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PSEUDOCOMPACT DENSE SUBGROUPS WITHOUT NON-TRIVIAL CONVERGENT SEQUENCES OF SOME COMPACT GROUPS

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Abstract. It is shown that if κ and λ are two cardinal numbers such that $\lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$, then $\{0,1\}^\kappa$ contains a dense pseudocompact subgroup without non-trivial convergent sequences. As a direct consequence of this result, we can see that if M is a model of ZFC in which $M \models \mathfrak{c} < \aleph_\omega < 2^\mathfrak{c}$, then $M \models \{0,1\}^{\aleph_\omega}$ contains a dense pseudocompact subgroup without non-trivial convergent sequences. On the other hand, it is known that if $\aleph_\omega$ is a strong limit cardinal number, then every dense subgroup of $\{0,1\}^{\aleph_\omega}$ contains a non-trivial convergent sequence. We also give some conditions to guarantee that $\{0,1\}^\kappa$ contains a countably compact dense subgroup without non-trivial convergent sequences, where κ is a cardinal number such that $\lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$ for some cardinal number λ. We also prove that it is consistent with the axioms of ZFC that the compact group $\{0,1\}^{\omega_1}$ contains a countably compact dense subgroup without non-trivial convergent sequences and the continuum $\mathfrak{c}$ takes any possible value.

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1. Introduction

In this paper, our spaces will be Tychonoff and the topological groups will be Hausdorff (hence, they will be Tychonoff). The symbol $\mathbb{T}$ will stand for the unit circle equipped with the complex multiplication and the topology inherited from $\mathbb{R}^2$. The identity element of an arbitrary Abelian group will be simply denoted by 0; we make only one exception for $\mathbb{T}$ whose identity element is denoted by 1. The order of an element $g \in G$ is denoted by $o(g)$. A finite subset $\{g_0, ..., g_k\}$ of an Abelian group G is said to be *independent* if $g_i \neq 0$ for all $i \leq k$, and $n_0 g_o + ... + n_k g_k = 0$, where $n_i \in \mathbb{Z}$ for all $i \leq k$, imply that $n_i g_i = 0$, for each $i \leq k$. An infinite subset L of G is called *independent* if every finite subset of L is independent. If $\emptyset \neq A \subseteq I$, then $\pi_A : \prod_{i \in I} X_i \to \prod_{i \in A} X_i$ will stand for the projection map. If X is a set and κ is an infinite cardinal number, then $[X]^\kappa = \{A \in \mathcal{P}(X) : |A| = \kappa\}$ and the meaning of the symbol $[X]^{<\kappa}$ is clear. A subspace Y of a space X is called G_δ-*dense* if every nonempty G_δ subset of X meets Y. If m and n are integer numbers, the symbol $m|n$ should be read as m *divides* n.

In 1980, Eric K. van Douwen [7] assuming the existence of a countably compact Boolean group without non-trivial convergent sequences, constructed, in ZFC, two countably compact groups whose product is not countably compact: The existence of these two countably compact groups is unknown in ZFC. Also, the problem of the existence in ZFC of a countably compact group without non-trivial convergent sequences is still unsolved. All known examples are constructed either by using some set-theoretic axiom compatible with ZFC ([7], [13], [21], and [22]), or by assuming the existence of selective ultrafilters on ω ([10]). Concerning the problem about the existence of a dense pseudocompact subgroup without non-trivial convergent sequences of a compact group, S. M. Sirota gave the first construction of such a topological group [18]; Jan J. Dijkstra and Jan van Mill stated several conditions for the existence of pseudocompact dense subgroups without non-trivial convergent sequences of some compact groups [3] (see also [17] and [24]); M. G. Tkachenko constructed a dense pseudocompact subgroup G of $\{0,1\}^{\mathfrak{c}}$ such that every homeomorphism of any countable subgroup of G to a compact group K extends to a continuous homeomorphism from G to K (in particular, G does not have non-trivial convergent sequences) [19]; and a classification, under the assumption of GCH, of certain compact Abelian groups in which every dense pseudocompact subgroup has a non-trivial convergent sequence is given in [9].

In the second section, we prove that it is independent from the axioms of ZFC that the compact group $\{0,1\}^{\aleph_\omega}$ admits a dense pseudocompact subgroup without non-trivial convergent sequences. In the third section, some conditions are given to see that $\{0,1\}^\kappa$ contains a countably compact dense subgroup without non-trivial convergent sequences, whenever $\lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$ for some cardinal number λ. V. I. Malykhin and L. B. Shapiro showed that it is consistent with ZFC that $\{0,1\}^{\omega_1}$ contains a pseudocompact dense subgroup without non-trivial convergent sequences and the continuum $\mathfrak{c}$ takes any possible value [17]. We improved this result by replacing "pseudocompact" by "countably compact." The last section is devoted to the study of some infinite powers of the circle $\mathbb{T}$.

2. Pseudocompact dense subgroups of some powers of $\{0,1\}$

In this section, our goal is to establish the independence from ZFC of the statement "$\{0,1\}^{\aleph_\omega}$ admits a dense pseudocompact subgroup without non-trivial convergent sequences."

The first lemma is precisely the combinatorial property which lies behind the proof of Hewitt-Marczewski-Pondiczery's Theorem (see [8, Theorem 2.3.15]). To prove the lemma we need the following notion.

For a space (X,τ), $P(X)$ will denote the *P-modification* of the topology τ which is the topology on X whose base is $\{\bigcap_{n<\omega} U_n : \forall n < \omega(U_n \in \tau)\}$. For a space X, we know that $P(X)$ is a P-space of weight $\leq w(X)^\omega$ which is discrete iff $\psi(x,X) = \omega$ for every $x \in X$.

Lemma 2.1. *If $\omega \leq \lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$, then there is $\mathcal{B} \in [\mathcal{P}(\kappa)]^\lambda$ such that for every $\{\theta_n : n < \omega\} \in [\kappa]^\omega$ of pairwise distinct elements there is a pairwise disjoint family $\{B_n : n < \omega\}$ of elements of $\mathcal{B}$ such that $\theta_n \in B_n$, for every $n < \omega$.*

Proof: Let us consider the space $X = \{0,1\}^\lambda$. Since $cf(\lambda) > \omega$, it is clear that $P(X)$ is a non-discrete, Hausdorff P-space that has weight $\lambda^\omega = \lambda$. Let Y be a non-discrete subspace of $P(X)$ of size κ and let $\mathcal{B}$ be a base for Y with $|\mathcal{B}| = \lambda$. Fix $\{y_n : n < \omega\} \in [Y]^\omega$ such that $y_n \neq y_m$ for $n < m < \omega$. Since Y is a Hausdorff P-space, we may find a pairwise disjoint family $\{B_n : n < \omega\}$ of elements of $\mathcal{B}$ such that $y_n \in B_n$, for each $n < \omega$. This shows that $\mathcal{B}$ is the desired family. □

