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RETRACTS OF SIGMA-PRODUCTS OF HILBERT CUBES

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ABSTRACT. We consider the sigma-product of the ω_1 -power of the Hilbert cube. This space is characterized among its retracts as the only one without G_δ -points.

1. INTRODUCTION

Sigma-products and their subspaces have been extensively studied by topologists and functional analysts for several decades. We refer the reader to [4] where a comprehensive survey of the related results from both topology and functional analysis are discussed in detail.

Recall that the sigma-product $\Sigma(X, *)$ of an uncountable collection $\{X_t: t \in T\}$ of spaces with base points $*_t \in X_t$, $t \in T$, is the subspace of the product $X = \prod\{X_t: t \in T\}$ defined as

$$\Sigma(X, *) = \{\{x_t: t \in T\} \in X: |\{t \in T: x_t \neq *_t\}| \leq \omega\}.$$

We are interested in the case when each X_t is a copy of the Hilbert cube I^ω , $|T| = \omega_1$, and $*_t$ is the point (in I^ω) all coordinates of which equal to 0. The corresponding sigma-product is denoted by Σ .

Our main result (Theorem 3.1) states that if a retract of Σ does not contain G_δ -points, then it is homeomorphic to Σ .

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2. AUXILIARY LEMMAS

Terminology, notation, and results related to inverse spectra and absolute retracts used here can be found in [1]. One of the main concepts we need below is that of ω -spectra $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$. These are ω -continuous inverse spectra consisting of metrizable compact spaces X_α , surjective projections $p_\alpha^\beta: X_\beta \rightarrow X_\alpha$, $\alpha \leq \beta$, and an ω -complete indexing set A . This essentially means that A contains supremums of countable chains and that for any such chain $\{\alpha_n: n \in \omega\}$ the space X_α , where $\alpha = \sup\{\alpha_n: n \in \omega\}$, is naturally homeomorphic to the limit of the inverse sequence $\mathcal{S}_\alpha = \{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}}, \omega\}$.

Recall also that a map $p: X \rightarrow Y$ of compact spaces is soft if, for any compactum B , its closed subset A and maps g and h , such that the following diagram (of undotted arrows) commutes

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ g \uparrow & \searrow k & \uparrow h \\ A & \xrightarrow{i} & B \end{array}$$

there exists a map $k: B \rightarrow X$ (the dotted arrow) such that $k|_A = g$ and $fk = h$.

The prime example of a soft map is the projection $X \times I^\omega \rightarrow X$.

Finally, recall that for a given map $p: X \rightarrow Y$, a closed subset $F \subseteq X$ is a fibered Z -set in X (with respect to p) if, for any open cover $\mathcal{U} \in \text{cov}(X)$, there exists a map $f_{\mathcal{U}}: X \rightarrow X$ such that $pf_{\mathcal{U}} = p$ (i.e., $f_{\mathcal{U}}$ acts fiberwise), $f_{\mathcal{U}}(X) \cap F = \emptyset$, and $f_{\mathcal{U}}$ is \mathcal{U} -close to id_X .

Lemma 2.1. *Let a non-metrizable compact space X be represented as the limit of an ω -spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$ with soft limit projections $p_\alpha: X \rightarrow X_\alpha$. Suppose that F is a closed subset of X containing no closed G_δ -subsets of X . Then for each $\alpha \in A$, there exists $\beta \in A$, with $\beta > \alpha$, such that there is a map $i_\alpha^\beta: X_\alpha \rightarrow X_\beta$ satisfying the following two properties:*

- (1) $p_\alpha^\beta i_\alpha^\beta = \text{id}_{X_\alpha}$,
- (2) $i_\alpha^\beta(X_\alpha) \cap p_\beta(F) = \emptyset$.

