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AN UNCOUNTABLE COLLECTION OF MUTUALLY INCOMPARABLE PLANAR FANS

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Abstract. In this paper, we show an uncountable collection of mutually incomparable fans in the plane.

1. Introduction

A continuum means a nonempty compact connected metric space and a map means a continuous function. An arc means a space homeomorphic to the closed unit interval $[0,1]$. A continuum X is arcwise connected provided each two points of X are contained in some arc contained in X. A continuum X is unicoherent if $A \cap B$ is connected for each subcontinua A and B such that $A \cup B = X$. X is hereditarily unicoherent if every subcontinuum of X is unicoherent. A dendroid is an arcwise connected hereditarily unicoherent continuum. A fan is a dendroid with only one ramification point. We say that two continua are comparable by continuous maps if one of those continua can be mapped onto the other. Otherwise, the continua are incomparable. In 1961, B. Knaster [2] asked for an uncountable family of continuously incomparable dendroids. Recently, this question was answered independently in [3] by Piotr Minc and in [4]. These dendroids are fans, but it is not clear if they are planable. Minc presented his example during the Spring Topology Conference held in Greensboro, North Carolina, in March 2006

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and Andrew Lelek asked if there exists such an example with planar dendroids. The same question was posed again in a session of the Hyperspace Seminar, conducted by Professor Alejandro Illanes at the University of Mexico (UNAM). In this paper, we answer the question by constructing an uncountable family of incomparable fans in the plane. Our family of fans is a modification of the collection of Elsa continua constructed by Marwan M. Awartani in [1]. We construct a family $\{F_\alpha : \alpha \in \mathcal{C}\}$, $\mathcal{C} \subset 2^N$, $\mathcal{C}$ uncountable and every F_α is a fan such that if $\alpha \neq \beta$ and $a, \beta \in \mathcal{C}$ then F_α and F_β are not comparable.

In the second section of this paper, we recall the examples given by Awartani and introduce some notations that will be used in the construction of the new family of fans.

In the third section, we construct our examples and we obtain the main result, Theorem 3.8, using a modification of the lemmas given in [1].

2. Awartani's examples

Definition 2.1. Let 2^N denote the set of all sequences of zeros and ones, and let $\{a_i\}$, $\{b_i\}$ be two elements in 2^N. Then

(1) $\{a_i\}$ is said to vertically dominate $\{b_i\}$ if $a_i \geq b_i$ eventually.
(2) $\{a_i\}$ is said to dominate $\{b_i\}$ if there exists an integer j_0 such that $a_i \geq b_{i+j_0}$ eventually.
(3) $\{a_i\}$ and $\{b_i\}$ are called incomparable if neither of them dominates the other.

The proofs of the following results are in [1].

Lemma 2.2. *Let $\{a_i\}$, $\{b_i\}$ be two elements in 2^N. If $\{a_{i_j}\}_{j=1}^{\infty}$ is a tail of the sequence of zeros in $\{a_i\}$, and if for some integer j_0, $b_{i_j+j_0}$ is eventually zero, then $\{a_i\}$ dominates $\{b_i\}$.*

Lemma 2.3. *2^N contains continuum many elements no one of which vertically dominates any other.*

Lemma 2.4. *2^N contains continuum many elements none of which is eventually constant and no one of which dominates any other.*

Let $\mathcal{C} \subset 2^N$, $\mathcal{C}$ uncountable, such that if $\alpha \neq \beta$ and $a, \beta \in \mathcal{C}$, no one dominates the other.

With each $\alpha = \{\alpha_i\} \in 2^N$, we associate an Awartani example (as in Figure 1) E_α as a compactification of the ray J_α with an arc as remainder.