Lemma 2.2. *Let κ be a cardinal number such that $\lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$ for some infinite cardinal λ and let G_α be a topological group with $w(G_\alpha) \leq \lambda$, for every $\alpha < \kappa$. Then the product $\prod_{\alpha<\kappa} G_\alpha$ contains a G_δ-dense subgroup H of size λ.*

Proof: For every $\alpha < \kappa$, let D_α be a G_δ-dense subgroup of G_α of size λ. Now, pick $\mathcal{B} \in [\mathcal{P}(\kappa)]^\lambda$ satisfying all the conditions of Lemma 2.1. We enumerate the set $\{f : \omega \to \mathcal{B} : f(n) \cap f(m) = \emptyset$ provided that $n < m < \omega\}$ as $\{f_\xi : \xi < \lambda\}$. Let us consider the set

$$A = \{x \in \prod_{\alpha<\kappa} G_\alpha : \exists \xi < \kappa \ \forall \alpha \notin \bigcup_{n<\omega} f_\xi(n)(x(\alpha) = 0)$$
$$\text{and } \pi_{f_\xi(n)}(x) \in \prod_{\alpha \in f_\xi(n)} D_\alpha, \text{ for all } n < \omega\}.$$

By using the properties of $\mathcal{B}$, it is not difficult to see that A is a G_δ-dense subset of $\prod_{\alpha<\kappa} G_\alpha$ and it is clear that $\langle A \rangle$ has size λ. We put $H = \langle A \rangle$. □

Next, we generalize Lemma 2.1 from [9].

Lemma 2.3. *Let κ and λ be two infinite cardinal numbers such that $\lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$. Let $\{G_\alpha : \alpha < \kappa\}$ be a collection of topological Abelian groups such that $w(G_\alpha) \leq \lambda$, for every $\alpha < \kappa$, and the following condition holds:*

(*) *If there are $1 \leq n < \omega$, $\beta < \lambda$, and $g \in G_\beta$ such that $o(g) \not| \, n$, then for every $X \in [\lambda]^{<\lambda}$, there are $\alpha \in \lambda \setminus X$ and $h \in G_\alpha$ such that $o(h) \not| \, n$.*

Then, for each $y \in \prod_{\alpha<\kappa} G_\alpha$, there is a G_δ-dense, independent subset A of $\prod_{\alpha<\kappa} G_\alpha$ of size λ such that whenever $S, T \in [A]^\omega$ are disjoint we can find $\gamma < \lambda$ satisfying $x(\gamma) = 0$, for all $x \in S$, and

$x(\gamma) = y(\gamma)$, for all $x \in T$. In addition, if $y(\alpha)$ has infinite order, for every $\alpha < \kappa$, then $\langle A \rangle$ is a free Abelian group.

Proof: For the case when $\kappa = \lambda$, the statement is Lemma 2.1 from [9]. Suppose that $\lambda < \kappa$. From Lemma 2.2, we know that the topological group $\prod_{\alpha \in \kappa \setminus \lambda} G_\alpha$ contains a G_δ-dense subgroup H of size λ. Enumerate H as $\{h_\xi : \xi < \lambda\}$. Now let us consider the product $\prod_{\alpha < \lambda} G_\alpha$. We will follow the main idea of the proofs of Lemma 2.1 from [9] and Theorem 3 from [3]. For each $\alpha < \lambda$, let D_α be a G_δ-dense subset of G_α with $|D_\alpha| \leq \lambda$. Enumerate $D = \bigcup_{I \in [\lambda]^\omega} \prod_{\alpha \in I} D_\alpha$ as $\{x_{\mu,\nu} : \mu, \nu < \lambda\}$ in such a way that $x_{\xi,\zeta} = x_{\xi,\eta}$, for every $\xi, \zeta, \eta < \lambda$. Let $\{(E_{\mu,\nu}, \sigma_{\mu,\nu}) : \mu, \nu < \lambda\}$ be an enumeration of all pairs (E, σ) where $E \in [\lambda \times \lambda]^{<\omega} \setminus \{\emptyset\}$ and $\sigma : E \to \mathbb{Z}$ is a function, and let $\{(S_{\mu,\nu}, T_{\mu,\nu}) : \mu, \nu < \lambda\}$ be an enumeration of all pairs (S, T) where $S, T \in [\lambda \times \lambda]^\omega$ are disjoint. We may assume that $(S_{\mu,\nu} \cup T_{\mu,\nu}) \cap E_{\mu,\nu} = \emptyset$, for every $\mu, \nu < \lambda$. Proceeding as in the proof of Lemma 2.1 from [9], for every $\mu, \nu < \lambda$, we can extend each $x_{\mu,\nu}$ to λ such that there is $\gamma < \lambda$ such that the following conditions hold:

1. $x_{\xi,\zeta}(\gamma) = 0$, for all $(\xi, \zeta) \in S_{\mu,\nu}$,
2. $x_{\xi,\zeta}(\gamma) = y(\gamma)$, for all $(\xi, \zeta) \in T_{\mu,\nu}$, and
3. if there are $\xi_0, \zeta_0 \in E_{\mu,\nu}$, $\beta < \lambda$, and $g \in G_\beta$ such that $o(g) \not| \sigma_{\mu,\nu}((\xi_0, \zeta_0))$, then $\sum_{(\xi,\zeta) \in E_{\mu,\nu}} \sigma_{\mu,\nu}((\xi, \zeta)) x_{\xi,\zeta}(\gamma) \neq 0$.

Next, we complete the extension of $x_{\xi,\zeta}$ by defining $x_{\xi,\zeta}(\theta) = 0$ for those coordinates $\theta < \lambda$ where $x_{\xi,\zeta}$ has not been defined yet, for each $\xi, \zeta < \lambda$. We can also guarantee that $x_{\xi,\zeta} \neq x_{\xi',\zeta'}$ whenever $(\xi, \zeta) \neq (\xi', \zeta')$ and $\xi, \zeta, \xi', \zeta' < \lambda$. Notice that $\{x_{\xi,\zeta} : \xi, \zeta < \lambda\}$ is an independent set. Finally, we define $A = \{(x_{\xi,\zeta}, h_\zeta) : \xi, \zeta < \lambda\}$. From the construction, we see that A satisfies all the requirements. □

Theorem 2.4. *Let κ and λ be two infinite cardinal numbers such that $\lambda = \lambda^\omega \leq \kappa \leq 2^\lambda$. Then $\{0, 1\}^\kappa$ contains a dense pseudocompact subgroup without non-trivial convergent sequences.*

