Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 31, No. 1, 2007
Pages 115-123
http://topology.auburn.edu/tp/

ON COUNTABLE-TO-ONE IMAGES OF METRIC SPACES

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Abstract. In this paper, it is proved that weak-open (almost-open, respectively), countable-to-one images of metric spaces and weak-open (almost-open, respectively), s-images of metric spaces are coincident, which gives some answers to a question on countable-to-one images of metric spaces.

1. Introduction

A map $f : X \longrightarrow Y$ is a countable-to-one map (an s-map, respectively) if the fiber $f^{-1}(y)$ is countable (separable, respectively) in X for each $y \in Y$. The following question is investigated by Chuan Liu and Shou Lin in [10].

Question 1.1. Are quotient countable-to-one images of metric spaces and quotient s-images of metric spaces coincident?

The answer for Question 1.1 is negative. Chuan Liu and Shou Lin gave an pseudo-open, s-image of a metric space which is not any quotient, countable-to-one image of a metric space (see [10, Example 6]). Furthermore, they proved the following theorem.

Theorem 1.2 ([10, Theorem 1]). *The following are equivalent for a space X.*

2000 *Mathematics Subject Classification.* 54C10; 54E40; 54D55.
Key words and phrases. almost-open maps, countable-to-one maps, pseudo-open maps, quotient maps, s-maps, weak-open maps.
This project was supported by NSFC (No.10571151 and 10671173).

(1) *X is an open s-image of a metric space.*
(2) *X is an open, countable-to-one image of a metric space.*

Note that weak-open (almost-open, respectively) maps, which play an important role in the study of images of metric spaces [6], [7], [9], and [12], are between open maps and quotient maps. As a further investigation of Question 1.1 and Theorem 1.2, it is natural to pose the following question.

Question 1.3. Are weak-open (almost-open, respectively) s-images of metric spaces weak-open (almost-open, respectively) countable-to-one images of metric spaces?

In this paper, the question is answered affirmatively and some related results are obtained. Throughout this paper, all spaces are assumed to be T_2 and all maps are continuous surjections. Denote the set of all natural numbers by $\mathbb{N}$. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}_x$ be a family of subsets of X. $\mathcal{P}_x$ is a network at x in X, if $x \in \bigcap \mathcal{P}_x$ and for each neighborhood U of x in X there is $P \in \mathcal{P}_x$ such that $P \subset U$. We refer the reader to [2] for notations and terminology not explicitly given here.

2. Definitions and remarks

Definition 2.1 ([3])**.** Let X be a space.

(1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to x, then $\{x_n\}$ is eventually in P.

(2) Let $P \subset X$. P is called a sequentially open subset in X if P is a sequential neighborhood of x in X for each $x \in P$.

(3) X is called a sequential space if each sequentially open subset in X is open in X.

(4) X is called a Fréchet-space if for each $P \subset X$ and for each $x \in \overline{P}$, there is a sequence $\{x_n\}$ in P converging to x, where $\overline{P}$ is the closure of P.

Remark 2.2. The following are well known.

(1) P is a sequential neighborhood of x in X iff each sequence $\{x_n\}$ converging to x is frequently in P.

(2) The intersection of finitely many sequential neighborhoods of x in X is a sequential neighborhood of x in X.

(3) First countable $\Longrightarrow$ Fréchet $\Longrightarrow$ sequential.

Definition 2.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that for each $x \in X$, the following conditions (a) and (b) are satisfied.

(a) $\mathcal{P}_x$ is a network at x in X.

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \bigcap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is called a weak base [1] of X if whenever $G \subset X$, G is open in X iff for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$. $\mathcal{P}_x$ is called a weak neighborhood network (wn-network) at x in X.

(2) $\mathcal{P}$ is called an sn-network [4] of X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. $\mathcal{P}_x$ is called an sn-network at x in X.

Remark 2.4. For a space, weak base $\Longrightarrow$ sn-network. An sn-network for a sequential space is a weak base [8].

Definition 2.5. Let $f : X \longrightarrow Y$ be a map.

(1) f is called almost-open [7] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x in X, then $f(U)$ is a neighborhood of y in Y.