Proof: Let $\alpha_0 = \alpha$ and $x_0 \in X_{\alpha_0}$. Since $p_{\alpha_0}^{-1}(x_0)$ is closed and G_δ in X , it follows from the assumption that $p_{\alpha_0}^{-1}(x_0) \setminus F \neq \emptyset$. Take

an index $\alpha_1 \in A$ such that $\alpha_1 > \alpha_0$ and $(p_{\alpha_0}^{\alpha_1})^{-1}(x_0) \setminus p_{\alpha_1}(F) \neq \emptyset$. Let $x_1 \in (p_{\alpha_0}^{\alpha_1})^{-1}(x_0) \setminus p_{\alpha_1}(F)$. The softness of the projection $p_{\alpha_0}^{\alpha_1}: X_{\alpha_1} \rightarrow X_{\alpha_0}$ guarantees the existence of a map $i_0^1: X_{\alpha_0} \rightarrow X_{\alpha_1}$ such that $p_{\alpha_0}^{\alpha_1} i_0^1 = \text{id}_{X_{\alpha_0}}$ and $i_0^1(x_0) = x_1$. Let

$$V_1 = \{x \in X_{\alpha_0} : i_0^1(x) \notin p_{\alpha_1}(F)\}.$$

Note that $x_0 \in V_1$ and consequently, V_1 is a non-empty open subset of X_{α_0} .

Let $\gamma < \omega_1$. Suppose that for each λ , $1 \leq \lambda < \gamma$, we have already constructed an index $\alpha_\lambda \in A$, an open subset $V_\lambda \subseteq X_{\alpha_0}$, and a section $i_0^\lambda: X_{\alpha_0} \rightarrow X_{\alpha_\lambda}$ of the projection $p_{\alpha_0}^{\alpha_\lambda}: X_{\alpha_\lambda} \rightarrow X_{\alpha_0}$, satisfying the following conditions:

- (i) $\alpha_\lambda < \alpha_\mu$, whenever $\lambda < \mu < \gamma$;
- (ii) $\alpha_\mu = \sup\{\alpha_\lambda : \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal;
- (iii) $V_\lambda \subsetneq V_\mu$, whenever $\lambda < \mu < \gamma$;
- (iv) $V_\mu = \cup\{V_\lambda : \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal;
- (v) $i_0^\mu = \Delta\{i_0^\lambda : \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal;
- (vi) $i_0^\lambda = p_{\alpha_\lambda}^{\alpha_\mu} i_0^\mu$, whenever $\lambda < \mu < \gamma$;
- (vii) $V_\lambda = \{x \in X_{\alpha_0} : i_0^\lambda(x) \notin p_{\alpha_\lambda}(F)\}$.

We shall construct the index α_γ , the open subset $V_\gamma \subseteq X_{\alpha_0}$, and the section $i_0^\gamma: X_{\alpha_0} \rightarrow X_{\alpha_\gamma}$ of the projection $p_{\alpha_0}^{\alpha_\gamma}: X_{\alpha_\gamma} \rightarrow X_{\alpha_0}$.

Suppose that γ is a limit ordinal. By (i), $\{\alpha_\mu : \mu < \gamma\}$ is a countable chain in A and we let (recall that the indexing set A of \mathcal{S} is an ω -complete set and therefore contains supremums of countable chains of its elements)

$$\alpha_\gamma = \sup\{\alpha_\mu : \mu < \gamma\} \in A.$$

By the ω -continuity of the spectrum \mathcal{S} , the compactum X_γ is naturally homeomorphic to the limit of the inverse sequence $\{X_{\alpha_\mu}, p_{\alpha_\lambda}^{\alpha_\mu}, \lambda, \mu < \gamma\}$. Consequently, by (vi), the diagonal product

$$i_0^\gamma = \Delta\{i_0^\mu : \mu < \gamma\}: X_{\alpha_0} \rightarrow X_{\alpha_\gamma}$$

is well-defined and satisfies corresponding conditions (v) and (vi). Let

$$V_\gamma = \{x \in X_{\alpha_0} : i_0^\gamma(x) \notin p_{\alpha_\gamma}(F)\}.$$

Note that $V_\gamma = \cup\{V_{\alpha_\mu} : \mu < \gamma\}$. Then, corresponding conditions (vii), (iii), and (iv) are satisfied.

