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ON SOME NEW DIMENSION-LIKE FUNCTIONS

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Abstract. In this paper, we generalize the dimension-like functions defined by D. N. Georgiou, S. D. Iliadis, and A. Megaritis in *Dimension-like functions and universality* [to appear in Topology and its Applications], and give some basic properties of these new dimension-like functions including sum and product theorems and the property of universality.

1. Introduction

All spaces are considered to be T_0-spaces of weight $\leq \tau$, where τ is a fixed infinite cardinal. The least cardinal greater than τ is denoted by τ^+ and the first infinite cardinal is denoted by ω. We also denote by ν a fixed cardinal such that $\omega \leq \nu \leq \tau$. The cardinal of a set X is denoted by $|X|$. The class of all ordinals is denoted by $\mathcal{O}$. We also consider two symbols: -1 and ∞. It is assumed that $-1 < \alpha < \infty$ for every $\alpha \in \mathcal{O}$, $-1(+)\alpha = \alpha(+)(-1) = \alpha$ for every $\alpha \in \mathcal{O} \cup \{-1, \infty\}$, and $\infty(+)\alpha = \alpha(+)\infty = \infty$ for every $\alpha \in \mathcal{O} \cup \{\infty\}$, where by $(+)$ we denote the natural sum of Hessenberg (see [5]).

By a *class of subsets*, we mean a class $\mathbb{K}$ consisting of pairs (K, X), where K is a subset of a space X.

By a *class of bases*, we mean a class $\mathbb{B}$ consisting of pairs (B^X, X), where B^X is a base of cardinality less than or equal to τ containing

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the sets $\emptyset$ and X for the space X. A base B of a space X is said to be *an* $\mathbb{B}$-*base* if $(B, X) \in \mathbb{B}$.

In [6], two dimensions, dm and Dm, are introduced and studied. In [1], transfinite extensions of these dimensions, denoted by $trdm$ and $trDm$, are given. Below, we give the definition of these transfinite extensions (using the original notations dm and Dm).

Definition 1.1. (See, for example, [2].) Let A and B be two disjoint subsets of a space X. We say that a subset L of X *separates* A and B if there exist two open subsets U and W of X such that (a) $A \subseteq U$, $B \subseteq W$, (b) $U \cap W = \emptyset$, and (c) $X \setminus L = U \cup W$.

Definition 1.2. (See [6] and [1].) We denote by dm and Dm the *dimension-like functions* (or *dimensions*) with the class of all spaces as domain and the class $\mathcal{O} \cup \{-1, \infty\}$ as range satisfying the following conditions.

(1) $dm(X) = Dm(X) = -1$ if and only if $X = \emptyset$.

(2) $Dm(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if for any pair of distinct points x and y of X there exists a subset L of X which separates the singletons $\{x\}$ and $\{y\}$ such that $dm(L) < \alpha$.

(3) $dm(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if $X = \cup\{Q_i^X : i \in \omega\}$ such that the subset Q_i^X of X is closed and $Dm(Q_i^X) \leq \alpha$, $i \in \omega$.

Therefore, $dm(X) = \infty$ (respectively, $Dm(X) = \infty$) if and only if the inequality $dm(X) \leq \alpha$ (respectively, $Dm(X) \leq \alpha$) is not true for every $\alpha \in \mathcal{O}$. In particular, $Dm(X) = \infty$ if X is not Hausdorff.

We note that the dimension Dm does not have the universality property (at least in the class of the separable metrizable spaces). Indeed, it is easy to see that $Dm(X) = 0$ if and only if the space X is totally disconnected. However, in the class of all separable metrizable totally disconnected spaces there are no universal elements (see [7]).

In [3], we give modifications of dimensions dm and Dm. These new dimension-like functions are denoted by $dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}$ and $Dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}$, where $\mathbb{E}$ is a class of spaces, $\mathbb{K}$ is a class of subsets, and $\mathbb{B}$ is a class of bases. We also prove that the dimensions $dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}$ and $Dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}$ have the universality property. However, we do not study other standard properties of these dimensions.

Here, we introduce and study dimension-like functions $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, where $\nu \leq \tau$ is an infinite cardinal. These dimension-like functions generalize the dimension-like functions in [3]: for $\nu = \omega$, we have $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E}} = dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\omega}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E}} = Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\omega}$.