Proof: By applying Lemma 2.3 to the constant function $y \in \{0, 1\}^\kappa$ with $y(\xi) = 1$, for all $\xi < \kappa$, we obtain a G_δ-dense, independent subset A of $\{0, 1\}^\kappa$ of size λ such that whenever $S, T \in [A]^\omega$ are disjoint, we can find $\gamma < \kappa$ satisfying that $x(\gamma) = 0$, for all $x \in S$, and $x(\gamma) = 1$, for all $x \in T$. Put $H = \langle A \rangle$. It follows from Theorem 1.2 of [2] that H is pseudocompact and dense in $\{0,1\}^\kappa$. Now, let $(a_n)_{n<\omega}$ be a sequence in H such that $a_n \neq a_m$, provided that $n < m < \omega$, and let us assume that $a_n \to 0$. Then, there is $\{E_n : n < \omega\} \subseteq [\lambda]^{<\omega} \setminus \{\emptyset\}$ such that $a_n = \sum_{\xi \in E_n} x_\xi$, for every $n < \omega$. Our assumption implies that $E_n \neq E_m$ whenever $n < m < \omega$. Without loss of generality, we may assume that either $|E_n|$ is odd, for all $n < \omega$, or $|E_n|$ is even, for all $n < \omega$. First, let us suppose that $|E_n|$ is odd, for all $n < \omega$. By putting $N = \bigcup_{n<\omega} E_n$ and taking any $M \in [\lambda \setminus N]^\omega \setminus \{\emptyset\}$, we can find $\theta < \kappa$ such that $x_\xi(\theta) = 0$, for all $\xi \in M$, and $x_\zeta(\theta) = 1$, for all $\zeta \in N$. Then $a_n(\theta) = 1$ for each $n < \omega$, but this is impossible since $a_n(\theta) \to 0$. Thus, we must have that $|E_n|$ is even, for all $n < \omega$. It is clear that we can find a strictly increasing sequence $(n_k)_{k<\omega}$ of positive integers and $P, Q \in [\lambda]^\omega \setminus \{\emptyset\}$ so that $|P \cap E_{n_k}| = 1$ for all $k < \omega$ and $Q = (\bigcup_{k<\omega} E_{n_k}) \setminus P$. Let us consider the subsequence $(a_{n_k})_{k<\omega}$. Then, we can find $\lambda < \kappa$ such that $x_\xi(\lambda) = 0$, for all $\xi \in P$, and $x_\zeta(\lambda) = 1$, for all $\zeta \in Q$. Again, we have that $a_{n_k}(\lambda) = 1$ for each $k < \omega$, but this is impossible since $a_{n_k}(\lambda) \to 0$. Therefore, G does not have any non-trivial convergent sequence. □

It follows from Theorem 2.4 that if M is a model of ZFC in which the inequality $\mathfrak{c} < \aleph_\omega < 2^\mathfrak{c}$ holds, then $M \models \{0,1\}^{\aleph_\omega}$ contains a dense pseudocompact subgroup without non-trivial convergent sequences. Contrary to this, if $\aleph_\omega$ is a strong limit cardinal number, then, by Theorem 1 from [3], we have that every dense subgroup of $\{0,1\}^{\aleph_\omega}$ contains a non-trivial convergent sequence.

3. Countably compact subgroups of some powers of $\{0,1\}$

In this section, under the assumption of the existence of a selective ultrafilter on ω, we shall construct a countably compact dense subgroup without non-trivial convergent sequences of some infinite powers of $\{0,1\}$.

Here, we shall use the following notation.

Let λ be an infinite cardinal with $\lambda = \lambda^\omega$. Let $\{I_0, I_1\}$ be a partition of λ in subsets of size λ such that $\omega \subseteq I_1$. Let $\{I_d : d \in \bigcup_{s \in [\lambda]^{<\omega}} \{0,1\}^s\}$ be a partition of I_1 in subsets of cardinality λ. Fix an enumeration $\{f_\alpha : \alpha \in I_0\}$ of all one-to-one functions $f : \omega \to [\lambda]^{<\omega} \setminus \{\emptyset\}$. Without loss of generality, we may assume that $\bigcup_{n<\omega} f_\alpha(n) \subseteq \alpha+\omega$, for every $\alpha<\lambda$. As in [1], we say that a space X is *card-homogeneous* if $|V| = X$ for every nonempty open subset V of X.

Lemma 3.1. *Let λ be an uncountable cardinal number. If the subset $\{x_\alpha : \alpha < \lambda\}$ of $\{0,1\}^\lambda$ satisfies*

(1) *$\{x_\alpha : \alpha < \lambda\}$ is an independent subset of $\{0,1\}^\lambda$;*
(2) *x_α is an accumulation point of the set $\{\sum_{\mu\in f_\alpha(n)} x_\mu : n < \omega\}$, for every $\alpha \in I_0$;*
(3) *the group $\langle\{x_\alpha : \alpha < \lambda\}\rangle$ does not contain non-trivial convergent sequences ,*

then $\{0,1\}^\lambda$ contains a card-homogeneous dense independent subset $\{y_\alpha : \alpha < \lambda\}$ satisfying conditions $(1) - (3)$.

Proof: By adding coordinates λ-many times, we may assume that

1' For every $F \in [\lambda]^{<\omega} \setminus \{\emptyset\}$,
$$|\{\theta < \lambda : \sum_{\mu\in F} x_\mu(\theta) \neq 0\}| = \lambda.$$

3' For every one-to-one function $f : \omega \to [\lambda]^{<\omega} \setminus \{\emptyset\}$,
$$|\{\theta < \lambda : \text{the sequence} \left(\textstyle\sum_{\mu\in f(n)} x_\mu(\theta)\right)_{n<\omega} \text{ does not converge in } \{0,1\}\}| = \lambda.$$

According to condition (2), for every $\alpha \in I_0$, we can find $p_\alpha \in \omega^*$ such that
$$x_\alpha = p_\alpha - lim_{n\to\omega} \sum_{\mu\in f_\alpha(n)} x_\mu.$$

Now, let $\{d_\alpha : \alpha \in I_1\}$ be an enumeration of $\bigcup_{s\in[\lambda]^{<\omega}} \{0,1\}^s$ in such a way that each element appears λ-many times. Put $y_0 = x_0$ and suppose that we have defined $y_\beta \in \{0,1\}^\lambda$, for every $\beta < \alpha < \lambda$. If $\alpha \in I_0$, then we define
$$(*) \qquad y_\alpha = p_\alpha - lim_{n\to\omega} \sum_{\mu\in f_\alpha(n)} y_\mu.$$

If $\alpha \in I_1$, then we define $y_\alpha = x_\alpha|_{\lambda\setminus dom(d_\alpha)} \cup d_\alpha$. It is not difficult to prove that $\{y_\alpha : \alpha < \lambda\}$ is dense in $\{0,1\}^\lambda$. Condition 1' guarantees that (1) holds for $\{y_\alpha : \alpha < \lambda\}$. It follows from $(*)$ that y_α is an accumulation point of the set $\{\sum_{\mu\in f_\alpha(n)} y_\mu : n < \omega\}$, for every $\alpha \in I_0$. The third condition follows directly from 3'. □

Theorem 3.2. *Let λ be a cardinal number such that $\lambda = \lambda^\omega$. If the subset $\{x_\alpha : \alpha < \lambda\}$ of $\{0,1\}^\lambda$ satisfies*

(1) *$\{x_\alpha : \alpha < \lambda\}$ is an independent subset of $\{0,1\}^\lambda$;*
(2) *x_α is an accumulation point of the set $\{\sum_{\mu \in f_\alpha(n)} x_\mu : n < \omega\}$, for every $\alpha \in I_0$;*
(3) *the group $\langle \{x_\alpha : \alpha < \lambda\}\rangle$ does not contain any non-trivial convergent sequence,*

then $\{0,1\}^\kappa$ contains a countably compact dense subgroup without non-trivial convergent sequences, for every cardinal number κ with $\lambda \leq \kappa \leq 2^\lambda$.