Next consider the case $\gamma = \mu + 1$. In case $V_\mu = X_{\alpha_0}$, the desired β is α_μ . Suppose that $V_\mu \neq X_{\alpha_0}$ and let

$$x_\mu = i_0^\mu(z) \in i_0^\mu(X_{\alpha_0}) \subseteq X_{\alpha_\mu},$$

where $z \in X_{\alpha_0} \setminus V_\mu$. Since $p_{\alpha_\mu}^{-1}(x_\mu)$ is closed and G_δ in X , we have $p_{\alpha_\mu}^{-1}(x_\mu) \setminus F \neq \emptyset$ (note that X_{α_μ} is a metrizable compactum). Choose an index $\alpha_\gamma \in A$ so that $\alpha_\gamma > \alpha_\mu$ and $(p_{\alpha_\mu}^{\alpha_\gamma})^{-1}(x_\mu) \setminus p_{\alpha_\gamma}(F) \neq \emptyset$.

Softness of the projection $p_{\alpha_\mu}^{\alpha_\gamma}: X_{\alpha_\gamma} \rightarrow X_{\alpha_\mu}$ guarantees the existence of a map $i_\mu^\gamma: X_{\alpha_\mu} \rightarrow X_{\alpha_\gamma}$ such that $p_{\alpha_\mu}^{\alpha_\gamma} i_\mu^\gamma = \text{id}_{X_{\alpha_\mu}}$ and $i_\mu^\gamma(x_\mu) = z'$, where $z' \in (p_{\alpha_\mu}^{\alpha_\gamma})^{-1}(x_\mu) \setminus p_{\alpha_\gamma}(F)$. Let $i_0^\gamma = i_\mu^\gamma i_0^\mu$ and $V_\gamma = \{x \in X_{\alpha_0}: i_0^\gamma(x) \notin p_{\alpha_\gamma}(F)\}$. Note that $V_\mu \subseteq V_\gamma$ and $z' \in V_\gamma \setminus V_\mu$. This completes the construction of the needed objects in the case $\gamma = \mu + 1$.

Thus, the construction can be carried out for each $\lambda < \omega_1$, and we obtain a strictly increasing collection $\{V_\lambda: \lambda < \omega_1\}$ of open subsets of the metrizable compactum X_{α_0} . Clearly, this collection must stabilize, which means that there is an index $\lambda_0 < \omega_1$ such that $V_\lambda = V_{\lambda_0}$ for any $\lambda \geq \lambda_0$. By construction, this is only possible if $V_{\lambda_0} = X_{\alpha_0}$. Let $\beta = \alpha_{\lambda_0}$ and $i_\alpha^\beta = i_{\alpha_0}^{\lambda_0}$. Clearly, $i_\alpha^\beta(X_\alpha) \cap p_\beta(F) = \emptyset$. \square

We also need the following statement.

Lemma 2.2. *Let a non-metrizable compact space X be represented as the limit of an ω -spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$ with soft limit projections $p_\alpha: X \rightarrow X_\alpha$. Suppose that F is a closed subset of X containing no closed G_δ -subsets of X . Then for each $\alpha \in A$, there exists an index $\beta \in A$, with $\beta > \alpha$, such that $p_\beta(F)$ is a fibered Z -set in X_β with respect to the projection $p_\alpha^\beta: X_\beta \rightarrow X_\alpha$.*

