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# PERFECT PREIMAGES AND SMALL DIAGONAL

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Abstract. M. Hušek defines a space $X$ to have a *small diagonal* if each uncountable subset of $X^2$ disjoint from the diagonal has an uncountable subset whose closure is disjoint from the diagonal. It is known that the existence of a perfect preimage of $\omega_1$ which has a small diagonal is independent of the usual axioms of set-theory. In this note we prove that a perfect preimage of $\omega_1$ which is scattered will not have a small diagonal.

## 1. Introduction

We refer the reader to Gary Gruenhage's interesting article [2] for more background on spaces with small diagonal (see also [6]). In particular, Gruenhage proves that, consistent with CH, each countably compact space with a small diagonal is metrizable; hence, no countably compact preimage of $\omega_1$ could have a small diagonal. On the other hand, the authors prove in [1] that it follows from $\diamondsuit^+$ (a strengthening of CH) that there is a space with a small diagonal which maps perfectly onto $\omega_1$. In this paper, we prove (in ZFC) that there is no scattered space with a small diagonal which maps perfectly onto $\omega_1$.

M. Hušek [3], of course, originally asked about small diagonals for compact and $\omega_1$-compact spaces. The main open question in this area is whether every compact space with a small diagonal is

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metrizable. This statement has been shown to be consistent; for example, it follows from each of CH and PFA. A counterexample will have to have a continuous image which does not have a small diagonal [1, Proposition 18]; hence, we will consider preimages of those spaces that do not have a small diagonal. We offer the following problem as another interesting question about spaces with a small diagonal that may be easier to resolve in ZFC.

**Question 1.** If a compact space $X$ maps onto the Alexandroff double of the unit interval or of the compact double arrow space, will $X$ not have a small diagonal?

Using Hušek's result [3] that a compact non-metrizable space which has a small diagonal must have weight larger than $\omega_1$ and I. Juhász and Z. Szentmiklóssy's result [4] that it must have countable tightness, the authors showed the following.

**Proposition 2.** [1, Corollary 5] *If a compact space has a small diagonal, then it is metrizable if each of its separable subspaces is metrizable.*

In fact, we should have stated the following strengthening because it uses the same proof.

**Proposition 3.** *If a compact non-metrizable space has a small diagonal, then it has a countable discrete subset whose closure is not metrizable.*

*Proof:* Assume that no countable discrete subset of $X$ is dense. Inductively select points $x_\alpha$ not in the closure of $D_\alpha = \{x_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$. Juhasz and Szentmiklóssy [4] have shown that $X$ will have countable tightness (because it is compact and has a small diagonal). Therefore, $Y = \bigcup_{\alpha<\omega_1} \overline{D_\alpha}$ will be compact and have a small diagonal. If each $\overline{D_\alpha}$ is metrizable, $Y$ will have net-weight, hence weight, equal to $\aleph_1$. By Hušek's result, $Y$ should be metrizable, which, clearly, it is not. □

Therefore, if a compact space with a small diagonal maps onto the Alexandroff double, then the preimage of the non-isolated points will not be metrizable. In fact, more generally, Gruenhage [2, Corollary 2.5] has shown that if a non-metrizable compact space with a small diagonal maps onto a metric space, one of the fibers will be non-metrizable.

## 2. Perfect preimages of $\omega_1$

Recall the following reformulation of a space having a small diagonal.

**Proposition 4.** *A space $X$ has a small diagonal iff for each uncountable family of pairs of points of $X$, $\{(x_\alpha, y_\alpha) : \alpha \in \omega_1\}$, there is an uncountable $A \subset \omega_1$ such that each point $x$ of $X$ has a neighborhood $U_x$ satisfying that $|U_x \cap \{x_\alpha, y_\alpha\}| \leq 1$ for all $\alpha \in A$.*

The following is a simple generalization.