2. Relations between $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, and other dimensions

Definition 2.1. (See [3].) A class $\mathbb{E}$ of spaces is said to be $\mathbb{B}$-*hereditarily-separated*, where $\mathbb{B}$ is a class of bases, if for every element X of $\mathbb{E}$ there exists an $\mathbb{B}$-base $B^X = \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1} and U_{δ_2} of B^X with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$ there exists a subset L of X separating the sets $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ and belonging to $\mathbb{E}$.

We note that if $\mathbb{E}$ is $\mathbb{B}$-hereditarily-separated, then $\emptyset \in \mathbb{E}$. This follows by the fact that the empty set is the unique subset of X separating the elements $\emptyset$ and X of B^X.

In addition, the class $\mathbb{E}$ consisting of the empty space only is $\mathbb{B}$-hereditarily-separated for every class $\mathbb{B}$ of bases.

Definition 2.2. Let $\mathbb{B}$ be a class of bases, $\mathbb{E}$ an $\mathbb{B}$-hereditarily-separated class of spaces, and $\mathbb{K}$ a class of subsets with $(X, X) \in \mathbb{K}$ for every space X. We denote by $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ the *dimension-like functions* with the class of all spaces as domain and the class $\mathcal{O} \cup \{-1, \infty\}$ as range satisfying the following conditions:

(1) $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$ if and only if $X \in \mathbb{E}$.

(2) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists an $\mathbb{B}$-base $B^X = \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1}, U_{δ_2} of B^X with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$ there exists a subset L of X separating $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ with $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L) < \alpha$.

(3) $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if $X = \cup\{Q^X_i : i \in \nu\}$ such that (a) the subset Q^X_i of X is closed; (b) $(Q^X_i, X) \in \mathbb{K}$; and (c) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q^X_i) \leq \alpha$.

Therefore, $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$ (respectively, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$) if and only if for every $\alpha \in \mathcal{O}$ the inequality $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$ (respectively, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$) is not true. In particular, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \infty$ if there are no $\mathbb{B}$-bases for X.

Remark. (1) In order that the above definition be well defined, we need to show that if for a space X we have $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1$, then $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$ and $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq 0$. For dimension-like function $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ this follows immediately by the fact that $X \in \mathbb{E}$ and the class $\mathbb{E}$ is $\mathbb{B}$-hereditarily-separated. For dimension-like function $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, this follows immediately by the properties (a) $X = \cup\{Q^X_i : i \in \nu\}$ where $Q^X_i = X$; (b) $(X, X) \in \mathbb{K}$; and (c) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = -1 \leq 0$.

(2) For $\nu = \omega$, the dimension-like functions $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ coincide with the dimension-like functions $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E}}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E}}$, respectively, which are introduced in [3].

(3) In what follows, whenever we consider the dimension-like functions $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, it is supposed that $\mathbb{B}$ is a class of bases, $\mathbb{E}$ is an $\mathbb{B}$-hereditarily-separated class of spaces, and $\mathbb{K}$ is a class of subsets with $(X, X) \in \mathbb{K}$ for every space X. In the case where $\mathbb{E} = \{\emptyset\}$ instead of $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}$, we write $dm^{\mathbb{K},\mathbb{B}}_{\nu}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\nu}$, respectively.

Theorem 2.3. *For every space X we have*

$$dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X). \tag{1}$$

Proof: Let $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \alpha \in \{-1, \infty\} \cup \mathcal{O}$. The inequality (1) is clear if $\alpha = -1$ or $\alpha = \infty$. Suppose that $\alpha \in \mathcal{O}$. We have

$$X = \cup\{Q^X_i : i \in \nu\},$$

where $Q^X_i = X$. Since $(Q^X_i, X) = (X, X) \in \mathbb{K}$ and $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q^X_i) = Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$, then condition (3) of Definition 2.2 implies that $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \alpha$. □

Theorem 2.4. *For every space X,*

$$Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1, \infty\} \cup \tau^+$$

and, therefore,

$$dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \in \{-1,\infty\} \cup \tau^{+}.$$

Proof: Suppose that the theorem is not true. Let α be the minimal element of $\mathcal{O} \setminus \tau^{+}$ such that there exists a space X with $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) = \alpha$. Let $B^{X} = \{U_{\delta} : \delta \in \tau\}$ be the $\mathbb{B}$-base for X mentioned in condition (2) of Definition 2.2.