(2) f is called weak-open [6], [12] if there is a weak base $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ of Y, and for each $y \in Y$, there is $x \in f^{-1}(y)$ satisfying the condition that for each open neighborhood U of x in X, $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

(3) f is called sn-open if there is an sn-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y, and for each $y \in Y$, there is $x \in f^{-1}(y)$ satisfying the condition that for each open neighborhood U of x in X, $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

(4) f is called 1-sequence-covering [8], [11] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(5) f is called quotient [2] if $f^{-1}(U)$ is open in X iff U is open in Y.

(6) f is called pseudo-open [1] if for each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ in X, $f(U)$ is a neighborhood of y in Y.

Remark 2.6. For a map, the following hold.

(1) Almost-open $\Longrightarrow$ weak-open $\Longrightarrow$ sn-open.
(2) Weak-open $\Longrightarrow$ quotient [12, Proposition 3.2].
(3) Almost-open $\Longrightarrow$ pseudo-open $\Longrightarrow$ quotient [7, Remark 1.2.2].
(4) Quotient maps preserve sequential spaces [7, Lemma 1.4.3(2)].
(5) Pseudo-open maps preserve Fréchet spaces [7, Lemma 1.4.3(3)].

3. Relations among maps

The following lemma is obtained by having gained some enlightenment from [5, Lemma 2.9].

Lemma 3.1. *Let $f : X \longrightarrow Y$ be a map and $\{y_n\}$ be a sequence converging to y in Y. If $\{B_n\}$ is a decreasing network at some $x \in f^{-1}(y)$ in X and $\{y_n\}$ is eventually in $f(B_k)$ for each $k \in N$, then there is a sequence $\{x_n\}$ converging to x in X such that each $x_n \in f^{-1}(y_n)$.*

Proof: Let $\{B_n\}$ be a decreasing network at some $x \in f^{-1}(y)$ in X, and let $\{y_n\}$ be eventually in $f(B_k)$ for each $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $y_n \in f(B_k)$ for all $n > n_k$, so $f^{-1}(y_n) \bigcap B_k \neq \emptyset$ for each $n > n_k$. Without loss of generality, we can assume $1 < n_k < n_{k+1}$.

For each $n \in \mathbb{N}$, pick

$$x_n \in \begin{cases} f^{-1}(y_n) & n < n_1 \\ \\ f^{-1}(y_n) \bigcap B_k & n_k \leq n < n_{k+1} \end{cases}$$

Then $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$. It is easy to show that $\{x_n\}$ converges to x. □

Proposition 3.2. *Let $f : X \longrightarrow Y$ be a map. Then the following hold.*

(1) *If f is 1-sequence-covering, then f is sn-open.*
(2) *If f is sn-open and X is first countable, then f is 1-sequence-covering.*
(3) *If f is sn-open and Y is sequential, then f is weak-open.*
(4) *If f is sn-open, quotient and X is sequential, then f is weak-open.*

Proof: (1) Let f be 1-sequence-covering. For each $y \in Y$, we construct an sn-network $\mathcal{P}_y$ at y in Y as follows.

Let $y \in Y$, then there is $x \in f^{-1}(y)$ such that the condition in Definition 2.5(4) is satisfied. Let $\mathcal{B}_x$ be a neighborhood base at x in X and $\mathcal{P}'_y = \{f(B) : B \in \mathcal{B}_x\}$. Then each element of $\mathcal{P}'_y$ is a sequential neighborhood of y in Y. In fact, let $P' = f(B) \in \mathcal{P}'_y$, where $B \in \mathcal{B}_x$. Since the condition in Definition 2.5(4) is satisfied, whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$. Note that B is a neighborhood of x. $\{x_n\}$ is eventually in B, so $\{y_n\} = \{f(x_n)\}$ is eventually in $f(B) = P'$. Thus, P' is a sequential neighborhood of y in Y. Put $\mathcal{P}_y = \{\bigcap \mathcal{F}_y : \mathcal{F}_y\ is\ a\ finite\ subfamily\ of\ \mathcal{P}'_y\}$, then each element of $\mathcal{P}_y$ is a sequential neighborhood of y in Y from Remark 2.2(2). Also, it is not difficult to check that $\mathcal{P}_y$ satisfies conditions (a) and (b) in Definition 2.3. Thus, we obtain that $\mathcal{P}_y$ is an sn-network at y in Y.

