induction that $M_{n_j} = [c, c_{S_{k_{j+1}}}]$ and $k_j = R(k_{j+1})$ for all $j < i$.

This leaves us with two cases:

Case A. There are i arbitrarily large such that $M_{n_{i-1}} = [c, c_{S_{k_i}}]$, and therefore $M_{n_{i-1}} = [c, c_{S_{k_i}}]$ for all i , or

Case B. $L_{n_{i-1}} = [c, c_{S_{k_i}}]$ for all i sufficiently large.

We tackle Case B first. Let us call $\mathcal{T}_{n_{i-1}} = [z_{R(k_i)}, c_{S_{k_i}}]$ the *tip* of $\mathcal{H}_{n_{i-1}}$. We claim that $f^{n_{i-1}}$ is monotone on $\mathcal{T}_{n_{i-1}}$.

If $k_{i-1} = R(k_i)$, then

$$z_{R(1+k_i)} \in [c, c_{S_{1+k_i}}] = f^{S_{k_{i-1}}}(\mathcal{T}_{n_{i-1}}) \subset \mathcal{H}_{n_{i-1}}$$

is the closest precritical point with lowest index in $f^{S_{k_{i-1}}}(\mathcal{T}_{n_{i-1}})$. So to prove the claim, we need to show that $S_{R(1+k_i)} \geq n_{i-2}$. Using (1) and Lemma 2.1(4), respectively, we find for any j ,

$$\begin{aligned} S_{2+k_j} &= S_{1+k_j} + S_{R(1+k_j)} \\ &\geq S_{1+k_j} + S_{2+R^2(k_j)} \geq S_{1+k_j} + S_{2+k_{j-2}}. \end{aligned}$$

Using this repeatedly, we obtain

$$\begin{aligned} S_{2+k_{i-2}} &\geq S_{1+k_{i-2}} + S_{2+k_{i-4}} \\ &\geq S_{1+k_{i-2}} + S_{1+k_{i-4}} + S_{2+k_{i-6}} \\ &\quad \vdots \quad \quad \quad \vdots \\ &\geq S_{1+k_{i-2}} + S_{1+k_{i-4}} + S_{1+k_{i-6}} + S_{1+k_{i-8}} + \dots \end{aligned}$$

Therefore, using Lemma 2.1 and (1) once more,

$$\begin{aligned} n_{i-2} &= S_{k_{i-2}} + S_{k_{i-3}} + S_{k_{i-4}} + S_{k_{i-5}} + S_{k_{i-6}} + S_{k_{i-7}} \dots \\ &\leq S_{k_{i-2}} + S_{R(k_{i-2})} + S_{k_{i-4}} + S_{R(k_{i-4})} + S_{k_{i-6}} + S_{R(k_{i-6})} \dots \\ &\leq S_{1+k_{i-2}} + S_{1+k_{i-4}} + S_{1+k_{i-6}} \dots \\ &\leq S_{2+k_{i-2}} \leq S_{1+R^2(k_i)} \leq S_{R(1+k_i)}, \end{aligned}$$

as claimed. If $k_{i-1} < R(k_i)$, then the same computation gives strict inequality $n_{i-2} < S_{R(1+k_i)}$.

We can picture the behavior of $f^{n_i-n_1} : L_{n_i} \rightarrow I$ as a strip of paper that is folded over and over again; see Figure 3 with the property that whenever a piece of strip is folded, only one subpiece is folded again.

It follows that the number of branches of $f^{n_i-n_1}|_{L_{n_i}}$ grows linear in i . Let us distinguish between ‘‘long branches’’ (i.e., the ones whose image contains c), and ‘‘short ones,’’ whose image does not contain c . Then there is an easy recurrence relation between their cardinalities l_i and s_i :

$$\begin{pmatrix} l_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} l_i \\ s_i \end{pmatrix} = \begin{pmatrix} l_i + 1 \\ l_i + s_{i-1} - 1 \end{pmatrix}.$$

This gives $l_i + s_i = i + \frac{(i-1)(i-2)}{2} = \frac{i(i-1)}{2} + 1$. Furthermore, if $k_j = R(k_{j+1})$ for all $j < i$, then the parts of L_{n_i} that are folded at step n_j map to

$$f^{n_{j+1}}(\mathcal{T}_{n_{j+1}}) = f^{n_j}([c, c_{S_{1+k_{j+2}}}] = [c_{S_{Q(2+k_{j+2})}}, c_{S_{2+k_{j+2}}}]$$