Denote by P the set of all pairs $(\delta_1, \delta_2) \in \tau \times \tau$ with

$$\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset.$$

For every $(\delta_1, \delta_2) \in P$, let $L(\delta_1, \delta_2)$ be a subset of X separating the sets $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ with

$$dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2) < \alpha.$$

First, we suppose that $\beta(\delta_1, \delta_2) < \tau^{+}$ for every $(\delta_1, \delta_2) \in P$. Since $|P| \leq \tau$, there exists an ordinal $\beta \in \tau^{+}$ such that $\beta(\delta_1, \delta_2) < \beta$ for every $(\delta_1,\, \delta_2) \in P$. Then, $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) < \beta$ and, by condition (2) of Definition 2.2, $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(X) \leq \beta$, which is a contradiction.

Now, we suppose that there exists $(\delta_1, \delta_2) \in P$ such that $\tau^{+} \leq \beta(\delta_1, \delta_2)$. Since $dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2)$, there exist closed subsets $Q_i^{L(\delta_1,\delta_2)}$ of $L(\delta_1, \delta_2)$, $i \in \nu$, such that

(*a*) $L(\delta_1, \delta_2) = \cup\{Q_i^{L(\delta_1,\delta_2)} : i \in \nu\}$;

(*b*) $(Q_i^{L(\delta_1,\delta_2)}, L(\delta_1, \delta_2)) \in \mathbb{K}$; and

(*c*) $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q_i^{L(\delta_1,\delta_2)}) = \beta_i \leq \beta(\delta_1, \delta_2) < \alpha$.

If $\beta_i < \tau^{+}$ for all $i \in \nu$, then there exists an ordinal $\beta \in \tau^{+}$ such that $\beta_i \leq \beta$, which means that $Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q_i^{L(\delta_1,\delta_2)}) \leq \beta$. Therefore,

$$dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(L(\delta_1, \delta_2)) \leq \beta < \tau^{+} \leq \beta(\delta_1, \delta_2),$$

which is a contradiction. Thus, there exists $i \in \nu$ such that

$$\tau^{+} \leq Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu}(Q_i^{L(\delta_1,\delta_2)}) < \alpha.$$

The last relation contradicts the choice of the ordinal α, completing the proof of the theorem. □

Theorem 2.5. *Let X be a regular space X. Then, $dm(X) \leq dm^{\mathbb{K},\mathbb{B}}_{\omega}(X)$ and $Dm(X) \leq Dm^{\mathbb{K},\mathbb{B}}_{\omega}(X)$.*

Proof: We prove the theorem by induction. If

$$dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = Dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = -1,$$

then $X = \emptyset$ and $dm(X) = Dm(X) = -1$ proving the theorem.

Suppose that the relation $Dm(X) \leq Dm_{\omega}^{\mathbb{K},\mathbb{B}}(X)$ is true for all spaces X with $Dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) < \alpha$ and the relation $dm(X) \leq dm_{\omega}^{\mathbb{K},\mathbb{B}}(X)$ is true for all spaces X with $dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) < \alpha$, where $\alpha \in \mathcal{O}$.

First, we suppose that X is a space with $Dm_{\nu}^{\mathbb{K},\mathbb{B}}(X) = \alpha$. We need to prove $Dm(X) \leq \alpha$. Consider two arbitrary distinct points x and y of X. There exists an $\mathbb{B}$-base $B^X = \{U_\delta : \delta \in \tau\}$ for X such that for every two elements U_{δ_1} and U_{δ_2} of B^X with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$ there exists a subset L of X separating the sets $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ with $dm_{\omega}^{\mathbb{K},\mathbb{B}}(L) < \alpha$.

Since X is regular, there exist U_{δ_1} and $U_{\delta_2} \in B^X$ such that $x \in \mathrm{Cl}(U_{\delta_1})$, $y \in \mathrm{Cl}(U_{\delta_2})$, and $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$. Thus, there exists a subset L of X separating the sets $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ with $dm_{\omega}^{\mathbb{K},\mathbb{B}}(L) = \beta < \alpha$. Obviously, L separates the singletons $\{x\}$ and $\{y\}$. Moreover, by inductive assumption

$$dm(L) \leq dm_{\omega}^{\mathbb{K},\mathbb{B}}(L) = \beta < \alpha.$$

So, by condition (2) of Definition 1.2, $Dm(X) \leq \alpha$.

