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# NON-SMOOTHNESS IS NOT COUNTABLE IN THE CLASS OF SEMI-SMOOTH DENDROIDS

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Abstract. J. J. Charatonik and Carl Eberhart (*On smooth dendroids*, Fund. Math. **67** (1970), 297–322) asked if the property of non-smoothness is finite in the class of semi-smooth dendroids. We answer this question in the negative by showing an uncountable family $\mathcal{A}$ of semi-smooth non-smooth subdendroids such that if $X \in \mathcal{A}$ and $A$ is a semi-smooth non-smooth dendroid of $X$, then $A$ cannot be embedded in $Y$ for any $Y \in \mathcal{A}$ different from $X$. In addition, we also show that we can choose $\mathcal{A}$ to be a family of planar rational fans with closed set of end points.

## 1. Introduction

A *continuum* is a compact, connected, metric space. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. A dendroid is said to be *smooth* (*semi-smooth*) if there exists a point $p \in X$ such that for every $a \in X$ and each convergent sequence $\{a_n\}_{n\in\mathbb{N}} \subset X$, with $a_n \to a$, we have $\lim pa_n = pa$ ($\limsup pa_n$ is an arc).

A dendroid is called a *fan* provided that it has exactly one ramification point. A continuum is said to be *rational* provided that

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each of its points has arbitrarily small neighborhoods with countable boundaries.

Given a continuum $X$, a subcontinuum $A$ of $X$, and $\varepsilon > 0$, denote by $N_\varepsilon(A) = \{p \in X : \text{there is } a \in A \text{ such that } d(a,p) < \varepsilon\}$. Also given a continuum $X$, define $C(X) = \{A \subset X : A \text{ is a non-empty subcontinuum of } X\}$; we consider this hyperspace with the Hausdorff metric, $H$, defined by $H(A,B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)\}$.

A property $\mathcal{P}$ is said to be *finite (countable) in the class* $\mathcal{A}$ provided that there is a finite (countable) set $\mathcal{F}$ of members of $\mathcal{A}$ such that a member $X$ of $\mathcal{A}$ has property $\mathcal{P}$ if and only if $X$ contains a homeomorphic copy of some member of $\mathcal{F}$.

In [1, p. 307], J. J. Charatonik and Carl Eberhart ask if the property of non-smoothness is finite in the class of semi-smooth dendroids; they even show nine semi-smooth non-smooth dendroids and ask if these nine dendroids satisfy the required property. The purpose of this paper is to answer this question negatively (even if you ask for the property to be countable) by showing an uncountable family $\mathcal{A}$ of semi-smooth non-smooth dendroids such that if $X \in \mathcal{A}$ and $A$ is a semi-smooth non-smooth subdendroid of $X$, then $A$ cannot be embedded in $Y$ for any $Y \in \mathcal{A}$ different from $X$. It is important to note that, in addition, we can choose $\mathcal{A}$ to be a family of planar rational fans with closed set of end points.

## 2. Construction of the example

If $p, q \in \mathbb{R}^2$, let $pq$ denote the straight line segment that joins $p$ and $q$ in $\mathbb{R}^2$. Denote $\theta = (0,0)$ and $r = \left(0, \frac{1}{2}\right)$.

Consider the unit square $I^2 \subset \mathbb{R}^2$. Define $a_0 = (0,1)$, $b_0 = (1,0)$, and, for each $n \in \mathbb{N}$, $a_n = \left(0, \dfrac{1}{2n+1}\right)$, $b_n = \left(1, \dfrac{1}{2n}\right)$.

For each natural number $n$, we define an arc $S_n$ from $a_0$ to $b_0$, which is composed of $2n+1$ straight line segments,

$$S_n = \left(\bigcup_{i=0}^{n-1} (a_i b_{i+1} \cup a_{i+1} b_{i+1})\right) \cup a_n b_0.$$

For each pair of numbers $n, m \in \mathbb{N}$, consider the square

$$C_{n,m} = \left[\frac{1}{2^n}, \frac{3}{2^{n+1}}\right] \times \left[\frac{1}{m+1}, \frac{1}{2^n}\left(\frac{2^n m + 1}{m(m+1)}\right)\right],$$

and the following homeomorphism $h_{n,m} : I^2 \to C_{n,m}$, which will be used to transform arcs $S_n$,

$$h_{n,m}(x,y) =$$
$$\left(x\left(\frac{3}{2^{n+1}}\right) + (1-x)\left(\frac{1}{2^n}\right), y\left(\frac{1}{2^n}\left(\frac{2^n m+1}{m(m+1)}\right)\right) + (1-y)\left(\frac{1}{m+1}\right)\right).$$

Also, for each pair of numbers $n, m \in \mathbb{N}$ and for each $j \in \mathbb{N}\cup\{0\}$, define $a^j_{n,m} = h_{n,m}(a_j)$ and $b^j_{n,m} = h_{n,m}(b_j)$. Notice that, for each $j \in \mathbb{N}\cup\{0\}$, $\pi_1(a^j_{n,m}) = \dfrac{1}{2^n}$ and $\pi_1(b^j_{n,m}) = \dfrac{3}{2^{n+1}}$ (where $\pi_1$ is the projection onto the first coordinate).