Put $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$, then $\mathcal{P}$ is an sn-network of Y. For each $y \in Y$, $x \in f^{-1}(y)$ is chosen as above. Then whenever U is a neighborhood of x, there is $B \in \mathcal{B}_x$ such that $x \in B \subset U$, so $f(B) \in \mathcal{P}_y$ and $f(B) \subset f(U)$. This proves that f is sn-open.

(2) Let f be sn-open and X be first countable, and let $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ be a sn-network of Y described in Definition 2.5(3). For each $y \in Y$, there is $x \in f^{-1}(y)$, which satisfies the condition in Definition 2.5(3). Let $\{y_n\}$ be a sequence converging to y in Y. Since X is first countable, there is a countable decreasing neighborhood base $\{B_n\}$ at x in X. For each $n \in \mathbb{N}$, there is $P_n \in \mathcal{P}_y$ such that $P_n \subset f(B_n)$. Note that P_n is a sequential neighborhood of y in Y. So $\{y_n\}$ is eventually in $f(B_n)$. By Lemma 3.1, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$. This proves that f is 1-sequence-covering.

(3) It is holds by Remark 2.4.

(4) Let $f : X \longrightarrow Y$ be sn-open, quotient and X be sequential. Then Y is sequential by Remark 2.6(4). So f is weak-open by the above (3). □

The following two theorems are obtained from Remark 2.6 and Proposition 3.2.

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a map. If X is first countable, then the following are equivalent.*

(1) *f is a weak-open map.*

(2) *f is an sn-open, quotient map.*
(3) *f is a 1-sequence-covering, quotient map.*

Theorem 3.4. *Let $f : X \longrightarrow Y$ be a map. If X is first countable and Y is sequential, then the following are equivalent.*

(1) *f is a weak-open map.*
(2) *f is an sn-open map.*
(3) *f is a 1-sequence-covering map.*

Now we investigate almost-open maps.

Lemma 3.5. *Let X be a Fréchet space and $x \in X$. If P is a sequential neighborhood of x in X, then $x \in P^\circ$, where P° is the interior of P.*

Proof: Let P be a sequential neighborhood of x in X. If $x \notin P^\circ$, then $x \in X - P^\circ = \overline{X - P}$. There is a sequence $\{x_n\}$ in $X - P$ converging to x because X is Fréchet. This contradicts that P is a sequential neighborhood of x in X. So $x \in P^\circ$. □

Proposition 3.6. *Let $f : X \longrightarrow Y$ be a map.*

(1) *If f is sn-open and Y is Fréchet, then f is almost-open.*
(2) *If f is sn-open, pseudo-open and X is Fréchet, then f is almost-open.*

Proof: (1) Let f be sn-open and Y be Fréchet. Then there is an sn-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y such that for each $y \in Y$, there is $x \in f^{-1}(y)$ satisfying the condition in Definition 2.5(3). Whenever U is a neighborhood of x in X, then $P \subset f(U)$ for some $P \in \mathcal{P}_y$. Note that P is a sequential neighborhood of y in Y. So $f(U)$ is a sequential neighborhood of y in Y. By Lemma 3.5, $y \in (f(U))^\circ$, so $f(U)$ is a neighborhood of y in Y. This proves that f is almost-open.

(2) Let $f : X \longrightarrow Y$ be sn-open, pseudo-open and X be Fréchet. Then Y is Fréchet by Remark 2.6(5). So f is almost-open by (1) above. □

The following two theorems are obtained from Remark 2.6, Proposition 3.2, and Proposition 3.6.

Theorem 3.7. *Let $f : X \longrightarrow Y$ be a map. If X is first countable, then the following are equivalent.*

(1) *f is an almost-open map.*

(2) *f is a weak-open, pseudo-open map.*
(3) *f is an sn-open, pseudo-open map.*
(4) *f is a 1-sequence-covering, pseudo-open map.*

Theorem 3.8. *Let $f : X \longrightarrow Y$ be a map. If X is first countable and Y is Fréchet, then the following are equivalent.*

(1) *f is an almost-open map.*
(2) *f is a weak-open map.*
(3) *f is an sn-open map.*
(4) *f is a 1-sequence-covering map.*

4. Answers to Question 1.3

Proposition 4.1. *The following are equivalent for a space X.*

(1) *X is a 1-sequence-covering, countable-to-one image of a metric space.*
(2) *X is a 1-sequence-covering, s-image of a metric space.*

Proof: (1) $\Longrightarrow$ (2) is obvious. We prove that (2) $\Longrightarrow$ (1).