Now, we suppose that $dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = \alpha$. We need to prove that $dm(X) \leq \alpha$. We have

$$X = \cup\{Q_i^X : i \in \omega\},$$

where (a) Q_i^X is closed subset of X; (b) $(Q_i^X, X) \in \mathbb{K}$; and (c) $Dm_{\omega}^{\mathbb{K},\mathbb{B}}(Q_i^X) \leq \alpha$. By the preceding, $Dm(Q_i^X) \leq \alpha$, $i \in \omega$. So, by condition (3) of Definition 1.2, $dm(X) \leq \alpha$.

The theorem is trivial if $dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = Dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = \infty$. □

Theorem 2.6. *Let X be a normal space with $Dm_{\nu}^{\mathbb{K},\mathbb{B}}(X) \neq \infty$. Then, we have*

$$Dm_{\nu}^{\mathbb{K},\mathbb{B}}(X) \leq Ind(X).$$

Proof: Let $Ind(X) = \alpha \in \{-1, \infty\} \cup \mathcal{O}$. We prove the theorem by induction on α. The theorem is clear if $\alpha = -1$ or $\alpha = \infty$. Let $\alpha \in \mathcal{O}$ and suppose that the theorem is true for all spaces X with $Ind(X) < \alpha$. We prove the theorem for a space X with $Ind(X) = \alpha$.

Let $B^X = \{U_\delta : \delta \in \tau\}$ be an arbitrary $\mathbb{B}$-base for X. The existence of such a base follows by the condition $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) \neq \infty$. Let $U_{\delta_1}, U_{\delta_2} \in B^X$ with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$. Since $Ind(X) = \alpha$, there exists a closed set L of X which separates $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ such that $Ind(L) < \alpha$. By inductive assumption,

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(L) \leq Ind(L) < \alpha.$$

By Theorem 2.3, we have

$$dm_\nu^{\mathbb{K},\mathbb{B}}(L) \leq Dm_\nu^{\mathbb{K},\mathbb{B}}(L).$$

This means that

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X) \leq \alpha.$$

□

Corollary 2.7. *Let X be a normal space with $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) \neq \infty$. Then, we have*

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X) \leq Ind(X).$$

Theorem 2.8. *Let X be a compact Hausdorff space with $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) \neq \infty$. Then, the condition $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) = 0$ is equivalent to $Ind(X) = 0$.*

Proof: By Theorem 2.6, it is sufficient to prove that if $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) = 0$, then $Ind(X) = 0$.

Let $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) = 0$ and let $B^X = \{U_\delta : \delta \in \tau\}$ be an $\mathbb{B}$-base for X such that for every two elements U_{δ_1} and U_{δ_2} of B^X with $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$, the empty set separates the sets $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$. It is easy to verify that this property implies that the empty set separates any two disjoint closed subsets of X, which means that $Ind(X) = 0$. □

Theorem 2.9. *Let X be a compact Hausdorff space with $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) \neq \infty$. Then, the condition $dm_\omega^{\mathbb{K},\mathbb{B}}(X) = 0$ is equivalent to $Ind(X) = 0$.*

Proof: If $Ind(X) = 0$, then by Corollary 2.7,

$$dm_\omega^{\mathbb{K},\mathbb{B}}(X) = 0.$$

Conversely, let $dm_\omega^{\mathbb{K},\mathbb{B}}(X) = 0$. Then,

$$X = \cup\{Q_i^X : i \in \omega\}$$

such that (a) the subset Q_i^X of X is closed; (b) $(Q_i^X, X) \in \mathbb{K}$; and (c) $Dm_{\omega}^{\mathbb{K},\mathbb{B}}(Q_i^X) \leq 0$. Since Q_i^X is compact and $Dm_{\mathbb{E}}^{\mathbb{K},\mathbb{B}}(Q_i^X) \leq 0$ by Theorem 2.8, $Ind(Q_i^X) \leq 0$. Therefore, $Ind(X) = 0$. (See, for example, [2].) □

Corollary 2.10. *Let X be a compact Hausdorff space such that $Dm_{\nu}^{\mathbb{K},\mathbb{B}}(X) \neq \infty$. Then, the following conditions are equivalent:*

(a) $Dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = 0$; (b) $dm_{\omega}^{\mathbb{K},\mathbb{B}}(X) = 0$; and (c) $Ind(X) = 0$.