Let $\mathcal{K} = \{k_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. For each pair of numbers $n, m \in \mathbb{N}$, define $S^m_{k_n} = h_{n,m}(S_{k_n})$. Define

$$X_{\mathcal{K}} = \theta r \cup \theta b_0 \cup \left(\bigcup_{m\in\mathbb{N}}\left(\bigcup_{n\in\mathbb{N}}\left(S^m_{k_n} \cup a^0_{n,m} b^0_{n+1,m}\right)\right)\right) \text{ (see Figure 1).}$$

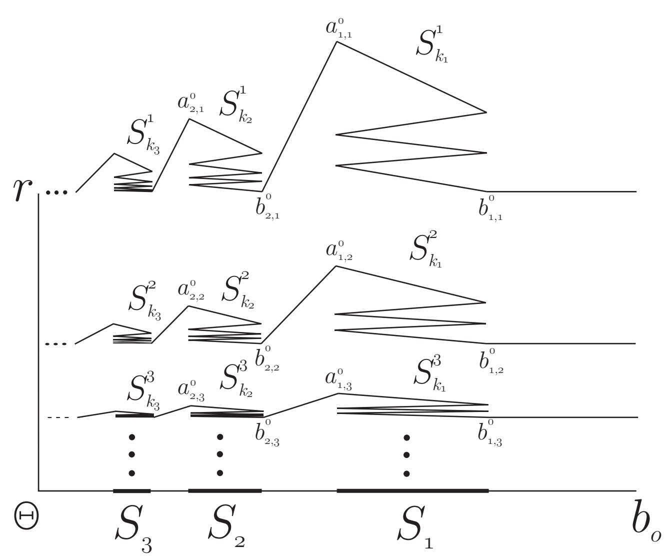


FIGURE 1

**Remark 2.1.** If $\mathcal{K}$ is a sequence of natural numbers, then $X_{\mathcal{K}}$ is a semi-smooth non-smooth dendroid.

## 3. An uncountable family of semi-smooth non-smooth dendroids

**Definition 3.1.** Let $\mathcal{K} = \{k_n\}_{n\in\mathbb{N}}$ and $\mathcal{L} = \{l_n\}_{n\in\mathbb{N}}$ sequences of positive integers. We say that $\mathcal{K}$ and $\mathcal{L}$ are eventually disjoint if there exists $N \in \mathbb{N}$ such that for every $n, m \geq N$, $k_n \neq l_m$.

**Remark 3.2.** By [2, Problem 118, p. 101], there exists an uncountable family $\Lambda$ of eventually disjoint sequences of positive integers.

**Definition 3.3.** Let $\Lambda$ be a family as in Remark 2.1. Define

$$\mathcal{A} = \{X_{\mathcal{K}} : \mathcal{K} \in \Lambda\}.$$

Note that $\mathcal{A}$ is an uncountable family of semi-smooth non-smooth dendroids.

**Theorem 3.4.** *Let $\mathcal{K} = \{k_n\}_{n\in\mathbb{N}}$ and $\mathcal{L} = \{l_n\}_{n\in\mathbb{N}}$ eventually disjoint sequences of natural numbers. If $A$ is a non-locally connected subcontinuum of $X_{\mathcal{K}}$, then $A$ cannot be embedded in $X_{\mathcal{L}}$.*

*Proof:* Denote by $B_{\mathcal{K}}$ the arc $\theta b_0$ in $X_{\mathcal{K}}$ and by $B_{\mathcal{L}}$ the arc $\theta b_0$ in $X_{\mathcal{L}}$. In addition, for each $n \in \mathbb{N}$, denote by $S_n^{\mathcal{K}}$ the set $\lim_{m\to\infty} S_{k_n}^m = \left[\left(\frac{1}{2^n}, 0\right), \left(\frac{3}{2^{n+1}}, 0\right)\right]_{X_{\mathcal{K}}}$ and by $S_n^{\mathcal{L}}$ the set $\lim_{m\to\infty} S_{l_n}^m = \left[\left(\frac{1}{2^n}, 0\right), \left(\frac{3}{2^{n+1}}, 0\right)\right]_{X_{\mathcal{L}}}$.