Let $f : M \longrightarrow X$ be a 1-sequence-covering, s-map, where M is a metric space with a metric d. For each $x \in X$, since f is 1-sequence-covering, let $a_x \in f^{-1}(x)$ such that whenever $\{x_n\}$ is a sequence converging to x in X, there is a sequence $\{a_n\}$ converging to a_x in M with each $a_n \in f^{-1}(x_n)$, and let D_x denote a countable dense subset of $f^{-1}(x)$ because $f^{-1}(x)$ is separable. Put $D = \bigcup\{D_x \bigcup\{a_x\} : x \in X\}$, and $g = f|D : D \longrightarrow X$, that is, g is the restriction of f on D. It is clear that g is a countable-to-one map from the metric space D onto the space X. We only need to prove that g is 1-sequence-covering. Let $x \in X$, then $a_x \in D$. Whenever $\{x_n\}$ is a sequence converging to x in X, there is a sequence $\{a_n\}$ converging to a_x in M with each $a_n \in f^{-1}(x_n)$. For each $n \in \mathbb{N}$, since D_{x_n} is a dense subset of $f^{-1}(x_n)$ and $a_n \in f^{-1}(x_n)$, we can choose $b_n \in D_{x_n}$ such that $d(a_n, b_n) < 1/n$. Then $\{b_n\}$ is a sequence in D with each $b_n \in g^{-1}(x_n)$. It suffices to prove that $\{b_n\}$ converges to a_x in D. Whenever $\varepsilon > 0$, there is $k_1 \in \mathbb{N}$ such that $1/k_1 < \varepsilon/2$. Since $\{a_n\}$ converges to a_x in M, there is $k_2 \in \mathbb{N}$ such that $d(a_n, a_x) < \varepsilon/2$. Put $k = \max\{k_1, k_2\}$. If $n > k$, then $d(b_n, a_x) \leq d(b_n, a_n) + d(a_n, a_x) < 1/n + \varepsilon/2 < 1/k_1 + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This proves that $\{b_n\}$ converges to a_x in D. □

Remark 4.2. Recall a map $f : M \longrightarrow X$ is sequence-covering [8] if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$. This evokes an interesting question: Is a sequence-covering, s-image of a metric space a sequence-covering, countable-to-one image of a metric space? The answer for the question is negative. In fact, let X be the space in [10, Example 6], which is a closed image of a separable metric space, then X is an $\aleph_0$-space; hence, it is a sequence-covering image of a separable metric space. But suppose that X is a sequence-covering, countable-to-one image of a metric space. Since X is a sequential space, then X is a quotient countable-to-one image of a metric space. Hence, X is $\aleph_0$-weakly first-countable, and this is a contradiction.

Now we answer Question 1.3.

Theorem 4.3. *The following are equivalent for a space X.*

(1) *X is a weak-open, countable-to-one image of a metric space.*
(2) *X is a weak-open, s-image of a metric space.*

Proof: (1) $\Longrightarrow$ (2) is obvious. We prove that (2) $\Longrightarrow$ (1).

Let X be a weak-open, s-image of a metric space. Then X is a 1-sequence-covering, quotient, s-image of a metric space by Theorem 3.3. X is sequential because quotient maps preserve sequential spaces. By Proposition 4.1, X is a 1-sequence-covering, countable-to-one image of a metric space. By Theorem 3.4, X is a weak-open, countable-to-one image of a metric space. □

Theorem 4.4. *The following are equivalent for a space X.*

(1) *X is an almost-open, countable-to-one image of a metric space.*
(2) *X is an almost-open, s-image of a metric space.*

Proof: (1) $\Longrightarrow$ (2) is obvious. We prove that (2) $\Longrightarrow$ (1).

Let X be an almost-open, s-image of a metric space. Then X is a 1-sequence-covering, pseudo-open, s-image of a metric space by Theorem 3.7. X is Fréchet because pseudo-open maps preserve Fréchet spaces. By Proposition 4.1, X is a 1-sequence-covering, countable-to-one image of a metric space. By Theorem 3.8, X is an almost-open, countable-to-one image of a metric space. □

Acknowledgment. The author thanks Professor S. Lin and Professor Z. Yun for their valuable amendments and suggestions and the referee for his/her valuable report and correction of Remark 4.2.

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