3. The sum and product theorems and the property of universality

Theorem 3.1. *Let $\mathbb{K}$ be a class of subsets such that $(K, X) \in \mathbb{K}$ for every space X and for every closed subset K of X. If a space X is the union of closed subsets F_i, $i \in \nu$, such that $dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(F_i) \leq \alpha \in \mathcal{O}$, then*

$$dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(X) \leq \alpha.$$

Proof: Since $dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(F_i) \leq \alpha$, condition (3) of Definition 2.2 implies that $F_i = \cup\{Q_j^i : j \in \nu\}$ such that for every $j \in \nu$ we have (a) the subset Q_j^i of F_i is closed; (b) $(Q_j^i, F_i) \in \mathbb{K}$; and (c) $Dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(Q_j^i) \leq \alpha$. Since the subset Q_j^i of X is closed in X, we have $(Q_j^i, X) \in \mathbb{K}$. Also, $X = \cup\{Q_j^i : i, j \in \nu\}$. The above conditions mean that

$$dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(X) \leq \alpha.$$

□

Definition 3.2. It is said that

(a) a class $\mathbb{K}$ of subsets and

(b) a class $\mathbb{B}$ of bases

are *closed with respect to the products* if we have, respectively,

(a) $(Q^X \times Q^Y, X \times Y) \in \mathbb{K}$ for every (Q^X, X), $(Q^Y, Y) \in \mathbb{K}$, and

(b) $(B^{X\times Y}, X \times Y) \in \mathbb{B}$ for every (B^X, X), $(B^Y, Y) \in \mathbb{B}$, where $B^{X\times Y} = \{U^X \times U^Y : U^X \in B^X, U^Y \in B^Y\}$.

Theorem 3.3. *For any two spaces X and Y, we have*

$$Dm_{\nu}^{\mathbb{K},\mathbb{B}}(X \times Y) \leq Dm_{\nu}^{\mathbb{K},\mathbb{B}}(X)(+)Dm_{\nu}^{\mathbb{K},\mathbb{B}}(Y) \quad (2)$$

and

(3) $\quad dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) \le dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y),$

where $\mathbb{K}$ *and* $\mathbb{B}$ *are classes closed with respect to the products.*

Proof: We prove the theorem by induction. If

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = -1$$

or

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = -1,$$

then X and Y are empty and, therefore,

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) = -1 \text{ or } dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) = -1,$$

respectively.

Suppose that the inequality (2) is true for any spaces X and Y with

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) < \alpha,$$

and the inequality (3) is true for any spaces X and Y with

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y) < \alpha,$$

where α is a fixed ordinal.

First, we consider two spaces X and Y with

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = \alpha.$$

We need to prove that

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) \le \alpha.$$

If $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) = -1$ or $Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = -1$, then $X \times Y = \emptyset$ and, therefore,

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) = -1 < \alpha.$$

Let $Dm_\nu^{\mathbb{K},\mathbb{B}}(X) = \beta$ and $Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = \gamma$, where β, $\gamma \in \mathcal{O}$. Then, there exist $\mathbb{B}$-bases $B^X = \{U_\delta : \delta \in \tau\}$ and $B^Y = \{V_\delta : \delta \in \tau\}$ for X and Y, respectively, such that condition (2) of Definition 2.2 is satisfied.

Since the class $\mathbb{B}$ is closed with respect to the products, the set $B^{X\times Y} = \{U_\delta \times V_{\delta'} : \delta, \delta' \in \tau\}$ is an $\mathbb{B}$-base of $X \times Y$. Suppose that $U_{\delta_1} \times V_{\delta'_1}$, $U_{\delta_2} \times V_{\delta'_2} \in B^{X\times Y}$, and

$$\mathrm{Cl}(U_{\delta_1} \times V_{\delta'_1}) \cap \mathrm{Cl}(U_{\delta_2} \times V_{\delta'_2}) = \emptyset.$$