Suppose to the contrary that there is an embedding $h : A \to X_{\mathcal{L}}$. Since $A$ is non-locally connected, we have that $A$ contains a nondegenerate subarc of $B_{\mathcal{K}}$; hence, $h(A)$ contains a nondegenerate subarc of $B_{\mathcal{L}}$. Hence, there are $u, v \in \mathbb{N}$ such that for each $n \geq u$, $S_n^{\mathcal{K}} \subset A$ and $A$ is non-locally connected at each point of $S_n^{\mathcal{K}}$; and for each $n \geq v$, $S_n^{\mathcal{L}} \subset h(A)$ and $h(A)$ is non-locally connected at each point of $S_n^{\mathcal{L}}$.

We can suppose, without loss of generality, that

$$\left(\bigcup_{n=1}^{u-1} S_n^{\mathcal{K}}\right) \cap A = \emptyset \text{ and } \left(\bigcup_{n=1}^{v-1} S_n^{\mathcal{L}}\right) \cap h(A) = \emptyset.$$

Let $\widehat{h} : C(A) \to C(h(A))$ be the induced map given by $\widehat{h}(B) = h(B)$ for each $B$ in $C(A)$. Since $h$ is an embedding, it is easy to see that $\widehat{h}$ is a homeomorphism ([3, Exercise 77.29]). Notice that $h(\theta) = \theta$.

Take $n \geq u + 1$.

Claim 1. $h\left(S_n^{\mathcal{K}}\right) = S_j^{\mathcal{L}}$ for some $j \geq v$.

Since $A$ is non-locally connected at each point of $S_n^{\mathcal{K}}$, we have that

$$h\left(S_n^{\mathcal{K}}\right) \subset B_{\mathcal{L}} - \{\theta\}.$$

We show that

$$h\left(S_n^{\mathcal{K}}\right) \subset \bigcup_{i=v}^{\infty} S_i^{\mathcal{L}}.$$

Suppose there is $y \in h\left(S_n^{\mathcal{K}}\right) - \bigcup\limits_{i=v}^{\infty} S_i^{\mathcal{L}}$; then there exists $x \in S_n^{\mathcal{K}}$ such that $h(x) = y$.

Let $\varepsilon = dist\left(y, \bigcup\limits_{i=v}^{\infty} S_i^{\mathcal{L}}\right)$. Take $J$ a nondegenerate arc in $S_n^{\mathcal{K}}$ such that $x \in J$ and $diam(h(J)) < \dfrac{\varepsilon}{2}$. Let

$$\mu = \min\left\{dist\left(h(J), \bigcup_{i=v}^{\infty} S_i^{\mathcal{L}}\right), dist\left(h\left(S_n^{\mathcal{K}}\right), \theta r\right), diam\ h(J)\right\}.$$

Since $\widehat{h}$ is uniformly continuous, there exists $\delta > 0$ such that

$$\text{if } H(E,F) < \delta, \text{ then } H(\widehat{h}(E), \widehat{h}(F)) < \frac{\mu}{2}.$$

Since $A$ is not locally connected at each point of $S_n^{\mathcal{K}}$, there exists $m \in \mathbb{N}$ such that $S_{k_n}^m \subset A$ and $H(S_{k_n}^m, S_n^{\mathcal{K}}) < \delta$. Take two disjoint arcs $J_1, J_2 \subset S_{k_n}^m$ such that $H(J, J_1) < \delta$ and $H(J, J_2) < \delta$. We have that there are $t, N \in \mathbb{N}$ such that the components of $N_{\frac{\mu}{2}}(h(J))$ that do not contain $y$ are contained in $\bigcup\limits_{m \geq N} a_{t,m}^0 b_{t+1,m}^0$. Notice that $h(J_1)$ and $h(J_2)$ are two disjoint arcs such that $h(J) \subset N_{\frac{\mu}{2}}(h(J_1))$ and $h(J) \subset N_{\frac{\mu}{2}}(h(J_2))$. Hence, since $h(J_1), h(J_2) \subset N_{\frac{\mu}{2}}(h(J))$, it follows that $h(J_1)$ and $h(J_2)$ are in different components of $N_{\frac{\mu}{2}}(h(J))$. On the other hand, $h(S_{k_n}^m)$ is a connected set that contains both $h(J_1)$ and $h(J_2)$ and is contained in $N_{\frac{\mu}{2}}(h\left(S_n^{\mathcal{K}}\right))$ which contradicts the fact that the only arc that connects $h(J_1)$ and $h(J_2)$ passes through $\theta r$. Hence, $h\left(S_n^{\mathcal{K}}\right) \subset \bigcup\limits_{i=v}^{\infty} S_i^{\mathcal{L}}$. Since $\bigcup\limits_{i=v}^{\infty} S_i^{\mathcal{L}}$ is a union of disjoint arcs, there is $j \geq v$ such that $h\left(S_n^{\mathcal{K}}\right) \subset S_j^{\mathcal{L}}$. Similarly, using $h^{-1}$ and $\widehat{h^{-1}}$, we can show that $S_j^{\mathcal{L}} \subset h\left(S_n^{\mathcal{K}}\right)$. So, we have concluded the proof of Claim 1.