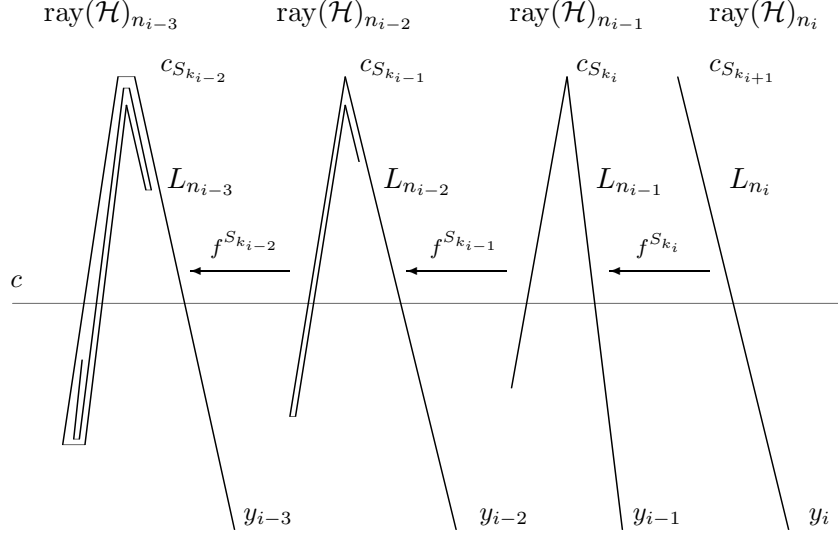


FIGURE 3. Impression how the critical projections of $\text{ray}(\mathcal{H})$ in Case B are parametrized. The sequence of points $\{y_i\}$ is explained in the proof of Theorem 1.2.

These are nested intervals, and their diameters tend to 0 as $j \rightarrow \infty$. Therefore, the limit set of $\text{ray}(\mathcal{H})$ is a spiralling arc; it is the set of points $x \in \mathcal{H}$ that belong to L_{n_i} infinitely often (in fact, every other i).

This shows that \mathcal{H} is an arc + ray continuum. We postpone the proof that \mathcal{H} is $\sin \frac{1}{x}$ -curves until we have treated Case A: $M_{n_{i-1}} = [c, c_{S_{k_i}}]$ (and hence, $k_{i-1} = R(k_i)$) for all i . By (2), $L_{n_{i-1}} = [c, c_{S_{1+k_{i+1}}}]$; see Figure 4.

We claim that $f^{n_{i-1}}|_{L_{n_{i-1}}}$ is monotone. Since $z_{R(1+k_{i+1})}$ is the closest precritical point of lowest index in $L_{n_{i-1}}$, this claim follows from $S_{R(1+k_{i+1})} \geq n_{i-1}$, but this is the same as the computation of Case B with $i-2$ replaced by $i-1$. The image $f^{S_{R(1+k_{i+1})}}(L_{n_{i-1}}) = [c_{S_{R(1+k_{i+1})}}, c_{S_{2+k_{i+1}}}]$ is an interval whose length tends to 0 as $i \rightarrow \infty$. Since $f^{n_{i-1}}(L_{n_{i-1}})$ is a preimage of this, also $|f^{n_{i-1}}(L_{n_{i-1}})| \rightarrow 0$ as $i \rightarrow \infty$. This shows (cf. Figure 4), that $\{f^{n_i}(L_{n_i})\}_{i \text{ even}}$ and $\{f^{n_i}(L_{n_i})\}_{i \text{ odd}}$ are nested sequence of intervals converging to

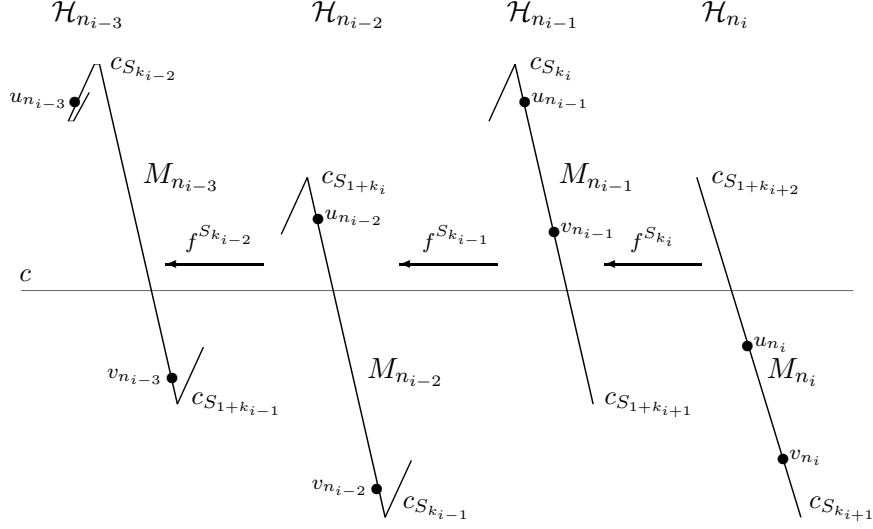


FIGURE 4. Impression of the critical projections of \mathcal{H} in Case A. The points u_{n_j} and v_{n_j} feature in the proof that the subcontinuum of Case B is a $\sin \frac{1}{x}$ -curve.