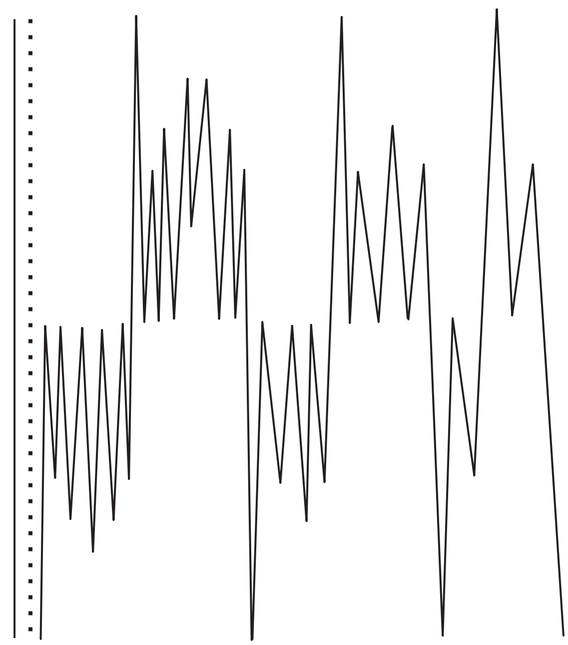

FIGURE 1. E_α with $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1$, ... Notice that each 1 in a sequence shifts one local minimum to the $\frac{2}{3}$ level.

Now, we will describe J_α, as the graph of a piece wise linear function from $(0,1]$ to $[0,1]$, with $(1,0) = a_1$ as an end point. We consider every point in J_α by the first coordinate. Moreover if a and b are two points in J_α, then $[a,b]$ denotes the subarc of J_α from a to b.

Let m_α and M_α denote the set of minima and the set of maxima of J_α and let $V_\alpha = m_\alpha \cup M_\alpha$.

Let $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be two sequences in $(0,1]$ such that $\lim x_i = 0 = \lim y_i$ and for every i, $y_{i+1} < x_{i+1} < y_i < x_i$.

We will obtain J_α as the union of subarcs $[x_i, y_i]$ and $[y_i, x_{i+1}]$, in such a way that

1. if $\alpha_i = 0$, then $[x_i,y_i] \cap V_\alpha$ has the following properties:
   a) $|[x_i, y_i] \cap M_\alpha| = 2i$ and $|[x_i, y_i] \cap m_\alpha| = 2i - 1$.

b) If $\{M_{i,j} : 1 \leq j \leq 2i\}$ is an enumeration from right to left of the elements of the set $[x_i, y_i] \cap M_\alpha$, then $\pi_2(M_{i,2i}) = 1$ and $\pi_2(M_{i,j}) = \pi_2(M_{i,2i-j}) = \frac{j+2}{j+3}$, $1 \leq j \leq i$.

c) $\pi_2([x_i, y_i] \cap m_\alpha) = \frac{1}{2}$;

2. if $\alpha_i = 1$, then $[x_i, y_i] \cap V_\alpha$ has the following properties:

a) $|[x_i, y_i] \cap M_\alpha| = 2i + 1$ and $|[x_i, y_i] \cap m_\alpha| = 2i$.

b) If $\{M_{i,j} : 1 \leq j \leq 2i + 1\}$ is an enumeration from right to left of the elements of the set $[x_i, y_i] \cap M_\alpha$, then $\pi_2(M_{i,2i+1}) = 1$ and $\pi_2(M_{i,j}) = \pi_2(M_{i,2i+1-j}) = \frac{j+2}{j+3}$, $1 \leq j \leq i$.

c) If $\{m_{i,j} : 1 \leq j \leq 2i\}$ is an enumeration from right to left of the elements of the set $[x_i, y_i] \cap m_\alpha$, then $\pi_2(m_{i,j}) = \frac{2}{3}$ for $i = j$ and $\pi_2(m_{i,j}) = \frac{1}{2}$ for $i \neq j$;