Proof: According to Lemma 3.1, we may assume that the following condition holds.

(4) For every $d \in \bigcup_{s \in [\lambda]^{<\omega}} \{0,1\}^s$ and every $\alpha \in I_d$, $x_\alpha|_{dom(d)} = d$.

By the second condition, for every $\alpha \in I_0$, we can find $p_\alpha \in \omega^*$ such that

$$x_\alpha = p_\alpha - lim_{n \to \omega} \sum_{\mu \in f_\alpha(n)} x_\mu.$$

We know from Theorem 2.4 that $\{0,1\}^{\kappa \setminus \lambda}$ contains a G_δ-dense (dense suffices here) subgroup $\{h_\alpha : \alpha < \lambda\}$ of size λ without non-trivial convergent sequences. Now, for each $d \in \bigcup_{s \in [\lambda]^{<\omega}} \{0,1\}^s$, let $\{\alpha(d,\xi) : \xi < \lambda\}$ be a faithful enumeration of I_d. We shall define inductively a suitable set $\{z_\alpha : \alpha < \lambda\} \subseteq \{0,1\}^{\kappa \setminus \lambda}$. Suppose that $\{z_\beta : \beta < \alpha\}$ has been defined for $\alpha < \lambda$. If $\alpha \in I_0$, we let $z_\alpha = p_\alpha - lim_{n \to \omega} \sum_{\mu \in f_\alpha(n)} z_\mu$. If $\alpha \in I_1$, then there exists $d \in \bigcup_{s \in [\lambda]^{<\omega}} \{0,1\}^s$ such that $\alpha \in I_d$, and then we define $z_\alpha = h_\xi$, where $\xi < \lambda$ satisfies that $\alpha = \alpha(d,\xi)$. Next, we proceed to define $y_\alpha = (x_\alpha, z_\alpha)$, for every $\alpha < \lambda$. We claim that the group $H = \langle \{y_\alpha : \alpha < \lambda\}\rangle$ is the required one. Indeed, it is not difficult to see that H is countably compact. The density of H in $\{0,1\}^\kappa$ follows from the fact that $\{(x_{\alpha(d,\xi)}, h_\xi) : \xi < \lambda$ and $d \in \bigcup_{s \in [\lambda]^{<\omega}} \{0,1\}^s\}$ is dense in $\{0,1\}^\kappa$. Let $f : \omega \to [\lambda]^{<\omega} \setminus \{\emptyset\}$ be a one-to-one function and consider the sequence $(\sum_{\alpha \in f(n)} y_\alpha)_{n<\omega}$. Suppose that this sequence converges. Then, by clauses $(1)-(3)$, the sequence $(\sum_{\alpha \in f(n)} x_\alpha)_{n<\omega}$ must be trivial. Hence, we deduce that the sequence $(\sum_{\alpha \in f(n)} y_\alpha)_{n<\omega}$ is also trivial. □

Corollary 3.3. *If there is a selective ultrafilter on* ω*, then* $\{0,1\}^\kappa$ *contains a countably compact dense subgroup without non-trivial convergent sequences, for every cardinal number* κ *with* $\mathfrak{c} \leq \kappa \leq 2^{\mathfrak{c}}$*.*

Proof: Assume the existence of a selective ultrafilter on ω. We know from Example 2.4 of [10] that $\{0,1\}^{\mathfrak{c}}$ contains a countably compact subgroups of size $\mathfrak{c}$. The conclusion follows directly from Theorem 3.2. □

Next, we address our attention to the power space $\{0,1\}^{\omega_1}$.

Definition 3.4 (**A. Hajnal and I. Juhász** [11])**.** We say that $Y \subseteq \prod_{\alpha<\omega_1} X_\alpha$ is ω-HFD if for every $X \in [Y]^\omega$ there is $\beta < \omega_1$ such that for every basic open subset U of $\prod_{\alpha<\omega_1} X_\alpha$ with $\pi_\gamma[U] = X_\gamma$ for every $\gamma < \beta$, we have that $U \cap X \neq \emptyset$.

The following result is taken from [12].

Lemma 3.5. *Let* $X = \prod_{\alpha<\omega_1} X_\alpha$ *be a product of compact metric spaces each one of them having at least two points, and let* $Y \subseteq X$ *be* ω-HFD*. If* $\pi_{[0,\beta)}[Y] = \prod_{\alpha<\beta} X_\alpha$*, where* $\pi_{[0,\beta)} : X \to \prod_{\alpha<\beta} X_\alpha$ *is the projection map, for every* $\beta < \lambda$*, then* Y *is a countably compact dense subspace of* X *which does not contain non-trivial convergent sequences.*

Theorem 3.6. *Let* $X = \{x_\alpha : \alpha < \mathfrak{c}\} \subseteq \{0,1\}^{\omega_1}$ *be such that for every* $F \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$*, we have that* $\{\mu < \omega_1 : \sum_{\xi\in F} x_\xi(\mu) \neq 0\}$ *is unbounded in* ω_1 *and* $\langle X \rangle$ *is* ω-HFD*. Then* $\{0,1\}^{\omega_1}$ *contains a countably compact dense subgroup without non-trivial convergent sequences.*

Proof: Let $\{K_\beta : \beta < \omega_1\}$ be a partition of $\mathfrak{c}$ in subsets of size $\mathfrak{c}$. For each $\beta < \omega_1$, fix an enumeration $\{z_\alpha : \alpha \in K_\beta\}$ of $\{0,1\}^\beta$. Then, for every $\alpha < \mathfrak{c}$, we define $y_\alpha = z_\alpha \cup x_\alpha|_{[\beta,\omega_1)}$, where $\beta < \omega_1$ satisfies $\alpha \in K_\beta$. Put $Y = \{y_\alpha : \alpha < \mathfrak{c}\}$.

Claim 1. Y is independent.

Proof of Claim 1: Fix $F \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$. Let $E = \{\beta < \mathfrak{c} : F \cap K_\beta \neq \emptyset\}$. Since E is finite, we can find $\mu < \omega_1$ such that $max\{E\} < \mu$ and $\sum_{\xi\in F} x_\xi(\mu) \neq 0$. As $max\{E\} < \mu$, we must have that $x_\xi(\mu) = y_\xi(\mu)$, for every $\xi \in F$. Thus, we obtain that $\sum_{\xi\in F} y_\xi \neq 0$. So, Y is independent.