We have

$$\mathrm{Cl}(U_{\delta_1} \times V_{\delta'_1}) \cap \mathrm{Cl}(U_{\delta_2} \times V_{\delta'_2}) =$$
$$(\mathrm{Cl}(U_{\delta_1}) \times \mathrm{Cl}(V_{\delta'_1})) \cap (\mathrm{Cl}(U_{\delta_2}) \times \mathrm{Cl}(V_{\delta'_2})) =$$

$$(\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2})) \times (\mathrm{Cl}(V_{\delta'_1}) \cap \mathrm{Cl}(V_{\delta'_2})) = \emptyset.$$

If $\mathrm{Cl}(U_{\delta_1}) \cap \mathrm{Cl}(U_{\delta_2}) = \emptyset$, then there exists a subset L of X separating $\mathrm{Cl}(U_{\delta_1})$ and $\mathrm{Cl}(U_{\delta_2})$ with $dm_\nu^{\mathbb{K},\mathbb{B}}(L) < \beta$. Therefore, there exist two open subsets W_{δ_1} and H_{δ_2} of X such that (a) $Cl(U_{\delta_1}) \subseteq W_{\delta_1}$, $Cl(U_{\delta_2}) \subseteq H_{\delta_2}$; (b) $W_{\delta_1} \cap H_{\delta_2} = \emptyset$; and (c) $X \setminus L = W_{\delta_1} \cup H_{\delta_2}$.

Let $W = W_{\delta_1} \times Y$, $H = H_{\delta_2} \times Y$, and $P = L \times Y$. Then, we have

$\mathrm{Cl}(U_{\delta_1} \times V_{\delta'_1}) = \mathrm{Cl}(U_{\delta_1}) \times \mathrm{Cl}(V_{\delta'_1}) \subseteq W$,

$\mathrm{Cl}(U_{\delta_2} \times V_{\delta'_2}) = \mathrm{Cl}(U_{\delta_2}) \times \mathrm{Cl}(V_{\delta'_2}) \subseteq H$,

$W \cap H = \emptyset$, and $(X \times Y) \setminus P = W \cup H$,

which means that P separates $\mathrm{Cl}(U_{\delta_1} \times V_{\delta'_1})$ and $\mathrm{Cl}(U_{\delta_2} \times V_{\delta'_2})$ in the space $X \times Y$. By Theorem 2.3, $dm_\nu^{\mathbb{K},\mathbb{B}}(Y) \leq Dm_\nu^{\mathbb{K},\mathbb{B}}(Y)$. Therefore,

$$\begin{aligned} dm_\nu^{\mathbb{K},\mathbb{B}}(L)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y) &\leq dm_\nu^{\mathbb{K},\mathbb{B}}(L)(+)Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) \\ &< \beta(+)\gamma = \alpha. \end{aligned}$$

By inductive assumption,

$$dm_\nu^{\mathbb{K},\mathbb{B}}(P) = dm_\nu^{\mathbb{K},\mathbb{B}}(L \times Y) \leq dm_\nu^{\mathbb{K},\mathbb{B}}(L)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y) < \alpha.$$

Similar to the above, if $\mathrm{Cl}(V_{\delta'_1}) \cap \mathrm{Cl}(V_{\delta'_2}) = \emptyset$, then in the space $X \times Y$, we can construct a subset P' which separates the subsets $\mathrm{Cl}(U_{\delta_1} \times V_{\delta'_1})$ and $\mathrm{Cl}(U_{\delta_2} \times V_{\delta'_2})$ of $X \times Y$ such that

$$dm_\nu^{\mathbb{K},\mathbb{B}}(P') < \alpha.$$

Thus,

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) \leq \alpha. \tag{4}$$

Now, we consider two spaces X and Y with

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = \alpha.$$

We need to prove that

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) \leq \alpha.$$

If $dm_\nu^{\mathbb{K},\mathbb{B}}(X) = -1$ or $dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = -1$, then $X \times Y = \emptyset$ and, therefore,

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) = -1 < \alpha.$$

Suppose that $dm_\nu^{\mathbb{K},\mathbb{B}}(X) = \beta$ and $dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = \gamma$, where $\beta, \gamma \in \mathcal{O}$. Then, by condition (3) of Definition 2.2 we have

(a) $X = \cup\{Q_i^X : i \in \nu\}$, where Q_i^X is closed, $(Q_i^X, X) \in \mathbb{K}$, and $Dm_\nu^{\mathbb{K},\mathbb{B}}(Q_i^X) \leq \beta$ and

(b) $Y = \cup\{Q_i^Y : i \in \nu\}$, where Q_i^Y of Y is closed, $(Q_i^Y, Y) \in \mathbb{K}$, and $Dm_\nu^{\mathbb{K},\mathbb{B}}(Q_i^Y) \leq \gamma$.