3. for each $i \in \mathbb{N}$, $[y_i, x_{i+1}] \cap V_\alpha$ has the following properties:

a) $|[x_i, y_i] \cap M_\alpha| = 2i - 1$ and $|[x_i, y_i] \cap m_\alpha| = 2i$.

b) $\pi_2([x_i, y_i] \cap M_\alpha) = \frac{1}{2}$.

c) If $\{m_{i,j} : 1 \leq j \leq 2i\}$ is an enumeration from right to left of the elements of the set $[x_i, y_i] \cap m_\alpha$, then $\pi_2(m_{i,2i}) = 0$ and $\pi_2(m_{i,j}) = \pi_2(m_{i,2i-j}) = 1 - \frac{j+2}{j+3}$, $1 \leq j \leq i$.

Let $b_i = \begin{cases} M_{i,2i} & \text{if } \alpha_i = 0 \\ M_{i,2i+1} & \text{if } \alpha_i = 1 \end{cases}$ and $a_i = m_{i,2i}$, i.e., b_i and a_i are the points in J_α such that its second coordinate is one and zero, respectively.

$\{E_\alpha : \alpha \in \mathcal{C}\}$, is an uncountable family and every E_α is an Elsa continuum such that if $\alpha \neq \beta$ and $a, \beta \in \mathcal{C}$, then E_α and E_β are not comparable.

3. Construction

Let $\alpha \in 2^N$ and E_α be the Awartani example associated with α.

Though the fan we are interested in is F_α, we will construct a fan F'_α as an intermediate step with its only purpose being that it is easily visualizable.

We will construct a fan F_α as a union of arcs J_i^α. We start with the construction of a fan $F_\alpha^{'}$, as in the Figure 2, then we identify the harmonic fan drawn in the inferior half of this figure with a point a_α to obtain the fan F_α.

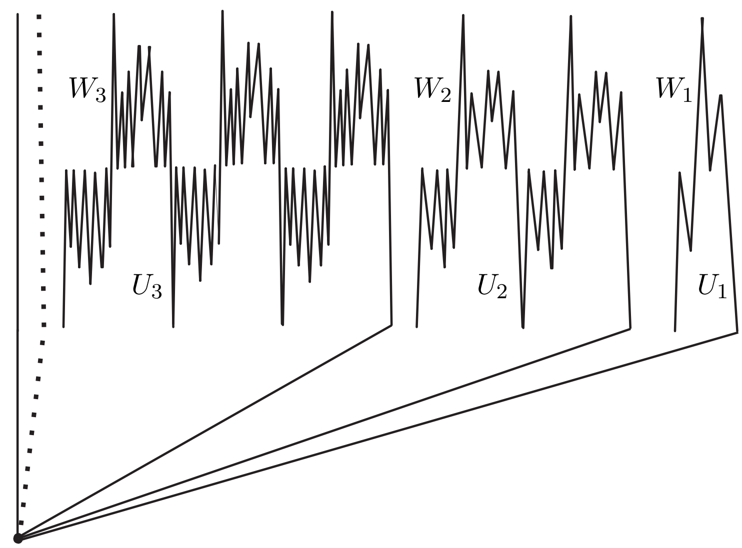

FIGURE 2. F'_α with $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = 1$, ...

Now we will describe F'_α. Let us denote the points in the plane: $(0,1) = c_\alpha$, $\left(0, \frac{1}{2}\right) = b_\alpha$, $(0,0) = a_\alpha$, and $(0,-1) = p_\alpha$. We define the arc E_i as the segment between p_α and the point in the plane $\left(\frac{1}{2^{i-1}}, 0\right)$.