**Lemma 5.** *If a space $X$ has a small diagonal and $\{F_\alpha : \alpha \in \omega_1\}$ is a family of finite subsets of $X$, then there is an uncountable $A \subset \omega_1$ such that each point $x \in X$ has a neighborhood $U_x$ satisfying that $U_x \cap F_\alpha$ has at most one element for each $\alpha \in A$.*

*Proof:* Let $n \in \omega$ be chosen so that $A_0 = \{\alpha : |F_\alpha| = n\}$ is uncountable. For each $\alpha \in A_0$, let $\{F_\alpha(i) : i < n\}$ be an enumeration of $F_\alpha$ and let $\{P_j : j < \binom{n}{2}\}$ enumerate all the two element subsets of $n$. Recursively apply Proposition 4 to select uncountable sets $A_{j+1} \subset A_j$ so that each $x \in X$ has a neighborhood $U_x$ satisfying $|U_x \cap \{F_\alpha(i) : i \in P_j\}| \leq 1$ for each $\alpha \in A_{j+1}$. Clearly, if $j = \binom{n}{2}$, then $A_j$ is the desired uncountable subset of $A_0$. □

It is nearly immediate now that no space with a small diagonal admits a finite-to-one perfect map onto $\omega_1$. We include this proof for the interest of the reader. Recall that a map is *perfect* if it is a closed map and the preimage of each point is compact.

**Corollary 6.** *If $f : X \to \omega_1$ is a perfect surjective map, and, for some stationary set $S \subset \omega_1$, $|f^{-1}(\alpha)|$ is finite for each $\alpha \in S$, then $X$ does not have a small diagonal.*

*Proof:* Let $S$ be a stationary set as in the statement of the Corollary 6. Since a countable union of non-stationary sets is again non-stationary, we may fix an integer $n$ so that $S_0 = \{\alpha \in S : |f^{-1}(\alpha)| = n\}$ is also stationary. For each $\alpha \in S_0$, choose a point $x_\alpha$ such that $f(x_\alpha) = \alpha + 1$ and let $F_\alpha = \{x_\alpha\} \cup f^{-1}(\alpha)$. Apply Lemma 5 to find an uncountable $A_0 \subset S_0$ such that each point $x \in X$ has a neighborhood $U_x$ satisfying $|U_x \cap F_\alpha| \leq 1$ for each $\alpha \in A_0$. Since $S_0$ is stationary, there is a $\lambda \in S_0$ that is a limit of $A_0$. By possibly shrinking the finitely many open sets, we can assume that $U_x \cap U_{x'}$ is empty for $x \neq x'$ with $f(x) = f(x') = \lambda$. Note that $F_\alpha \setminus \bigcup_{x \in f^{-1}(\lambda)} U_x$ is not empty for each $\alpha \in A_0$. It follows then that $A_0 \cap \lambda$ is contained in the image of the closed set $X \setminus \bigcup_{x \in f^{-1}(\lambda)} U_x$ while $\lambda$ is not. This implies that the map $f$ is not perfect. □

We will need to iterate the procedure from Lemma 5 in order to prove our main result. We adopt some notational conventions to do so. Suppose we fix a sequence $\{x_\alpha : \alpha \in \omega_1\}$ of points in a space $X$. For any finite set $F \subset \omega_1$, let us use $\widehat{F}$ to denote the corresponding finite set $\{x_\alpha : \alpha \in F\}$. Similarly, for any uncountable collection $\mathcal{F}$ of finite subsets of $\omega_1$, let $\widehat{\mathcal{F}} = \{\widehat{F} : F \in \mathcal{F}\}$ therefore be an uncountable collection of finite subsets of $X$.

Next, for any uncountable set $A \subset \omega_1$ and integer $n > 0$, let $\mathcal{F}_n^A$ denote the unique (canonical) partition of $A$ into sets of size $n$ such that $\max F < \min F'$ (or conversely) for $F \neq F' \in \mathcal{F}_n^A$. Finally, note that if $\mathcal{F}'$ is an uncountable subset of $\mathcal{F}_n^A$ and $B = \bigcup \mathcal{F}'$, then $B \subset A$ and $\mathcal{F}_n^B$ is a subfamily of $\mathcal{F}_n^A$ because $\mathcal{F}' = \mathcal{F}_n^B$.

As this notation builds up, the following simple fact is helpful.

**Lemma 7.** *Let $n, m$ be integers and let $A$ be an uncountable subset of $\omega_1$. Let $\mathcal{F}'$ be an uncountable subset of $\mathcal{F}_n^A$ and let $B = \bigcup \mathcal{F}'$. Then each member of $\mathcal{F}_{n \cdot m}^B$ is a union of $m$ many pairwise disjoint members of $\mathcal{F}'$.*

We can now prove the main theorem.