Claim 2. $\pi_{[0,\beta)}[Y] = \{0,1\}^\beta$, for every $\beta < \omega_1$.

Proof of Claim 2: This follows directly from the definition of Y.

Claim 3. $\langle Y \rangle$ is ω-HFD.

Proof of Claim 3: Let $\{a_n : n < \omega\} \subseteq \langle Y \rangle$. Choose a one-to-one function $f : \omega \to [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$ such that $a_n = \sum_{\xi \in f(n)} y_\xi$, for every $n < \omega$. Let $D = \{\beta < \omega_1 : (\bigcup_{n<\omega} f(n)) \cap K_\beta \neq 0\}$. We know that $\theta = \sup\{D\} < \omega_1$, because D is countable. Since $\langle X \rangle$ is ω-HFD, there exists an ordinal number μ with $\theta < \mu < \omega_1$ for which the set $\{b_n|_{\omega_1 \setminus \mu} : n < \omega\}$ is dense in $\{0,1\}^{\omega_1 \setminus \mu}$, where $b_n = \sum_{\xi \in f(n)} x_\xi$ for every $n < \omega$. As $\mu > \theta$, we have that $b_n|_{\omega_1 \setminus \mu} = a_n|_{\omega_1 \setminus \mu}$, for every $n < \omega$ and then $\{a_n|_{\omega_1 \setminus \mu} : n < \omega\}$ is dense in $\{0,1\}^{\omega_1 \setminus \mu}$. Thus, we have proved that Y is ω-HFD in $\{0,1\}^{\omega_1}$.

According to Lemma 3.5, $\langle Y \rangle$ is a countably compact dense subgroup of $\{0,1\}^{\omega_1}$ which does not contain any non-trivial convergent sequences. □

Theorem 3.7. *It is consistent with the axioms of ZFC that the compact group $\{0,1\}^{\omega_1}$ contains a countably compact dense subgroup without non-trivial convergent sequences and $\mathfrak{c}$ takes any possible value.*

Proof: It suffices to find a set $X = \{x_\alpha : \alpha < \mathfrak{c}\} \subseteq \{0,1\}^{\omega_1}$ satisfying all the conditions of Theorem 3.6. We start with a model M of GCH and let $\kappa > \omega_1$ be a regular cardinal with $\kappa = \kappa^\omega$. By using the Cohen forcing with the poset $Fn(\kappa \times \omega_1, 2)$, we obtain a generic function $E : \kappa \times \omega_1 \to 2$, where $E = \bigcup G$ and G is a $Fn(\kappa \times \omega_1, 2)$-generic set. Then, we define $x_\alpha(\gamma) = E(\alpha, \gamma)$, for every $(\alpha, \gamma) \in \kappa \times \omega_1$. Set $X = \{x_\alpha : \alpha < \kappa\}$. Since $\kappa \times \omega_1$ has cardinality κ, we can prove that $M \models \kappa = \mathfrak{c}$ (see [15, Lemma 5.14, p. 204]).

Claim 1. $\{\mu < \omega_1 : \sum_{\alpha \in F} x_\alpha(\mu) \neq 0\}$ is unbounded in ω_1, for every $F \in [\kappa]^{<\omega} \setminus \{\emptyset\}$.

Proof of Claim 1: Given $F \in [\kappa]^{<\omega} \setminus \{\emptyset\}$ and $\mu < \omega_1$, we define

$$D_{F,\mu} = \{p \in Fn(\kappa \times \omega_1, 2) : \exists \theta \in (\mu, \omega_1)(F \times \{\theta\} \subseteq dom(p)$$

and

$$\textstyle\sum_{\alpha \in F} p(\alpha, \theta) \neq 0)\}.$$

It is not hard to see that $D_{F,\mu}$ is dense in $Fn(\kappa\times\omega_1, 2)$. So, there is an ordinal number μ with $\mu < \theta < \omega_1$ such that

$$\sum_{\alpha\in F} x_\alpha(\theta) = \sum_{\alpha\in F} E(\alpha,\theta) \neq 0.$$

CLAIM 2. $\langle X \rangle$ is ω-HFD.

Proof of Claim 2: Let $\{a_n : n < \omega\} \subseteq \langle X \rangle$. Pick a one-to-one function $f : \omega \to [\kappa]^{<\omega} \setminus \{\emptyset\}$ such that $a_n = \sum_{\alpha\in f(n)} x_\alpha$, for every $n < \omega$. As $f \in M[G]$, according to Lemma 5.5 from [15, p. 206], there is a function $F : \omega \to \mathcal{P}(\{f(n) : n < \omega\})$ such that $F \in M$, $f(n) \in F(n)$, for all $n < \omega$, and $M \models \forall n < \omega(|F(n)| \leq \omega)$. Put $T = \bigcup_{n<\omega}(\bigcup F(n)) \in M$. Then, we have that $f \subseteq \omega \times [T]^{<\omega} \setminus \{\emptyset\}$ and $\omega \times [T]^{<\omega} \in M$. According to Lemma 2.2 from [15, p. 256], there exists $I_0 \subseteq \kappa\times\omega_1$ with $I_0 \in M$ and $M \models |I_0| \leq |\omega\times[T]^{<\omega}| \leq \omega$ such that $f \in M[G_0]$, where $G_0 = G \cap Fn(I_0, 2)$. Let $I_1 = (\kappa \times \omega_1) \setminus I_0$. By Lemma 2.1 from [15, p. 255], we know that $G_1 := G \cap Fn(I_1, 2)$ is $Fn(I_1)$-generic over $M[G_0]$ and $M[G] = M[G_0][G_1]$. Choose $\alpha < \omega_1$ such that $I_0 \subseteq \kappa \times \alpha$. Let $r : F \to \{0, 1\}$ be a function with $F \in [\omega_1 \setminus \alpha]^{<\omega} \setminus \{\emptyset\}$. As $f \in M[G_0]$, the set

$$D_r = \{p \in Fn(I_1, 2) : \exists k < \omega(dom(p) \supseteq f(k) \times F \\ \wedge \forall\beta \in F(\sum_{\xi\in f(k)} p(\xi,\beta) = r(\beta)))\}$$

lies in $M[G_0]$. We claim that D_r is $Fn(I_1, 2)$-dense in $M[G_0]$. Indeed, fix $q \in Fn(I_1, 2)$. Then, there exist $k < \omega$ and $\gamma \in f(k)$ such that $(\{\gamma\} \times F) \cap dom(q) = \emptyset$. We will extend q to p where $dom(p) = dom(q) \cup (f(k) \times F)$, $p(\xi, \beta) = q(\xi, \beta)$ if $(\xi, \beta) \in dom(q)$, $p(\xi, \beta) = 0$ if $(\xi, \beta) \in (f(k) \times F) \setminus (dom(q) \cup (\{\gamma\} \times F))$ and, for each $\beta \in F$, $p(\gamma, \beta)$ is defined such that

$$p(\gamma,\beta) + \sum_{\xi\in f(k)\setminus\{\gamma\}} p(\xi,\beta) = r(\beta).$$