If $A'_i = [a_i, a_{i+1}]$ is the arc between a_i and a_{i+1} contained in J_α, let I_1^α be a copy of A'_1 contained in the square $\left[\frac{3}{4}, 1\right] \times [0,1]$. (It means a contraction in the direction of the first coordinate of the graph J_α in $[a_2, a_1] \times [0,1]$ to $\left[\frac{3}{4}, 1\right] \times [0,1]$.) Then let I_2^α be the union of two copies of A'_2, contained in the square $\left[\frac{3}{2^3}, \frac{1}{2}\right] \times [0,1]$, in such a way that the first copy is denoted by A_2 and the second copy is denoted by $A_{1,2}$. In general, we define I_i^α as a union of i copies of A'_i contained in the square $\left[\frac{3}{2^{i+1}}, \frac{1}{2^{i-1}}\right] \times [0,1]$, in such a way that the first copy is denoted by A_i and the other copies are denoted by $A_{k,i}$ with $k \in \{1, 2, \cdots, i-1\}$. The arc J_i^α is the union of the arcs E_i and I_i^α, as in the Figure 2. So J_i^α is the arc with an end point in A_i and the other end point is p_α.

The sequence of subarcs defined in $\left[\frac{3}{2^{i+1}}, \frac{1}{2^{i-1}}\right] \times [0,1]$ converges to the arc between a_α and c_α, and the subarcs E_i converge to the arc between a_α and p_α. Let I_α be the arc between a_α and c_α, and I'_α be the arc between p_α and a_α.

We define the fan $F'_\alpha = I_\alpha \cup I'_\alpha \cup \bigcup_{i=1}^{\infty} J_i^\alpha$. F'_α contains the harmonic fan $F''_\alpha = I'_\alpha \cup \bigcup_{i=1}^{\infty} E_i$. The fan that we need is the quotient space $F_\alpha = F'_\alpha / F''_\alpha$, which is, finally, an identification of $I'_\alpha \cup \bigcup_{i=1}^{\infty} E_i$ with the point a_α, and we obtain this fan F_α with a_α as its vertex. We will say that F_α is the union $I_\alpha \cup \bigcup_{i=1}^{\infty} I_i^\alpha$ and $I_\alpha \cap I_i^\alpha = \{a_\alpha\}$.

We will denote some important arcs in every I_i^α.

For $i \in \mathbb{N}$, let K_i^α and L_i^α denote the subarcs of A_i, $[a_i, b_i]$, and $[b_i, a_{i+1}]$, respectively, in I_i^α. (We are using the notation of A'_α as the arc between a_i to a_{i+1} to denote these subarcs of A_i.) Let u_i^α denote the arc in I_i^α joining a_i with the first element of $K_i^\alpha \cap M_\alpha$. Similarly, let w_i^α denote the arc in I_i^α joining b_i with the first element of $L_i^\alpha \cap m_\alpha$.

In the same way, we could define the subarcs $K_{j,i}^\alpha$, $L_{j,i}^\alpha$, $u_{j,i}^\alpha$, and $w_{j,i}^\alpha$ in $A_{j,i}$.

The following lemma is easy to verify and gives important information about F_α.

Lemma 3.1. *For F_α, the following hold.*

(1) *If $t \in (a_\alpha, c_\alpha)$ and $t' \in I_\alpha$, then there exist sequences $\{t_i\}$ and $\{t'_i\}$, $t_i, t'_i \in I_i^\alpha$, converging to t and t', respectively, such that $\{d\,[t_i, t'_i]\}$ is bounded away from infinity where d denotes the length of the arc.*
(2) *Let $\{t_i\}$ and $\{t'_i\}$ be two sequences, t_i, $t'_i \in I_i^\alpha$, converging to a_α and c_α, respectively, then $\lim \{d\,[t_i, t'_i]\} = \infty$.*
(3) *For each $i \in \mathbb{N}$, let $p_i \in L_i^\alpha$; then $\lim \{d\,[p_i, p_{i+1}]\} = \infty$.*
(4) *Let $\{p_i\}$ and $\{q_i\}$ be two sequences, t_i, $t'_i \in I_i^\alpha$, both converging to a_α. If $\{d\,[p_i, q_i]\}$ is bounded away from zero, then $[a_\alpha, b_\alpha] \subset \lim\,[p_i, q_i]$.*

(5) *Let* $\{A_i\}$ *be a sequence of arcs* $A_i \subset I_i^\alpha$ *such that* $A_i \cap w_i^\alpha \neq \emptyset$ *eventually. If* $\lim A_i \supset [a_\alpha, b_\alpha]$*, then* $\lim (d(A_i)) = \infty$.

