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COUNTEREXAMPLES ON THE IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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Abstract. In this paper, we give negative answers to several questions on the images of locally separable, metric spaces.

1. Introduction

All spaces considered here are assumed to be regular T_1.

First, we recall some definitions.

The symbol $\mathbb{N}$ is the set of positive integers.

A continuous map is called an *s-map* if each fiber of the map is separable.

A continuous map $f : X \to Y$ is called *compact-covering* if every compact $K \subset Y$ is the image of a compact $C \subset X$.

Let $\mathcal{P}$ be a family of subsets of a space X. Then $\mathcal{P}$ is called a *cs-network* [5] if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in X$ and any neighborhood U of x, there exist $P \in \mathcal{P}$ and $m \in \mathbb{N}$ such that $\{x, x_n : n \geq m\} \subset P \subset U$. $\mathcal{P}$ is called a *cs*-network* [3] if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in X$ and any neighborhood U of x, there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x, x_{n_j} : j \in \mathbb{N}\} \subset P \subset U$. $\mathcal{P}$ is called a *k-network* [15] if for any compact set $K \subset X$ and an open set U with $K \subset U$, there exists a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that

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$K \subset \bigcup \mathcal{P}' \subset U$.

A space is called an $\aleph_0$-space [14] if it has a countable k-network.

In this paper, we give negative answers to the following questions.

Question 1.1. [9, Conjecture 5.1.3] Let X be a quotient s-image of a metric space. Is X a quotient s-image of a locally separable, metric space if each first countable subset of X is locally separable?

Question 1.2. [11, Question 3.4] Let X be a regular Fréchet space with a point-countable k-network. Is X a closed image of a locally separable, metric space if each first countable closed subspace of X is locally separable?

Question 1.3. [8, Question 6] Let X be a regular Fréchet space with a point-countable k-network. Is X a space with a point-countable k-network consisting of separable subsets of X if each first countable closed subspace of X is locally separable?

Question 1.4. [13, Question 1.2] Is it true that a space X is a closed s-image of a locally separable, metric space if and only if X is a regular Fréchet space with a point-countable cs^*-network, and each first countable closed subspace of X is locally separable?

Question 1.5. [12, Question 2.6(2)] Is a hereditarily separable, sequential space with a point-countable cs^*-network an $\aleph_0$-space?

Question 1.6. [10, Question 3.6] Let X be a quotient s-image of a metric space. Is X a quotient s-image of a locally compact, metric space if each first countable closed subset of X is locally compact?

Since a space X is a closed image of a locally separable, metric space if and only if X is a Fréchet space with a point-countable k-network consisting of separable subsets of X [16, Corollary 3.6], Question 1.2 and Question 1.3 are equivalent.

2. Counterexamples

First, we answer Question 1.1 through Question 1.4. We recall the space Y constructed in [18, Example 2.3]. Let P be a Bernstein set of the closed unit interval $\mathbb{I} = [0, 1]$ [2, p. 339]. In other words, P is an uncountable set which contains no uncountable closed set of $\mathbb{I}$. Let X be the space obtained from $\mathbb{I}$ by isolating the points of

P. The space X was considered in [4, Example 9.4]. Our space Y is the quotient space obtained from X by collapsing the set $X \setminus P$ to the one-point ∞.

Proposition 2.1. [18, Example 2.3] *The space Y has the following properties.*

(1) *Y is regular T_1 and Fréchet;*
(2) *Y has a point-countable closed family $\mathcal{P}$ which is both a cs-network (hence, cs^*-network) and a k-network;*
(3) *every first countable closed subset of Y is countable;*
(4) *Y does not have any star-countable k-network.*

We observe further properties of Y. By combining Proposition 2.1 with Theorem 2.2 below, we can easily see that Y is a counterexample for Question 1.1 through Question 1.4.

Theorem 2.2. *The space Y has the following properties, too.*

(a) *Y is a quotient s-image of a metric space;*
(b) *every first countable subset of Y is a discrete space or a countable space; hence, every first countable subset of Y is locally separable;*
(c) *Y does not have any point-countable cs^*-network of separable subsets; in particular, Y is not a quotient s-image of a locally separable, metric space;*
(d) *Y is not a closed image of a locally separable, metric space;*
(e) *Y does not have any point-countable k-network consisting of separable subsets.*

Proof: (a) It is proved in [4, p. 315] that every k-space with a point-countable k-network of closed sets is a quotient s-image of a metric space.

(b) Let A be a first countable subset of Y. If $\infty \notin A$, then A is discrete. Assume $\infty \in A$. Then A is closed in Y; hence, A is countable by Proposition 2.1(3).

(c) If Y has a point-countable cs^*-network of separable subsets, then Y has a point-countable k-network of separable subsets since each point-countable cs^*-network for a sequential space is a point-countable k-network [19, Corollary 1.5]. Hence, (e) implies (c).