This shows that D_r is $Fn(I_1, 2)$-dense in $M[G_0]$. Now, let us prove that $\{a_k|_{\omega_1\setminus\alpha} : k < \omega\}$ is dense in $\{0, 1\}^{\omega_1\setminus\alpha}$. Indeed, if $r : F \to \{0, 1\}$ is as above, then we can find $p \in G_1 \cap D_r$. So, there exists $k < \omega$ for which

$$a_k(\beta) = \sum_{\xi\in f(k)} x_\xi(\beta) = \sum_{\xi\in f(k)} E(\xi,\beta) = \sum_{\xi\in f(k)} p(\xi,\beta) = r(\beta),$$

for each $\beta \in dom(r)$. Therefore, $a_k|_{dom(r)} = r$, and we are done. Thus, we have proved the second claim. □

4. Countably compact dense subgroups of some powers of the circle

Our task in this last section will be the construction of countably compact dense subgroups without non-trivial convergent sequences of some infinite powers of $\mathbb{T}$. First, we state two general results and then we apply them to the compact group $\mathbb{T}^\kappa$.

The symbol $\oplus_\kappa \mathbb{Z}$ will denote the direct sum of κ-many copies of $\mathbb{Z}$, where κ is an infinite cardinal. The *support* of an element $F \in \oplus_\kappa \mathbb{Z}$ is the set $s(F) = \{\xi < \kappa : F(\xi) \neq 0\}$. In what follows, κ will be an infinite cardinal number with $\kappa^\omega = \kappa$ and we also assume that $\kappa = I_0 \cup I_1$, where $I_0 \cap I_1 = \emptyset$, $0 \in I_1$, and $|I_0| = |I_1| = \kappa$. Let $\{f_\alpha : \alpha \in I_0\}$ be an enumeration of all one-to-one functions $f : \omega \to \oplus_\kappa \mathbb{Z}$ such that $\bigcup_{n<\omega} s(f_\alpha(n)) \subseteq \alpha$, for every $\alpha < \kappa$.

Theorem 4.1. *Let G be a compact Abelian group. If $\{x_\alpha : \alpha < \kappa\} \subseteq G^\kappa$ satisfies*

(1) *for every $F \in \oplus_\kappa \mathbb{Z}$ with $s(F) \neq \emptyset$,*

$$|\{\mu < \kappa : \sum_{\xi \in s(F)} F(\xi) x_\xi(\mu) \neq 0\}| = \kappa;$$

(2) *for every $\alpha < \kappa$,*
$|\{\mu < \kappa :$ the sequence $(\sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi(\mu))_{n<\omega}$ does not converge in $G\}| = \kappa$;

(3) *for every $\alpha \in I_0$, x_α is an accumulation point of the set*

$$\{\sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi : n < \omega\},$$

then there is a countably compact dense subgroup of G^κ of size κ without non-trivial convergent sequences.

Proof: According to condition (3), for every $\alpha \in I_0$, we can find $p_\alpha \in \omega^*$ such that

$$x_\alpha = p_\alpha - lim_{n \to \omega} \sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi.$$

Let $\{I_t : t \in \bigcup_{s\in[\kappa]^{<\omega}} G^s\}$ be a partition of I_1 in subsets of size κ. Put $y_0 = x_0$ and suppose that y_β has been defined for each $\beta < \alpha < \kappa$. First, assume that $\alpha \in I_1$. Then, choose $t \in \bigcup_{s\in[\kappa]^{<\omega}} G^s$ such that $\alpha \in I_t$. Then, we proceed to define $y_\alpha = x_\alpha|_{\kappa\setminus dom(t)} \cup t$. Now, suppose that $\alpha \in I_0$. In this case, we put

$$(*) \qquad y_\alpha = p_\alpha - lim_{n\to\omega} \sum_{\xi\in s(f_\alpha(n))} f_\alpha(n)(\xi) y_\xi.$$

Claim. For every $\alpha < \kappa$, we have that

$$|\{\mu < \kappa : y_\alpha(\mu) \neq x_\alpha(\mu)\}| < \kappa.$$

Proof of the Claim: Let us proceed by transfinite induction. Suppose that the conclusion holds for every $\beta < \alpha$, where $\alpha < \kappa$. If $\alpha \in I_t$, for some $t \in \bigcup_{s\in[\kappa]^{<\omega}} G^s$, then we have that $y_\alpha = x_\alpha|_{\kappa\setminus dom(t)} \cup t$; hence, $\{\mu < \kappa : y_\alpha(\mu) \neq x_\alpha(\mu)\} \subseteq dom(t)$ which is a finite set. If $\alpha \in I_0$, then we know that $y_\alpha = p_\alpha - lim_{n\to\omega} \sum_{\xi\in s(f_\alpha(n))} f_\alpha(n)(\xi) y_\xi$. It then follows that

$$\{\mu < \kappa : y_\alpha(\mu) \neq x_\alpha(\mu)\} \subseteq \bigcup_{n<\omega} \bigcup_{\xi\in s(f_\alpha(n))} \{\nu < \kappa : y_\xi(\nu) \neq x_\xi(\nu)\}.$$

So, $|\{\mu < \kappa : y_\alpha(\mu) \neq x_\alpha(\mu)\}| < \kappa$.

It follows from the previous claim and the first condition that $\{y_\alpha : \alpha < \kappa\}$ is an independent subset of G^κ. Let us consider the subgroup $H = \langle \{y_\alpha : \alpha < \kappa\} \rangle$. By construction, we have that H is dense in G^κ. The countable compactness of H follows from $(*)$. The second condition and the claim assure us that H does not have non-trivial convergent sequences. □

Theorem 4.2. *If there exists an infinite countably compact free Abelian group without non-trivial convergent sequences, then there exists one that is dense in* $\mathbb{T}^{\mathfrak{c}}$.