(6) *Let* $p \in u_{k_i}^\alpha$ *and* $q \in w_{k_i}^\alpha$. *Then* $[p,q] \cap m_\alpha \cap \left(y = \frac{2}{3}\right) = \emptyset$ *(which means that the set of minima in the arc* $[p,q]$ *does not include* $\frac{2}{3}$*) iff* $\alpha_{k_i} = 0$.

Now, we will prove two lemmas which we will use. The proof is similar to the one appearing in the Awartani's paper.

Lemma 3.2. *Let* $h : F_\alpha \to F_\beta$ *be a surjective map and let* A *be an arc in* I_i^α. *If* B *is an arc in* $h(A)$*, then there exists an arc in* A *whose image is* B.

Proof: The result follows directly since the map $h|_A : A \to h(A)$ is simply a map between closed intervals. □

Lemma 3.3. *Let* $h : F_\alpha \to F_\beta$ *be a surjective map and let* $\{A_i\}$ *be a sequence of arcs in* $\bigcup_{i=1}^{\infty} I_i^\alpha$ *such that* $\limsup A_i \subseteq I_\alpha$. *The following hold.*

(1) *If* $\{d(h(A_i))\}$ *is bounded away from zero, then* $\{d(A_i)\}$ *is bounded away from zero.*

(2) *If* $\{d(A_i)\}$ *is bounded away from infinity, then* $\{d(h(A_i))\}$ *is bounded away from infinity.*

Proof: (1) We assume without loss of generality that $\{A_i\}$ is a convergent sequence. Since $\{d(h(A_i))\}$ is bounded away from zero, each $h(A_i)$ contains a pair of points p_i, q_i such that $\lim p_i \neq \lim q_i$. Hence, A_i contains a pair of points p_i', q_i' such that $\lim p_i' \neq \lim q_i'$, which implies that $\{d(A_i)\}$ is bounded away from zero.

(2) Suppose that $\lim \{d(h(A_i))\} = \infty$. Then for each $i \in \mathbb{N}$, $h(A_i)$ contains a collection $\{B_{i,j} : 1 \leq j \leq i\}$ of disjoint closed subarcs such that $\{d(B_{i,j}) : (i,j) \in \mathbb{N} \times \{1, 2, \ldots, i\}\}$ is bounded away from zero. For each $i \in \mathbb{N}$ and each j, $1 \leq j \leq i$, let $A_{i,j}$ be a subarc of A_i such that $h(A_{i,j}) = B_{i,j}$. This is possible by Lemma 3.2. It follows from (1) that $\{d(A_{i,j}) : (i,j) \in \mathbb{N} \times \{1, 2, \ldots, i\}\}$ is bounded away from zero, implying that $\lim d(A_i) = \infty$. □

Lemma 3.4. *Let* $h : F_\alpha \to F_\beta$ *be a surjective map. Then the following holds.*

(1) $h^{-1}\{a_\beta, c_\beta\} = \{a_\alpha, c_\alpha\}$.

(2) *If* $h(a_\alpha) = a_\beta$, *then* $h[a_\alpha, b_\alpha] = [a_\beta, b_\beta]$ *and* $h[b_\alpha, c_\alpha] = [b_\beta, c_\beta]$.
(3) *If* $h(a_\alpha) = c_\beta$, *then* $h[a_\alpha, b_\alpha] = [c_\beta, b_\beta]$ *and* $h[b_\alpha, c_\alpha] = [b_\beta, a_\beta]$.
(4) $h(b_\alpha) = b_\beta$.