**Theorem 8.** *If $X$ is a scattered space which maps perfectly onto $\omega_1$, then $X$ does not have a small diagonal.*

*Proof:* Assume that $f$ is a perfect mapping from $X$ onto $\omega_1$. Note that $X$ is locally compact since, for each $\lambda \in \omega_1$, the set $f^{-1}([0, \lambda])$ is compact. For each $\lambda \in \omega_1$, we will let $X_\lambda$ denote the points of $X$ that map to $\lambda$ and also note that $X_\lambda$ is compact and scattered. Assume towards a contradiction that $X$ has a small diagonal.

For each $\alpha \in \omega_1$, fix any point $x_\alpha \in X$ such that $f(x_\alpha) = \alpha$; thus, we have chosen a fixed sequence of points $\{x_\alpha : \alpha \in \omega_1\}$ as above. Recall that ${}^{<\omega}\mathbb{N}$ is the collection of all integer-valued functions with domain equal to some finite ordinal. We will inductively choose a collection, $\{A_t : t \in {}^{<\omega}\mathbb{N}\}$, of uncountable subsets of $\omega_1$. In addition, we will also have selected $\{\mathcal{W}_t : t \in {}^{<\omega}\mathbb{N}\}$ consisting of open covers of $X$. For each $\emptyset \neq t \in {}^{<\omega}\mathbb{N}$, let $\pi(t)$ denote the usual integer product $t(0) \cdot t(1) \cdots t(|t|-1)$, and let $\pi(\emptyset) = 1$.

To begin the induction, let $A_\emptyset$ denote the set $\omega_1$ and let $\mathcal{W}_\emptyset$ be any cover of $X$ by open sets. Suppose that $t \in {}^{<\omega}\mathbb{N}$ is such that $A_t$ has not been defined, but that (by induction) $A_{t'}$ and $\mathcal{W}_{t'}$ have been defined for all $t' \subset t$ in ${}^{<\omega}\mathbb{N}$. Let $t' = t \restriction (|t| - 1)$ be the immediate predecessor of $t$ and let $n$ denote the integer $\pi(t)$. We consider the family of finite sets $\mathcal{F} = \mathcal{F}_n^{A_{t'}}$ and the corresponding family $\widehat{\mathcal{F}}$ of finite subsets of $X$. By Lemma 5, there is an open cover $\mathcal{W}_t$ and an uncountable subcollection $\mathcal{F}'$ of $\mathcal{F}$ such that $W \cap \widehat{F}$ has at most one element for all $W \in \mathcal{W}_t$ and $F \in \mathcal{F}'$. We set $A_t = \bigcup \mathcal{F}'$; hence, $W \cap F$ has at most one element for all $W \in \mathcal{W}_t$ and $F \in \mathcal{F}^{A_t}_{\pi(t)}$. By Lemma 7, it follows by induction that for $t' \subset t$, each member of $\mathcal{F}^{A_t}_{\pi(t)}$ is a union of $\frac{\pi(t)}{\pi(t')}$ members of $\mathcal{F}^{A_{t'}}_{\pi(t')}$.

For each $t \in {}^{<\omega}\mathbb{N}$, the set of accumulation points in $\omega_1$ of the uncountable set $A_t$ will be a cub in $\omega_1$. Since the intersection of countably many cubs of $\omega_1$ is again a cub, we may choose a limit $\lambda \in \omega_1$ such that $A_t \cap \lambda$ is cofinal in $\lambda$ for each $t \in {}^{<\omega}\mathbb{N}$. Observe then that for each $F \in \mathcal{F}^{A_t}_{\pi(t)}$, with $\min F \in \lambda$, we also have $F \subset \lambda$ since $\lambda \cap (A_t \setminus \min F)$ is infinite and $\max F < \min F'$ for all $F' \in \mathcal{F}^{A_t}_{\pi(t)}$ such that $F' \setminus \min F$ is not empty.