points as $i \rightarrow \infty$. Using a parametrization analogous to the one of Lemma 2.2, we find that \mathcal{H} is a (spiralling) arc after all.

Remark. In fact, the arc of Case A is the bar of the arc + ray subcontinuum of Case B. This is because the subcontinua of Case A and Case B are the only ones with infinitely many critical projection times, and any Case A arc has a ray converging onto it as explained in the proof of Theorem 1.2 below.

We finish by proving that the subcontinuum \mathcal{H} from Case B is indeed a $\sin \frac{1}{x}$ -curve. A $\sin \frac{1}{x}$ -curve \mathcal{H} can be characterized (and thus distinguished from other arc + ray continua) by the following property: For all $u \neq v$ in the interior of $B := \text{bar}(\mathcal{H})$ and all disjoint neighborhoods $U \ni u$ and $V \ni v$, there are neighborhoods U', U'' with $u \in U'' \subset U' \subset U$ such that for every $x \in \text{ray}(\mathcal{H}) \cap U''$, if we follow the ray from x in at least one direction, we visit V before returning to U' . In other words, we can parametrize $\text{ray}(\mathcal{H})$ by $\varphi : \mathbb{R} \rightarrow \mathcal{H}$ such that $x = \varphi(0)$, and if $t_0 = \inf_{t>0} \varphi(t) \notin U'$, and $t_1 = \sup_{t>t_0} \varphi([t_0, t]) \cap U' = \emptyset$, then $\varphi([t_0, t_1]) \cap V \neq \emptyset$.

Take $u \neq v$ in the interior of B . Then there is i such that at critical projection $\pi_{n_i}(B)$, there are no “folds” overlapping at u_{n_i} and v_{n_i} , i.e., $\pi_{n_i}^{-1}(u_{n_i}) \cap B = u$ and $\pi_{n_i}^{-1}(v_{n_i}) \cap B = v$; see Figure 4. Take neighborhoods $U \ni u$ and $V \ni v$ disjoint but arbitrary otherwise. Let $J \subset B_{n_i}$ be a neighborhood of u_{n_i} such that $\pi_{n_i}^{-1}(\bar{J}) \cap B \subset U$ and $\#(\pi_{n_i}^{-1}(y) \cap B) = 1$ for each $y \in J$. Construct a “tubular neighborhood” of $\pi_{n_i}^{-1}(J)$ by setting

$$U' = \{x \in \mathcal{H} : \pi_{n_i}(x) \in J \text{ and } d(x, \pi_{n_i}^{-1}(J)) < \varepsilon\},$$

where ε is so small that $U' \subset U$ and

$$\{x \in \mathcal{H} : \pi_{n_i}(x) = v_{n_i} \text{ and } d(x, v) < \varepsilon\} \subset V.$$

Next let $U'' \ni u$ be so small that for every $x \in U'' \cap \text{ray}(\mathcal{H})$, when following the ray from x in either direction, we visit U' at least once more.

Note that the “folds” of both B and $\text{ray}(\mathcal{H})$ (taken sufficiently close to the bar) project to the same points under π_{n_i} , namely points of the form $f^{n_j - n_i}(c)$ for $j > i$. There are no such points between u_{n_i} and v_{n_i} , so $\pi_{n_i}^{-1}([v_{n_i}, u_{n_i}]) \cap \{x \in \mathcal{H} : d(x, B) < \varepsilon\}$ consists of a countable collection of arcs, each of which projects onto $[u_{n_i}, v_{n_i}]$ under π_{n_i} . It follows that for each $x \in \text{ray}(\mathcal{H}) \cap U''$, when following the ray from x in the direction of v , we visit V before returning to U'' . This completes the proof. \square

Proof of Theorem 1.2: Let \mathcal{H} be a proper subcontinuum that is not a point or an arc. From Theorem 1.1, we know that its sequence k_i satisfies $R(k_i) = k_{i-1}$ for all i sufficiently large. By applying \hat{f}^{-1} sufficiently often, we can assume that $R(k_i) = k_{i-1}$ for all i . Hence, \mathcal{H} corresponds to a backward R -orbit.