Proof: Let $\alpha_0 = \alpha$. Choose $\alpha_{k+1} > \alpha_k$ so that the projection $p_{\alpha_k}^{\alpha_{k+1}}: X_{\alpha_{k+1}} \rightarrow X_{\alpha_k}$ has a section $i_k^{k+1}: X_{\alpha_k} \rightarrow X_{\alpha_{k+1}}$ such that $i_k^{k+1}(X_{\alpha_k}) \cap p_{\alpha_{k+1}}(F) = \emptyset$. Let $\beta = \sup\{\alpha_k: k \in \omega\}$. Let us show that $p_\beta(F)$ is a fibered Z -set in X_β with respect to the projection p_α^β . Let $\mathcal{U} = \{U_i: i \in I\}$ be an open cover of X_β . Without loss of generality, we may assume that $U_i = (p_{\alpha_k}^\beta)^{-1}(U_i^k)$, $i \in I$, where $k \in \omega$ and U_i^k is open in X_{α_k} . Let $j: X_{\alpha_{k+1}} \rightarrow X_\beta$ be any section

of the projection $p_{\alpha_{k+1}}^\beta: X_\beta \rightarrow X_{\alpha_{k+1}}$. Consider the map $f_{\mathcal{U}} = j_k^{k+1} p_{\alpha_k}^\beta: X_\beta \rightarrow X_\beta$. Since

$$p_{\alpha_k}^\beta f_{\mathcal{U}} = p_{\alpha_k}^\beta j_k^{k+1} p_{\alpha_k}^\beta = p_{\alpha_k}^{\alpha_{k+1}} (p_{\alpha_{k+1}}^\beta j_k^{k+1}) p_{\alpha_k}^\beta = (p_{\alpha_k}^{\alpha_{k+1}} i_k^{k+1}) p_{\alpha_k}^\beta = p_{\alpha_k}^\beta,$$

it follows that $f_{\mathcal{U}}$ is \mathcal{U} -close to id_{X_β} . Also, $p_\alpha^\beta f_{\mathcal{U}} = p_\alpha^{\alpha_k} p_{\alpha_k}^\beta f_{\mathcal{U}} = p_\alpha^{\alpha_k} p_{\alpha_k}^\beta = p_\alpha^\beta$ (i.e., $f_{\mathcal{U}}$ acts fiberwise with respect to p_α^β). It remains only to note $f_{\mathcal{U}}(X_\beta) \cap p_\beta(F) = \emptyset$. \square

Lemma 2.3. *Let X be a pseudocompact space without G_δ -points. If βX (its Stone-Ćech compactification) is an absolute retract of weight ω_1 , then βX is homeomorphic to I^{ω_1} .*

Proof: By assumption, if $x \in X$, then x is not a G_δ -point in βX . Since X is pseudocompact, no point in $\beta X \setminus X$ is a G_δ -subset in βX (see [3, Exercise 6I.1]). Thus, βX has no G_δ -points. By Šćepin's theorem (see [1, Theorem 7.2.9]), $\beta X \approx I^{\omega_1}$. \square

3. MAIN RESULT

In this section we prove our main result.

Theorem 3.1. *Let X be a retract of Σ . Then the following conditions are equivalent.*

- (i) X is homeomorphic to Σ ;
- (ii) X has no G_δ -points.

Proof: The implication (i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (i). Let $|A| = \omega_1$. First, let us introduce some notation. If $C \subset B \subseteq A$, then $\pi_B: (I^\omega)^A \rightarrow (I^\omega)^B$ and $\pi_C^B: (I^\omega)^B \rightarrow (I^\omega)^C$ denote the corresponding projections. Similarly, by $\lambda_B: (I^\omega)^B \rightarrow (I^\omega)^A$ and $\lambda_C^B: (I^\omega)^C \rightarrow (I^\omega)^B$, we denote the sections of π_B and π_C^B defined as follows:

$$\lambda_B(\{x_t: t \in B\}) = (\{x_t: t \in B\}, \{0_t: t \in A \setminus B\})$$

and

$$\lambda_C^B(\{x_t: t \in C\}) = (\{x_t: t \in C\}, \{0_t: t \in B \setminus C\}).$$

Here, 0_t denotes the point in the t -th copy of the Hilbert cube, all coordinates of which are equal to 0. Note that $\Sigma = \bigcup \{\lambda_B((I^\omega)^B): B \in \exp_\omega A\}$.