We observe that

$$X \times Y = \cup\{Q_i^X \times Q_j^Y : i, j \in \nu\},$$

the subset $Q_i^X \times Q_j^Y$ of $X \times Y$ is closed, and

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(Q_i^X)(+)Dm_\nu^{\mathbb{K},\mathbb{B}}(Q_j^Y) \leq \beta(+)\gamma = \alpha,\ i,\ j \in \nu.$$

Since the class $\mathbb{K}$ is closed with respect to the products, we have

$$(Q_i^X \times Q_j^Y, X \times Y) \in \mathbb{K}.$$

Setting $X = Q_i^X$ and $Y = Q_j^Y$ into relation (4), we have

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(Q_i^X \times Q_j^Y) \leq \alpha,\ i,\ j \in \nu.$$

Thus, by condition (3) of Definition 2.2,

$$dm_\nu^{\mathbb{K},\mathbb{B}}(X \times Y) \leq \alpha.$$

Obviously, the theorem is true if

$$Dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)Dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = dm_\nu^{\mathbb{K},\mathbb{B}}(X)(+)dm_\nu^{\mathbb{K},\mathbb{B}}(Y) = \infty.$$

The proof of the theorem is complete. □

In the next theorem, we use the notions of saturated classes of spaces, of subsets, and of bases. These notions are introduced and studied in [4]. In particular, it is shown that any saturated class has a universal element.

Notation. For every ordinal α, we denote by $\mathbb{P}(dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}} \leq \alpha)$ and $\mathbb{P}(Dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}} \leq \alpha)$ the classes of all spaces X such that $dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(X) \leq \alpha$ and $Dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}(X) \leq \alpha$, respectively.

Theorem 3.4. *Let* $\mathbb{E}$, $\mathbb{K}$, *and* $\mathbb{B}$ *be saturated classes. Then, for every* $n \in \{-1\} \cup \omega$, *the classes* $\mathbb{P}(dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}} \leq n)$ *and* $\mathbb{P}(Dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}} \leq n)$ *are also saturated.*

The proof of this theorem is similar to the proof of Theorem 2.5 of [3].

Corollary 3.5. *For every $n \in \omega$ in the classes*

$$\mathbb{P}(dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq n) \text{ and } \mathbb{P}(Dm^{\mathbb{K},\mathbb{B}}_{\mathbb{E},\nu} \leq n)$$

there exist universal elements.

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References

[1] F. G. Arenas and M. L. Puertas, *A new approach to transfinite dimension*, Questions Answers Gen. Topology **17** (1999), no. 2, 227–232.

[2] Ryszard Engelking, *Dimension Theory*. Translated from the Polish and revised by the author. North-Holland Mathematical Library, 19. Amsterdam-Oxford-New York: North-Holland Publishing Co.; Warsaw: PWN—Polish Scientific Publishers, 1978.

[3] D. N. Georgiou, S. D. Iliadis, and A. Megaritis, *Dimension-like functions and universality*. To appear in Topology and its Applications.

[4] S. D. Iliadis, *Universal Spaces and Mappings*. North-Holland Mathematics Studies, 198. Amsterdam: Elsevier Science B.V., 2005.

[5] Kazimierz Kuratowski and Andrzej Mostowski, *Set Theory*. Translated from the 1966 Polish original. Second, completely revised edition. Studies in Logic and the Foundations of Mathematics, Vol. 86. Amsterdam-New York-Oxford: North-Holland Publishing Co.; Warsaw: PWN—Polish Scientific Publishers, 1976.

[6] A. K. O'Connor, *A new approach to dimension*, Acta Math. Hungar. **55** (1990), no. 1-2, 83–95.

[7] R. Pol, *There is no universal totally disconnected space*, Fund. Math. **79** (1973), no. 3, 265–267.

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