Proof: (1) It suffices to prove that $(a_\alpha, c_\alpha) \cap h^{-1}(c_\beta) = \emptyset = (a_\alpha, c_\alpha) \cap h^{-1}(a_\beta)$. Suppose that $(a_\alpha, c_\alpha) \cap h^{-1}(a_\beta) \neq \emptyset$ and let $t' \in [a_\alpha, c_\alpha]$, $t \in (a_\alpha, c_\alpha)$, be chosen so that $h(t') = c_\beta$ and $h(t) = a_\beta$. By Lemma 3.1(1), there are sequences $\{t_i\}$ and $\{t'_i\}$ in $\bigcup_{i=1}^{\infty} I_i^\alpha$, converging to t and t', respectively, such that $\{d[t_i, t'_i]\}$ is bounded away from infinity. Since $\lim h(t_i) = a_\beta$ and $\lim h(t'_i) = c_\beta$, it follows from Lemma 3.1(2) that $\lim d[h(t_i), h(t'_i)] = \infty$. This contradicts Lemma 3.3(1). Similarly, it can be shown that $(a_\alpha, c_\alpha) \cap h^{-1}(c_\beta) = \emptyset$.

(2) We only prove that $h[b_\alpha, c_\alpha] = [b_\beta, c_\beta]$ since the proof that $h[a_\alpha, b_\alpha] = [a_\beta, b_\beta]$ is similar and thus omitted. Since $h(a_\alpha) = a_\beta$, (1) implies that $h^{-1}(c_\beta) = c_\alpha$. Suppose that there exists $t \in (b_\alpha, c_\alpha)$ such that $h(t) < b_\beta$. Let T denote the sequence $\bigcup_{i=1}^{\infty} I_i^\alpha \cap (y = t)$ (it means its second coordinate is t). There exist two subsequences $\{p_i\}$ and $\{q_i\}$ of T such that for each $i \in \mathbb{N}$, p_i and q_i are adjacent in T and $[p_i, q_i] \cap M_\alpha = (y = 1)$, then, the second coordinate of each one of $h(p_i)$ and $h(q_i)$ is eventually in (a_β, b_β). If for an infinity of numbers i, $h[p_i, q_i] \cap (y = k) = \emptyset$, $k > \frac{1}{2}$, we obtain the following. Since $c_\alpha \in \lim [p_i, q_i]$, $h(c_\alpha) \in \lim [h(p_i), h(q_i)]$ and we obtain that $h(c_\alpha) \in [a_\beta, b_\beta]$, contradicting our assumption that $h^{-1}(c_\beta) = c_\alpha$. If $h[p_i, q_i] \cap (y = k) \neq \emptyset$, $k > \frac{1}{2}$, we obtain the following. $h(p_i) \in L_{k_i}^\beta$ and $h(q_i) \in L_{r_i}^\beta$ with $r_i \neq k_i$. It follows from Lemma 3.1(3) that $\lim d[h(p_i), h(q_i)] = \infty$ and consequently, $\lim (h[p_i, q_i]) = \infty$. This contradicts Lemma 3.3(2) since $\{d[p_i, q_i]\}$ is bounded away from infinity.

(3) This proof is similar to the proof of (2) and thus omitted.

(4) It follows directly from (2) and (3). □

Lemma 3.5. *Let* $h : F_\alpha \to F_\beta$ *be a surjective map. If neither* α *nor* β *is eventually constant, then the following hold.*

(1) *$h(a_\alpha) = a_\beta$; $h(c_\alpha) = c_\beta$; $h[a_\alpha, b_\alpha] = [a_\beta, b_\beta]$ and $h[b_\alpha, c_\alpha] = [b_\beta, c_\beta]$.*