Proof: Let G be an infinite countably compact free Abelian group without non-trivial convergent sequences. Without loss of generality, we can assume that G is a subgroup of $\mathbb{T}^\kappa$ for some cardinal κ. In addition, we may assume that $\{f_\alpha : \alpha \in I_0 \cap \mathfrak{c}\}$ enumerates all one-to-one functions $f : \omega \to \oplus_{\mathfrak{c}}\mathbb{Z}$. By using some ideas from [23], it is possible to get $\{x_\alpha : \alpha < \mathfrak{c}\} \subseteq G$ such that

a) $\{x_\alpha : \alpha < \mathfrak{c}\}$ is independent and,

b) for each $\alpha \in I_0 \cap \mathfrak{c}$, x_α is an accumulation point of the set
$$\{ \sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi : n < \omega \}.$$

Then, the subgroup $H = \langle \{x_\alpha : \alpha < \mathfrak{c}\} \rangle$ of G satisfies
$$\sum_{\xi \in s(F)} F(\xi) x_\xi \neq 0,$$
for every $F \in \oplus_\mathfrak{c} \mathbb{Z}$ with $s(F) \neq \emptyset$. Hence,

c) for every $F \in \oplus_\mathfrak{c} \mathbb{Z}$ with $s(F) \neq \emptyset$, there exists $\mu_F < \kappa$ such that
$$\sum_{\xi \in s(F)} F(\xi) x_\xi(\mu_F) \neq 0.$$

Observe that H does not have non-trivial convergent sequences. Thus, for each $\alpha \in I_0 \cap \mathfrak{c}$, the sequence
$$(\sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi)_{n<\omega}$$
does not converge in $\langle \{x_\alpha : \alpha < \mathfrak{c}\} \rangle$. So,

d) for each $\alpha \in I_0 \cap \mathfrak{c}$, there exists $\mu_\alpha < \kappa$ such that the sequence
$$(\sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi(\mu_\alpha))_{n<\omega}$$
does not converge in $\mathbb{T}$. Indeed, given $\mu < \kappa$, if the sequence
$$(\sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi(\mu))_{n<\omega}$$
converges in $\mathbb{T}$, then, by *b*), it must converge to $x_\alpha(\mu)$. Thus, if *d*) does not hold, then $(\sum_{\xi \in s(f_\alpha(n))} f_\alpha(n)(\xi) x_\xi)_{n<\omega}$ converges to x_α, but this is impossible. Now we shall transform the set $\{x_\alpha : \alpha < \mathfrak{c}\}$ in a subset of $\mathbb{T}^\mathfrak{c}$. To do that let $A = \{\mu_F : F \in \oplus_\mathfrak{c} \mathbb{Z}$ with $s(F) \neq \emptyset\} \cup \{\mu_\alpha : \alpha \in I_0 \cap \mathfrak{c}\}$ and fix a bijection $\sigma : \mathfrak{c} \longrightarrow A \times \mathfrak{c}$. For every $\alpha < \mathfrak{c}$, we define $y_\alpha : \mathfrak{c} \longrightarrow \mathbb{T}$ as $y_\alpha(\beta) = x_\alpha(\mu)$ where $\sigma(\beta) = (\mu, \xi)$, for each $\beta < \mathfrak{c}$. It is evident that $\{y_\alpha : \alpha < \mathfrak{c}\}$ satisfies conditions (1) - (3) of Theorem 4.1. (We add coordinates if it necessary as we did in Lemma 3.1.) Thus, there exists a countably compact free Abelian dense subgroup of $\mathbb{T}^\mathfrak{c}$ as required. □

Corollary 4.3. *If there are $\mathfrak{c}$ many RK-incomparable, selective ultrafilters in ω^*, then there exists a countably compact free Abelian dense subgroup of $\mathbb{T}^{\mathfrak{c}}$ without non-trivial convergent sequences.*

Proof: It was shown in [16] that there exists an infinite countably compact free Abelian group without non-trivial convergent sequences assuming the existence of $\mathfrak{c}$ many RK-incomparable, selective ultrafilters in ω^*. Applying Theorem 4.2, we obtain the required conclusion. □

If we require only pseudocompact subgroups, then we have the following.

Theorem 4.4. *Let G be a compact Abelian group. If $\{x_\alpha : \alpha < \kappa\} \subseteq G^\kappa$ satisfies*

(1) *for every $F \in \oplus_\kappa \mathbb{Z}$ with $s(F) \neq \emptyset$, there is $\mu < \kappa$ such that*
$$\sum_{\xi \in s(F)} F(\xi) x_\xi(\mu) \neq 0;$$
(2) *for every one-to-one function $f : \omega \to \oplus_\kappa \mathbb{Z}$, there is $\mu < \kappa$ such that the sequence $(\sum_{\xi \in s(f(n))} f(n)(\xi) x_\xi(\mu))_{n<\omega}$ does not converge in G,*

then there is a G_δ-dense independent subset A of G^κ such that $\langle A \rangle$ does not have non-trivial convergent sequences.

Proof: By repeating coordinates κ-many times, we may assume that conditions (1) and (2) hold for κ-many μ's. Let $\{K_d : d \in \bigcup_{s \in [\kappa]^\omega} G^s\}$ be a partition of κ in subsets of cardinality κ. For every $\alpha < \kappa$, choose $\alpha \in K_d$ and define $z_\alpha = d \cup x_\alpha|_{\kappa \setminus dom(d)}$. Set $A = \{z_\alpha : \alpha < \kappa\}$. It is evident that A is a G_δ-dense subset of G^κ.

Claim 1. The set $A = \{z_\alpha : \alpha < \kappa\}$ is independent.

Proof of Claim 1: Given $F \in \oplus_{2^{\mathfrak{c}}} \mathbb{Z}$ with $s(F) \neq \emptyset$, let $\gamma > \max\{dom(d) : s(F) \cap K_d \neq \emptyset\}$. From condition (1), we can find an ordinal number μ with $\gamma < \mu < \kappa$ such that
$$\sum_{\zeta \in s(F)} F(\zeta) z_\zeta(\mu) = \sum_{\zeta \in s(F)} F(\zeta) x_\zeta(\mu) \neq 0.$$
This shows that A is independent.

Claim 2. The group $\langle A \rangle$ does not have non-trivial convergent sequences.

Proof of Claim 2: Let $(a_n)_{n<\omega}$ be a sequence in $\langle A \rangle$. Without loss of generality, we may assume that there is a one-to-one continuous function $f : \omega \to \oplus_\kappa \mathbb{Z}$ such that $a_n = \sum_{\xi \in s(f(n))} f(n)(\xi) z_\xi$, for every $n < \omega$. Choose an ordinal number γ such that

$$\sup\{dom(d) : (\bigcup_{n<\omega} s(f(n))) \cap K_d \neq \emptyset \text{ and } d \in \bigcup_{s \in [\kappa]^\omega} G^s\} < \gamma < \kappa.$$

It follows from condition (2) that there is an ordinal number μ with $\gamma < \mu < \omega_1$ such that the sequence

$$(a_n)_{n<\omega} = \left(\sum_{\xi \in s(f(n))} f(n)(\xi) z_\xi(\mu) \right)_{n<\omega} = \left(\sum_{\xi \in s(f(n))} f(n)(\xi) x_\xi(\mu) \right)_{n<\omega}$$

does not converge in G. Hence, the sequence $(a_n)_{n<\omega}$ cannot converge in G^κ. □

Theorem 4.5. *Let $\kappa = \kappa^\omega$ be an infinite cardinal. The group $\mathbb{T}^\kappa$ contains a pseudocompact free Abelian dense subgroup without non-trivial convergent sequences.*

Proof: According to Lemma 2.6 from [9], $\mathbb{T}^\kappa$ contains a pseudocompact free dense subgroup without non-trivial convergent sequences of size 2^κ. Then, we apply Theorem 4.4. An alternative proof is the following.