Claim 2. If $h(S_n^{\mathcal{K}}) = S_j^{\mathcal{L}}$, then $k_n = l_j$.

Let $\varepsilon = \dfrac{diam(S_j^{\mathcal{L}})}{4}$. Since $\widehat{h}$ is uniformly continuous, there exists $\delta > 0$ such that if $H(E, F) < \delta$, then $H(\widehat{h}(E), \widehat{h}(F)) < \dfrac{\varepsilon}{2}$. Since $A$ is non-locally connected at each point of $S_n^{\mathcal{K}}$ there exists $m \in \mathbb{N}$ such that $S_{k_n}^m \subset A$ and $H(S_{k_n}^m, S_n^{\mathcal{K}}) < \delta$. By the construction of $S_{k_n}^m$, there are $2k_n + 1$ disjoint arcs $J_1, ..., J_{2k_n+1} \subset S_{k_n}^m$ such that $H(S_n^{\mathcal{K}}, J_i) < \delta$ for each $i \in \{1, ..., 2k_n + 1\}$. Since $h(J_1), ..., h(J_{2k_n+1}) \subset h(S_{k_n}^m) \subset N_{\frac{\varepsilon}{2}}(S_j^{\mathcal{L}})$, we have that $h(J_1), ...,$ $h(J_{2k_n+1})$ are $2k_n + 1$ disjoint arcs which lie in the same component of $N_{\frac{\varepsilon}{2}}(S_j^{\mathcal{L}})$. Notice that in each component of $N_{\frac{\varepsilon}{2}}(S_j^{\mathcal{L}})$ that does not contain $S_j^{\mathcal{L}}$ there are at most $2l_j + 1$ disjoint arcs within a distance less than $\dfrac{\varepsilon}{2}$ of $S_j^{\mathcal{L}}$. Hence, since $H(S_n^{\mathcal{K}}, J_i) < \delta$ for each $i \in \{1, ..., 2k_n + 1\}$, we have that $k_n \leq l_j$. Similarly, using $h^{-1}$ and $\widehat{h^{-1}}$, we can show that $l_j \leq k_n$. So, we have concluded the proof of Claim 2.

Now we are ready to finish the proof of the theorem. Let $n \geq u$. By Claim 1, there is a $j_n$ such that $h\left(S_n^{\mathcal{K}}\right) = S_{j_n}^{\mathcal{L}}$; hence, by Claim 2, $k_n = l_{j_n}$which contradicts the fact that $\mathcal{K}$ and $\mathcal{L}$ are eventually disjoint. Therefore, $A$ cannot be embedded in $X_{\mathcal{L}}$. □

**Theorem 3.5.** *The property of being non-smooth is neither finite nor countable in the class of semi-smooth dendroids.*

*Proof:* Suppose to the contrary that there is a countable family $\mathcal{M}$ of non-smooth semi-smooth dendroids, such that if $Y$ is a non-smooth semi-smooth dendroid, then $Y$ contains a homeomorphic copy of a member $X$ of $\mathcal{M}$.

Notice that the family $\mathcal{A}$ (Definition 3.3) of non-smooth semi-smooth dendroids is not countable, then there are $X_{\mathcal{K}}, X_{\mathcal{L}} \in \mathcal{A}$ and $X \in \mathcal{M}$ such that both $X_{\mathcal{K}}$ and $X_{\mathcal{L}}$ contain a homeomorphic copy of $X$, which contradicts Theorem 3.4. □

**Corollary 3.6.** *The property of being non-smooth is not countable in the class of semi-smooth planar rational fans with closed set of end points.*

*Proof:* For each member $X_{\mathcal{K}}$ of the family $\mathcal{A}$ (Definition 3.3, we can shrink the arc $r\theta$ to a point; i.e., applying a monotone mapping $\eta : X_{\mathcal{K}} \to Y_{\mathcal{K}}$ such that $\eta(r\theta)$ is the singleton $\theta$, while the partial mapping $\eta \mid (X_{\mathcal{K}} - r\theta)$ is a homeomorphism, we obtain a rational plane fan $Y_{\mathcal{K}}$ with closed set of endpoints that keeps the same properties of $X_{\mathcal{K}}$. Therefore, we can apply Theorem 3.5 to the family

$$\mathcal{B} = \{Y_{\mathcal{K}} = \eta(X_{\mathcal{K}}) : X_{\mathcal{K}} \in \mathcal{A}\}. \qquad \square$$

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