It is known in [12, Theorem 2.2] that if a space is a quotient s-image of a locally separable, metric space, then it has a point-countable cs^*-network of separable subsets. Hence, Y is not a quotient s-image of a locally separable, metric space.

(d) It is known in [16, Corollary 3.6] that a space is a closed image of a locally separable, metric space if and only if it is a Fréchet space with a star-countable k-network. Apply Proposition 2.1(4).

(e) It is known in [16, Theorem 3.5] that every Fréchet space with a point-countable k-network consisting of separable subsets has a star-countable k-network. Apply Proposition 2.1(4). □

We remark that the following two statements are incorrect in view of our example Y.

Statement 2.3. [12, Lemma 2.4] Suppose $\mathcal{P}$ is a point-countable collection of subsets of X, which is closed under finite intersections. Let

$$\mathcal{H} = \{P \in \mathcal{P} : P \text{ is a hereditarily separable subspace of } X\}.$$

Then $\mathcal{H}$ is a cs^*-network for X if and only if $\mathcal{P}$ is a cs^*-network for X and every first countable subspace of X is locally separable.

Statement 2.4. [12, Corollary 2.5] The following conditions are equivalent for a space X:

(i) X is a quotient s-image of a metric space and every first countable subspace of X is locally separable;
(ii) X is a sequential space with a point-countable cs^*-network consisting of hereditarily separable subspaces.

Indeed, let $\mathcal{P}$ be a point-countable cs^*-network for Y; see Proposition 2.1(2). Consider the family $\mathcal{P}' = \{P_0 \cap \cdots \cap P_n : P_i \in \mathcal{P}, n \in \omega\}$. A point-countable cs^*-network is closed under finite intersections. But $\mathcal{H} = \{P \in \mathcal{P}' : P \text{ is hereditarily separable}\}$ is not a cs^*-network by Theorem 2.2 (c). Thus, Statement 2.3 is incorrect.
By Theorem 2.2(a) and (b), Y satisfies condition (i) in Statement 2.4. But Y does not satisfy condition (ii) in Statement 2.4 by Theorem 2.2(c). Thus, Statement 2.4 is also incorrect.

Next, we answer Question 1.5. In [20, Example 1.6(2)], a space X_A is constructed as follows. Let A be an uncountable subset of the closed unit interval $\mathbb{I}$. For each $x \in A$, let $S_x = \{(x, 0)\} \cup \{(x, 1/n) : n \in \mathbb{N}\}$, and let

$$\mathcal{C} = \{S_x : x \in A\} \cup \{A \times \{0\}\} \cup \{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\},$$

where each element of $\mathcal{C}$ has the usual topology. Set

$$X_A = (A \times \{0\}) \cup (\cup\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\}),$$

and let its topology be determined by the cover $\mathcal{C}$. It is known in [12, Theorem 2.2] that if a space is a quotient s-image of a locally separable, metric space, then it has a point-countable cs^*-network. Since X_A is a quotient s-image of the locally separable, metric space $\oplus\{C : C \in \mathcal{C}\}$, it has a point-countable cs^*-network. Since the subspaces $A \times \{0\}$ and $\bigcup_{n \in \mathbb{N}} \mathbb{I} \times \{1/n\}$ of X_A have the usual Euclidean topology, X_A has a countable network; in particular, X_A is hereditarily separable. It was proved in [6, Example 4.1(7)] that X_A has not any star-countable k-network. Therefore, X_A is not an $\aleph_0$-space.

In general, X_A is not regular [17].

Definition 2.5 ([17])**.** A subset A of the real line is called a σ'-*set* if for every F_σ-set F of the real line, there is an F_σ-set H in the real line such that $F \cap H = \emptyset$ and $A \subset F \cup H$.

It is known that the Continuum Hypothesis implies the existence of an uncountable σ'-set [17].

Lemma 2.6. [17, Theorem 2.2] *Let A be a non-empty subset of $\mathbb{I}$. Then A is a σ'-set if and only if X_A is regular.*

Let A be an uncountable σ'-set in $\mathbb{I}$ under the Continuum Hypothesis. By Lemma 2.6 and the observations on X_A, X_A is a counterexample to Question 1.5.

Finally, we remark that Question 1.6 is also negative. By using a σ'-set, Huaipeng Chen in [1, Example 4.5] constructed a space X satisfying the following properties.

(1) X is regular T_1, and a quotient s-image of a metric space;
(2) every first countable closed subspace of X is locally compact;
(3) X is not a compact-covering s-image of any metric space.

It was proved in [7, Theorem 1] that each quotient s-image of a locally compact, metric space is a compact-covering quotient s-image of a locally compact, metric space. By property (3) above, Chen's space X is not a quotient s-image of a locally compact, metric space since X is not a compact-covering quotient s-image of a locally compact, metric space. This contradicts property (3) above. Hence, X is a counterexample to Question 1.6.

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