(2) *If $\{A_i\}$ is a sequence of arcs in $\bigcup_{i=1}^{\infty} I_i^\alpha$ such that $\lim A_i = [b_\alpha, c_\alpha]$ and $A_i \cap m_\alpha \cap \left(y = \frac{2}{3}\right) = \emptyset$, $\forall i \in \mathbb{N}$, then $h(A_i) \cap m_\beta \cap \left(y = \frac{2}{3}\right) = \emptyset$ eventually.*

Proof: (1) By the previous lemma, it suffices to prove that $h(a_\alpha) = a_\beta$. Suppose that $h(a_\alpha) = c_\beta$. It follows from the construction that there exist sequences $\{p_i\}$ and $\{q_i\}$ in $\bigcup_{i=1}^{\infty} I_i^\beta$ such that $\lim p_i = \lim q_i = c_\beta$, $\lim [p_i, q_i] = \left[\frac{2}{3}, c_\beta\right]$, and $\{d[p_i, q_i]\}$ is bounded away from zero. By Lemma 3.2, we may find a sequence $\{B_i\}$ of arcs in $\bigcup_{i=1}^{\infty} I_i^\alpha$ such that $h(B_i) = [p_i, q_i]$. Since $\{d[p_i, q_i]\}$ is bounded away from zero, it follows from Lemma 3.3(1) that $\{d(B_i)\}$ is bounded away from zero. Choose p_i' and q_i' in B_i such that $h(p_i') = p_i$ and $h(q_i') = q_i$. Since $h^{-1}(c_\beta) = a_\alpha$, it follows that $\lim p_i' = \lim q_i' = a_\alpha$. We assume without loss of generality that $\{B_i\}$ is a convergent sequence. Hence, by Lemma 3.1(4), $[a_\alpha, b_\alpha] \subset \lim B_i$. Since $\lim h(B_i) = \left[\frac{2}{3}, c_\beta\right]$, it follows that $h[a_\alpha, b_\alpha] \supset \left[\frac{2}{3}, c_\beta\right]$, contradicting Lemma 3.4(4). Hence, $h(c_\alpha) = c_\beta$ and $h(a_\alpha) = a_\beta$.

(2) The proof is similar and thus omitted. □

Lemma 3.6. *Let $h : F_\alpha \to F_\beta$ be a surjective map, and let $\{p_i\}$ and $\{q_i\}$ be sequences of points in F_α such that for each $i \in \mathbb{N}$, $p_i \in u_i^\alpha$ and $q_i \in w_i^\alpha$. If $\lim p_i = \lim q_i = b_\alpha$ and neither α nor β is eventually constant, then there exists an integer j_0 such that eventually $h(p_i) \in u_{i+j_0}^\beta$ and $h(q_i) \in w_{i+j_0}^\beta$.*

Proof: Since $h(b_\alpha) = b_\beta$, it follows that $\lim h(p_i) = \lim h(q_i) = b_\beta$.

First, we will show that $h(p_i) \in \bigcup_{j=1}^{\infty} u_j^\beta$ eventually. Suppose not, then we have the following cases.

Case 1. $\{d[h(p_i, M_{j_i})]\}$ converges to zero for some subsequences $\{M_{j_i}\}_{i=1}^{\infty}$ of M_β. Then it can be shown using the uniform continuity of h that there exists $\delta > 0$, such that $h[b_\alpha, b_\alpha + \delta] \subseteq [a_\beta, b_\beta]$ contradicting Lemma 3.5(1).

Case 2. $\{d\,[h\,(p_i, m_{j_i})]\}$ converges to zero for some subsequences $\{m_{j_i}\}_{i=1}^{\infty}$ of m_β. Then, using a reasoning similar to the above, it can be shown that there exists $\delta > 0$, such that $h\,[b_\alpha, b_\alpha - \delta] \subseteq [b_\beta, c_\beta]$ contradicting Lemma 3.5(1).