Let $\{f_\alpha : \omega \leq \alpha < \kappa\}$ be an enumeration of all one-to-one functions $f : \omega \to \oplus_\kappa \mathbb{Z}$ such that $\bigcup_{n<\omega} s(f_\alpha(n)) \subseteq \alpha$ for each infinite $\alpha < \kappa$. We partition κ in $A \cup B$ such that $|A| = |B| = \kappa$. For each infinite $\alpha < \kappa$, let $\chi_\alpha \in \oplus_\kappa \mathbb{Z}$ be such that $\chi_\alpha(\alpha) = 1$ and $\chi_\alpha(\mu) = 0$ if $\alpha \neq \mu$. Let $\{F_\alpha : \alpha \in A\}$ be an enumeration of all $F \in \oplus_\kappa \mathbb{Z}$ with $s(F) \neq \emptyset$ such that each F appears κ-many times in the enumeration. For each $\alpha \in A$, by using the lineal independence of $\{\chi_\alpha : \alpha < \kappa\}$, it is not difficult to construct a homomorphism $\phi_\alpha : \oplus_\kappa \mathbb{Z} \longrightarrow \mathbb{T}$ such that $\phi_\alpha(F_\alpha) \neq 0$. Now, partition B into $\{B_\alpha : \alpha < \kappa\}$ such that each B_α has cardinality κ. For each $\beta \in B$, let $\alpha < \kappa$ be the unique ordinal number such that $\beta \in B_\alpha$. Let E_β be a countable subset of κ such that $\alpha \in E_\beta$ and $s(f_\gamma(n)) \subseteq E_\beta$, for each $n \in \omega$ and for each $\gamma \in E_\beta$. By using an argument from [20] (also used in [14]), for each $\beta \in B$, we can find a homomorphism $\phi_\beta : \oplus_{E_\beta} \mathbb{Z} \longrightarrow \mathbb{T}$ such that the sequence $(\phi_\beta(f_\alpha(n)))_{n<\omega}$ does not converge in $\mathbb{T}$. Extend this homomorphism ϕ_β arbitrarily to $\oplus_\kappa \mathbb{Z}$.

We are ready to define the generators of our group. For each $\alpha < \kappa$, let $x_\alpha : \kappa \longrightarrow \mathbb{T}$ be defined by $x_\alpha(\mu) = \phi_\mu(\chi_\alpha)$. It is not hard to see that the set $\{x_\xi : \xi < \kappa\}$ satisfies all the conditions of Theorem 4.4. □

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References

[1] W. W. Comfort and Salvador Garca-Ferreira, *Resolvability: a selective survey and some new results*, Topology Appl. **74** (1996), no. 1-3, 149–167.

[2] W. W. Comfort and Kenneth A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. **16** (1966), 483–496.

[3] Jan J. Dijkstra and Jan van Mill, *Groups without convergent sequences*, Topology Appl. **74** (1996), no. 1-3, 275–282.

[4] Dikran Dikranjan and Dmitri Shakhmatov, *Algebraic Structure of Pseudocompact Groups*. Mem. Amer. Math. Soc. **133** (1998), no. 633.

[5] ———, *Forcing hereditarily separable compact-like group topologies on abelian groups*, Topology Appl. **151** (2005), no. 1-3, 2–54.

[6] Dikran Dikranjan and Mikhail Tkachenko, *Algebraic structure of small countably compact abelian groups*, Forum Math. **15** (2003), no. 6, 811–837.

[7] Eric K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), no. 2, 417–427.

[8] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[9] J. Galindo and S. Garcia-Ferreira, *Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences*, Topology Appl. **154** (2007), no. 2, 476–490.

[10] S. Garcia-Ferreira, A. H. Tomita, and S. Watson, *Countably compact groups from a selective ultrafilter*. Proc. Amer. Math. Soc. **133** (2005), no. 3, 937–943 (electronic).

[11] A. Hajnal and I. Juhász, *On hereditarily α-Lindelf and α-separable spaces. II*, Fund. Math. **81** (1973/74), no. 2, 147–158.

[12] ———, *A separable normal topological group need not be Lindelf*, General Topology and Appl. **6** (1976), no. 2, 199–205.

[13] Klaas Pieter Hart and Jan van Mill, *A countably compact topological group H such that $H \times H$ is not countably compact*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 811–821

[14] Piotr B. Koszmider, Artur H. Tomita, and S. Watson, *Forcing countably compact group topologies on a larger free abelian group*, Topology Proc. **25** (2000), Summer, 563–574 (2002).

[15] Kenneth Kunen, *Set Theory. An Introduction to Independence Proofs*. Studies in Logic and the Foundations of Mathematics, 102. Amsterdam-New York: North-Holland Publishing Co., 1980.

[16] Roberto E. Madariaga-Garcia and Artur Hideyuki Tomita, *Countably compact topological group topologies on free Abelian groups from selective ultrafilters*, Topology Appl. **154** (2007), no. 7, 1470-1480.

[17] V. I. Malykhin and L. B. Shapiro, *Pseudocompact groups without convergent sequences* (Russian), Mat. Zametki 37 (1985), no. 1, 103–109, 139. English translation: Math. Notes **37** (1985), no. 1-2, 59–62.

[18] S. M. Sirota, *The product of topological groups, and extremal disconnectedness* (Russian), Mat. Sb. (N.S.) **79(121)** (1969), 179–192. English translation: Math USSR-Sb. **8** (1969), no. 2, 169–180.

[19] M. G. Tkachenko, *Pseudocompact topological groups and their properties* (Russian), Sibirsk. Mat. Zh. **30** (1989), no. 1, 154–164; English translation: Siberian Math. J. **30** (1989), no. 1, 120–128.

[20] ———, *Countably compact and pseudocompact topologies on free abelian groups* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. **1990** (1990), no. 5, 68–75; English translation: Soviet Math. (Iz. VUZ) **34** (1990), no. 5, 79–86.

[21] Artur Hideyuki Tomita, *A group under* $\mathrm{MA}_{\mathrm{countable}}$ *whose square is countably compact but whose cube is not*, Topology Appl. **91** (1999), no. 2, 91–104.

[22] ———, *Countable compactness and finite powers of topological groups without convergent sequences*, Topology Appl. **146/147** (2005), 527–538.

[23] ———, *Square of countably compact groups without non-trivial convergent sequences*, Topology Appl. **153** (2005), no. 1, 107–122.

[24] Jack Widman, *Topological groups in which every dense subgroup has a convergent sequence*, Proceedings of the Eighth Prague Topological Symposium (1996). North Bay, ON: Topol. Atlas, 1997. 387–392 (electronic).

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