Let $X \subseteq \Sigma$ and $r: \Sigma \rightarrow X$ be a retraction. Recall that Σ is normal, pseudocompact, and $\beta\Sigma = (I^\omega)^A$ (see [2, problems 2.7.14, 3.10.E, and 3.12.23(c)]). Consequently, $\beta X = \text{cl}_{(I^\omega)^A} X$ and r has the extension $\tilde{r}: (I^\omega)^A \rightarrow \text{cl}_{(I^\omega)^A} X = Y$. Note that \tilde{r} is also a retraction and consequently, Y is a compact absolute retract. Note that X , as a retract of Σ , is pseudocompact. Therefore, by Lemma 2.3, $Y \approx I^{\omega_1}$.

For each $C, B \subseteq A$, with $C \subseteq B$, let $Y_B = \pi_B(Y)$, $p_B = \pi_B|_Y$, and $p_C^B = \pi_C^B|_{Y_B}$. Clearly, Y is the limit space of the ω -spectrum $\mathcal{S}_Y = \{Y_B, p_C^B, \exp_\omega A\}$. Since $Y \approx I^{\omega_1}$, Ščepin's spectral theorem (see [1, Theorem 1.3.4]) for ω -spectra ensures that there exists an ω -closed and cofinal subset $\mathcal{A} \subseteq \exp_\omega A$ such that

- (1) $Y = \lim \mathcal{S}_\mathcal{A}$, where $\mathcal{S}_\mathcal{A} = \{Y_B, p_C^B, \mathcal{A}\}$;
- (2) $Y_B \approx I^\omega$, whenever $B \in \mathcal{A}$;
- (3) $p_C^B: Y_B \rightarrow Y_C$ is a trivial fibration with fiber I^ω , whenever $C \subseteq B$, $C, B \in \mathcal{A}$.

Applying the same spectral theorem to the map $\tilde{r}: (I^\omega)^A \rightarrow Y$ and to the ω -spectra $\mathcal{S} = \{(I^\omega)^B, \pi_C^B, \mathcal{A}\}$ (whose limit is $(I^\omega)^A$), and to $\mathcal{S}_\mathcal{A}$, we can find an ω -closed and cofinal subset $\mathcal{B} \subseteq \exp_\omega A$ such that $\mathcal{B} \subseteq \mathcal{A}$, and for each $B \in \mathcal{B}$, there exists a retraction $r_B: (I^\omega)^B \rightarrow Y_B$ such that $p_B \tilde{r} = r_B \pi_B$.

Let $B \in \mathcal{B}$ and consider the composition $i_B = \tilde{r} \lambda_B|_{Y_B}: Y_B \rightarrow Y$. Note that $p_B i_B = p_B \tilde{r} \lambda_B|_{Y_B} = r_B \pi_B \lambda_B|_{Y_B} = r_B|_{Y_B} = \text{id}_{Y_B}$. In other words, i_B is a section of the projection p_B . If $C \subseteq B$ and $C, B \in \mathcal{B}$, we let $i_C^B = p_B i_C$. Note that i_C^B is a section of the projection p_C^B .

Next, we show that $X = \bigcup \{i_B(Y_B): B \in \mathcal{B}\}$. Indeed, let $x \in X$. Since $\Sigma = \bigcup \{\lambda_B((I^\omega)^B): B \in \mathcal{B}\}$, there is $B \in \mathcal{B}$ such that $x = \lambda_B(y)$ for some $y \in (I^\omega)^B$. But $y = \pi_B(\lambda_B(y)) = \pi_B(x) \subseteq \pi_B(X) \subseteq \pi_B(Y) = Y_B$. Consequently,

$$i_B(y) = \tilde{r}(\lambda_B(y)) \stackrel{\lambda_B(y) \in X}{=} \lambda_B(y) = x.$$