Case 3. $h\,(p_i) \in \bigcup_{j=1}^{\infty} w_j^\beta$ (with w_j^β in A_i or $A_{r,i}$) eventually. Let $m_i^* = u_i^\alpha \cap m_\alpha$. Since $\lim\,[m_i^*, p_i] = [a_\alpha, b_\alpha]$, Lemma 3.5(1) implies that $\lim h\,[m_i^*, p_i] = [a_\beta, b_\beta]$. Since $h\,[m_i^*, p_i] \cap \left(\bigcup_{j=1}^{\infty} w_j^\beta\right) \neq \emptyset$ eventually, it follows from Lemma 3.1(5) that $\lim d\,(h\,[m_i^*, p_i]) = \infty$. This contradicts Lemma 3.2(2) since $\{d\,[m_i^*, p_i]\}$ is bounded away from infinity. Hence, the only remaining possibility is for $h\,(p_i)$ to be eventually in $\bigcup_{j=1}^{\infty} u_j^\beta$.

The proof that $h\,(q_i) \in \bigcup_{j=1}^{\infty} w_j^\beta$ is similar and thus omitted.

Now, we will show that if $h\,(p_i) \in u_{j_i}^\beta$ eventually, then $h\,(q_i) \in w_{j_i}^\beta$ eventually. Suppose not, then $h\,(q_i) \in w_{k_i}^\beta$ where $k_i \neq j_i$ infinitely often. Then it can be deduced that $h\,[a_\alpha, b_\alpha] \cap (b_\beta, c_\beta) \neq \emptyset$ contradicting Lemma 3.5(1).

We finally prove that if $h\,(p_i) \in u_{j_i}^\beta$ eventually, then $j_{i+1} = j_i + 1$. Suppose that $j_{i+1} \neq j_i + 1$ infinitely often; then it can be deduced that $h\,[b_\alpha, c_\alpha] \cap (a_\beta, b_\beta) \neq \emptyset$, contradicting Lemma 3.5(1).

Then there exists an integer j_0 such that $h\,(p_i) \in u_{i+j_0}^\beta$ and $h\,(q_i) \in w_{i+j_0}^\beta$ eventually. This completes the proof. □

Theorem 3.7. *Let $h : F_\alpha \to F_\beta$ be a surjective map. If neither α nor β is eventually constant, then α dominates β.*

Proof: For each $i \in \mathbb{N}$, we choose $p_i \in u_i^\alpha$ and $q_i \in w_i^\alpha$ such that $\lim p_i = \lim q_i = b_\alpha$. By Lemma 3.6, there exists an integer j_0 such that eventually $h\,(p_i) \in u_{i+j_0}^\beta$ and $h\,(q_i) \in w_{i+j_0}^\beta$. Let $\{\alpha_{k_i}\}$ be a tail of the sequence of all zeros in α. By Lemma 2.2, it suffices to prove that $\beta_{k_i+j_0} = 0$ eventually. Since $\alpha_{k_i} = 0$, it follows from Lemma 3.1(6) that $[p_{k_i}, q_{k_i}] \cap m_\alpha \cap \left(y = \frac{2}{3}\right) = \emptyset$. By Lemma 3.5(2), $h\,[p_{k_i}, q_{k_i}] \cap m_\beta \cap \left(y = \frac{2}{3}\right) = \emptyset$ eventually and hence, $[h\,(p_{k_i})\,, h\,(q_{k_i})] \cap m_\beta \cap \left(y = \frac{2}{3}\right) = \emptyset$ eventually. Since $h\,(p_{k_i}) \in u_{k_i+j_0}^\beta$ and $h\,(q_{k_i}) \in w_{k_i+j_0}^\beta$, we conclude by Lemma 3.1(6) that $\beta_{k_i+j_0} = 0$ eventually. □

Using Lemma 2.4 and Theorem 3.7, we obtain the main result.

Theorem 3.8. *There exists an uncountable collection of planar fans, no member of which maps onto any other.*

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