Now we begin to inductively choose a finite sequence $t$ of integers (hence, $t \in {}^{<\omega}\mathbb{N}$) and a descending sequence of ordinals (which must therefore stop in finitely many steps). Let $\gamma_0$ denote the maximum non-empty scattering level of $X_\lambda$ (which must exist since $X_\lambda$ is compact and non-empty). Set $t(0)$ to be any integer greater than the finite number of points of $X_\lambda$ at scattering level $\gamma_0$. If we have defined the first $k$ elements of $t$, we will use $t \restriction k$ to denote that function, even though we don't yet know what $t$ is. Let $\mathcal{W}_0 \subset \mathcal{W}_{t \restriction 1}$ (with $|\mathcal{W}_0| < t(0)$) be a cover of those fewer than $t(0)$ many points at scattering level $\gamma_0$ of $X_\lambda$. Set $U_0 = \bigcup \mathcal{W}_0$ and note that $\widehat{F} \setminus U_0$ is not empty for each $F \in \mathcal{F}^{A_{t\restriction 1}}_{t(0)}$.

Assume now that we have defined $t(i), \gamma_i$ and $\mathcal{W}_i$ for $i < k$ such that $|\mathcal{W}_i| < t(i)$, $\mathcal{W}_i \subset \mathcal{W}_{t \restriction i+1}$, and $X_\lambda \setminus \bigcup\{\bigcup \mathcal{W}_i : i < k\}$ has scattering height less than $\gamma_{k-1}$. We continue as follows. Set $U = \bigcup\{\bigcup \mathcal{W}_i : i < k\}$; if $X_\lambda \setminus U$ is empty we stop. Otherwise, let $\gamma_k$ be the maximum non-empty scattering level of $X_\lambda \setminus U$ and let $t(k)$

be any integer larger than the cardinality of that level. Choose $\mathcal{W}_k \subset \mathcal{W}_{t\restriction k+1}$ to be any fewer than $t(k)$ many sets which covers that finite set of points of $X_\lambda \setminus U$ at scattering level $\gamma_k$.

Recall from above that we noted that for each $F \in \mathcal{F}^{A_{t\restriction 1}}_{t(0)}$, $\widehat{F} \setminus \bigcup \mathcal{W}_0$ is not empty. By Lemma 7, each $F \in \mathcal{F}^{A_{t\restriction 2}}_{\pi(t\restriction 2)}$ is a union of $t(1)$ many pairwise disjoint members of $\mathcal{F}^{A_{t\restriction 1}}_{t(0)}$. Therefore, since $|\mathcal{W}_1| < t(1)$, it follows that $\widehat{F}\setminus(\bigcup \mathcal{W}_0 \cup \bigcup \mathcal{W}_1)$ is not empty for each $F \in \mathcal{F}^{A_{t\restriction 2}}_{\pi(t\restriction 2)}$. By a straightforward induction, for each $F \in \mathcal{F}^{A_t}_{\pi(t)}$, we have that $\widehat{F} \setminus \bigcup\{\bigcup \mathcal{W}_i : i < |t|\}$ is not empty. We are now ready for our contradiction. Choose any sequence $\{F_n : n \in \omega\} \subset \mathcal{F}^{A_t}_{\pi(t)}$ such that $\{\min F_n : n \in \omega\}$ is cofinal in $\lambda$. Recall also that $\max F_n \in \lambda$ for each $n \in \omega$ as well. For each $n$, choose $y_n \in \widehat{F} \setminus \bigcup\{\bigcup \mathcal{W}_i : i < |t|\}$. It follows now that $\{f(y_n) : n \in \omega\}$ is cofinal in $\lambda$, while on the other hand, $\{y_n : n \in \omega\}$ is a closed subset of $X \setminus \bigcup\{\bigcup \mathcal{W}_i : i < |t|\}$ since $X_\lambda$ is contained in $\bigcup\{\bigcup \mathcal{W}_i : i < |t|\}$. □

**Question 9.** If a space $X$ has a small diagonal and maps perfectly onto a space $Y$ with point preimages being scattered, will $Y$ also have a small diagonal?

The formulation and proof of Theorem 8 can easily be strengthened to require only that the map be a closed map onto a stationary subset of $\omega_1$ and that point preimages are compact scattered rather than the whole space is scattered.

In addition, a compact scattered space with a small diagonal is easily shown to be countable and metrizable.

Combining these ideas yields the following result.

**Proposition 10.** *Suppose that a space $X$ maps onto a subset $S$ of $\omega_1$ by a closed mapping such that fibers are compact and scattered. Then $X$ has a small diagonal iff $S$ is not stationary and the point preimages are also countable.*

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