Conversely, given a backward R -orbit $\{k_i\}$, construct the subcontinuum \mathcal{H} of Case B with $n_1 = 0$, $n_i = \sum_{j=1}^i S_{k_j}$ (so $n_i - n_{i-1} = S_{k_i}$), and

$$\mathcal{H}_{n_i} = [cS_{k_{i+1}}, y_i] \ni c$$

such that $L_{n_i} = [c, cS_{k_{i+1}}]$ and $M_{n_i} = [c, y_i]$. The points y_i will be chosen inductively to satisfy

- $|y_i - c| > |cS_{k_{i+1}} - c|$, so M_{n_i} is larger than L_{n_i} , and $M_{n_i} \ni \hat{z}_{R(k_{i+1})}$;

- $\{z_{R(k_{i+1})-1}, \hat{z}_{R(k_{i+1})-1}\} \notin [c, y_i]$, so that \mathcal{H}_{n_i} maps indeed homeomorphically for $S_{k_i} = S_{R(k_{i+1})}$ iterates;
- $f^{S_{k_i}}(y_i) = y_{i-1}$.

Because Q is non-decreasing and $Q(k) \rightarrow \infty$,

$$|c_{S_{1+k_{i+1}}} - c| < |c_{S_{k_i}} - c| = |\hat{c}_{S_{k_i}} - c| < |c_{S_{Q(k_i)}} - c|.$$

The interval $[\hat{c}_{S_{1+k_{i+1}}}, \hat{z}_{R(k_{i+1})}]$ is mapped homeomorphically onto $[c_{S_{1+k_i}}, c_{S_{Q(k_i)}}]$ by $f^{S_{k_i}}$; see Figure 5. Therefore, for any point $y_{i-1} \in [\hat{c}_{S_{k_i}}, c_{S_{Q(k_i)}}]$, there is a point $y_i \in [\hat{c}_{S_{1+k_{i+1}}}, c_{S_{Q(k_{i+1})}}]$ such that $f^{S_{k_i}}(y_i) = y_{i-1}$. Thus, we can indeed find a sequence $\{y_i\}$ satisfying the conditions above. The resulting subcontinuum \mathcal{H} is a $\sin \frac{1}{x}$ -curve according to Theorem 1.1. \square

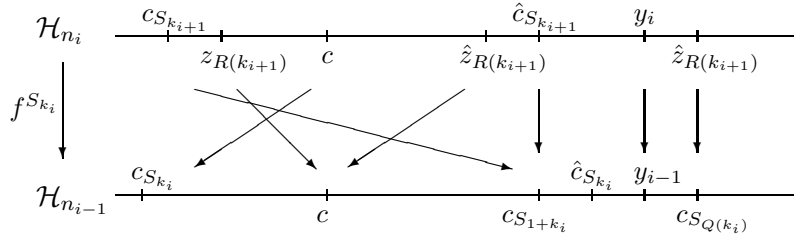


FIGURE 5. Points in \mathcal{H}_{n_i} and their images under $f^{S_{k_i}}$.

Proof of Proposition 1.3: Since $R(k) \rightarrow \infty$, $\#R^{-1}(k) < \infty$ for each k . Suppose that k_1 appears in infinitely many backward R -orbits. Using the pigeon hole principle, we can find an infinite backward R -orbit $k_1 \xleftarrow{R} k_2 \xleftarrow{R} \dots$ such that for each j , there is another infinite backward R -orbit that coincides with it up to at least k_j . Let \mathcal{H} and $\{\mathcal{H}^j\}_{j \in \mathbb{N}}$ be the corresponding type B subcontinua, as constructed in Theorem 1.2. Then \mathcal{H} and \mathcal{H}^j have the same critical projections times up till n_{j-1} . As $n_{j-1} - n_{j-2} \rightarrow \infty$, and hence $|\mathcal{H}_{n_k}|, |\mathcal{H}_{n_k}^j| \rightarrow 0$, this means that $\text{bar}(\mathcal{H}^j) \rightarrow \text{bar}(\mathcal{H})$.

Conversely, if there is a sequence $\{\mathcal{H}^j\}_j$ of disjoint subcontinua (having distinct sequences of critical projection times), such that $\text{bar}(\mathcal{H}^j) \rightarrow \text{bar}(\mathcal{H})$, then for each k , there is j_0 such that the critical projection times of \mathcal{H} and \mathcal{H}^j coincide up to k for all $j \geq j_0$. This

implies that k_1 belongs to infinitely many different backward R -orbits. \square

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