Next, we will construct a collection of countable subsets $\{A_\alpha: \alpha < \omega_1\} \subseteq \mathcal{B}$ of A and homeomorphisms $h_\alpha: Y_{A_\alpha} \rightarrow (I^\omega)^{A_\alpha}$ satisfying the following conditions:

- (i) $A_\alpha \subseteq A_\beta$, whenever $\alpha < \beta < \omega_1$;
- (ii) $A_\beta = \bigcup \{A_\alpha: \alpha < \beta\}$, whenever $\beta < \omega_1$ is a limit ordinal;

- (iii) for each $\alpha < \omega_1$, $\pi_{A_\alpha}^{A_{\alpha+1}} h_{\alpha+1} = h_\alpha p_{A_\alpha}^{A_{\alpha+1}}$; i.e., the following diagram is commutative

$$\begin{array}{ccc} Y_{A_{\alpha+1}} & \xrightarrow{h_{\alpha+1}} & (I^\omega)^{A_{\alpha+1}} \\ p_{A_\alpha}^{A_{\alpha+1}} \downarrow & & \downarrow \pi_{A_\alpha}^{A_{\alpha+1}} \\ Y_{A_\alpha} & \xrightarrow{h_\alpha} & (I^\omega)^{A_\alpha} \end{array}$$

- (iv) $h_\beta = \lim\{h_\alpha : \alpha < \beta\}$, whenever $\beta < \omega_1$ is a limit ordinal;
(v) $p_{A_\alpha}^{A_{\alpha+1}} : Y_{A_{\alpha+1}} \rightarrow Y_{A_\alpha}$ is a trivial fibration with fiber I^ω ;
(vi) $i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha})$ is a fibered Z -set in $Y_{A_{\alpha+1}}$ with respect to the projection $p_{A_\alpha}^{A_{\alpha+1}}$;
(vii) for each $\alpha < \omega_1$, $h_{\alpha+1} i_{A_\alpha}^{A_{\alpha+1}} = \lambda_{A_\alpha}^{A_{\alpha+1}} h_\alpha$; i.e., the following diagram commutes:

$$\begin{array}{ccc} Y_{A_{\alpha+1}} & \xrightarrow{h_{\alpha+1}} & (I^\omega)^{A_{\alpha+1}} \\ i_{A_\alpha}^{A_{\alpha+1}} \uparrow & & \uparrow \lambda_{A_\alpha}^{A_{\alpha+1}} \\ Y_{A_\alpha} & \xrightarrow{h_\alpha} & (I^\omega)^{A_\alpha} . \end{array}$$

Let A_0 be any element of \mathcal{B} and take any homeomorphism $h_0 : Y_{A_0} \rightarrow (I^\omega)^{A_0}$.

Let $\beta < \omega_1$. Suppose that for each $\alpha < \beta$, we have already constructed a countable set $A_\alpha \in \mathcal{B}$ and a homeomorphism $h_\alpha : Y_{A_\alpha} \rightarrow (I^\omega)^{A_\alpha}$ satisfying the above conditions for appropriate indices. We proceed by constructing these objects for the ordinal β .

If $\beta = \sup\{\alpha : \alpha < \beta\}$, then set $A_\beta = \cup\{A_\alpha : \alpha < \beta\}$ and $h_\beta = \lim\{h_\alpha : \alpha < \beta\}$. Then, all required conditions are clearly satisfied.

Now consider the case $\beta = \alpha + 1$. Since $i_{A_\alpha}(Y_{A_\alpha})$ is a metrizable compactum in Y , it cannot contain closed G_δ -subsets of Y (which contain copies of the Tychonov cube I^{ω_1}). Consequently, we can find, based on Lemma 2.2, an element $A_{\alpha+1} \in \mathcal{B}$, such that $A_\alpha \subseteq A_{\alpha+1}$, and $i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha}) = p_{A_{\alpha+1}}(i_{A_\alpha}(Y_{A_\alpha}))$ is a fibered Z -set with respect to the projection $p_{A_\alpha}^{A_{\alpha+1}}$. Since both projections $p_{A_\alpha}^{A_{\alpha+1}}$ and $\pi_{A_\alpha}^{A_{\alpha+1}}$ are trivial fibrations with fiber I^ω , there exists a homeomorphism $f : Y_{A_{\alpha+1}} \rightarrow (I^\omega)^{A_{\alpha+1}}$ such that $\pi_{A_\alpha}^{A_{\alpha+1}} f = h_{A_\alpha} p_{A_\alpha}^{A_{\alpha+1}}$.

Then the set $f(i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha}))$ is a fibered Z -set in $(I^\omega)^{A_{\alpha+1}}$ with respect to the projection $\pi_{A_\alpha}^{A_{\alpha+1}}$. Consider now another fibered Z -set in $(I^\omega)^{A_{\alpha+1}}$ (also with respect to the projection $\pi_{A_\alpha}^{A_{\alpha+1}}$) – namely, $\lambda_{A_\alpha}^{A_{\alpha+1}}((I^\omega)^{A_\alpha})$. There is a homeomorphism

$$g: f(i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha})) \rightarrow \lambda_{A_\alpha}^{A_{\alpha+1}}((I^\omega)^{A_\alpha})$$

which acts fiberwise (i.e., $\pi_{A_\alpha}^{A_{\alpha+1}}g = \pi_{A_\alpha}^{A_{\alpha+1}}$). The expression for g is

$$g = \lambda_{A_\alpha}^{A_{\alpha+1}} h_\alpha p_{A_\alpha}^{A_{\alpha+1}} f^{-1} | f(i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha})).$$

By the fibered Z -set unknotting theorem [5], g extends to a homeomorphism $G: (I^\omega)^{A_{\alpha+1}} \rightarrow (I^\omega)^{A_{\alpha+1}}$ such that $\pi_{A_\alpha}^{A_{\alpha+1}}G = \pi_{A_\alpha}^{A_{\alpha+1}}$ (i.e., G acts fiberwise). Then the required homeomorphism $h_{A_{\alpha+1}}$ is defined as the composition $Gf: Y_{A_{\alpha+1}} \rightarrow (I^\omega)^{A_{\alpha+1}}$. Straightforward verification shows that all the needed properties are satisfied.

This completes the inductive process. Now, let $h = \lim\{h_\alpha: \alpha < \omega_1\}$. It is easy to see that $h: Y \rightarrow I^A$ is a homeomorphism such that $h(i_{A_\alpha}(Y_{A_\alpha})) = \lambda_{A_\alpha}((I^\omega)^{A_\alpha})$ for each $\alpha < \omega_1$. Consequently, $h(X) = \Sigma$. \square

Corollary 3.2. *Let X be a retract of Σ . Then $X \times \Sigma$ is homeomorphic to Σ .*

Proof. Note that $X \times \Sigma$ is a retract of $\Sigma \times \Sigma \approx \Sigma$ and has no G_δ -points. \square

Corollary 3.3. *The following conditions are equivalent for a compact space X .*

- (i) $X \times \Sigma$ is homeomorphic to Σ ;
- (ii) X is a metrizable absolute retract.

Proof: (i) \Rightarrow (ii). Let $h: \Sigma \rightarrow X \times \Sigma$ be a homeomorphism and $\pi: X \times \Sigma \rightarrow X$ be the projection. Clearly, $r = \pi h: \Sigma \rightarrow X$ is a retraction. Since I^{ω_1} is the Stone-Ćech compactification of Σ (and since X is compact), r admits the extension $\tilde{r}: I^{\omega_1} \rightarrow X$. Therefore, X , as a retract of I^{ω_1} , is an absolute retract. Note also that X is separable (as an image of I^{ω_1}). But separable compact subspaces of Σ are metrizable.

(ii) \Rightarrow (i). Apply Corollary 3.2. \square

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