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A NOTE ABOUT SPECIAL ULTRAFILTERS ON ω

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ABSTRACT. In this note, we show how results by Krzysztof Ciesielski and Janusz Pawlikowski (*The Covering Property Axiom, CPA: A Combinatorial Core of the Iterated Perfect Set Model*. Cambridge Tracts in Mathematics, 164. Cambridge: Cambridge University Press, 2004) can be used to answer negatively a question by R. Michael Canjar (*On the generic existence of special ultrafilters*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 233–241) about the generic existence of selective ultrafilters and how this solution also answers in the negative a similar question about P-points. We also prove that if the covering number for the meager ideal is equal to the dominating number $\mathfrak{d}$, then every filter generated by less than $\mathfrak{d}$ of its members can be extended to $2^{\mathfrak{c}}$-many $\mathfrak{c}$-generated Q-points. This improves a theorem and a remark by Canjar (as above).

1. Introduction

We will use standard set theoretic notation. If $A, B \in [\omega]^\omega$ we write $A \subseteq^* B$ provided that $A \setminus B$ is finite. If $A \subseteq \omega \times \omega$, then $(A)_m = \{n < \omega\colon (m,n) \in A\}$ for every $m < \omega$. We say that a family of subsets of ω has the *strong finite intersection property* (SFIP) provided that the intersection of any finite subfamily is infinite.

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Given a family $\mathcal{A}$ of sets, we denote $\langle\mathcal{A}\rangle$ the filter generated by $\mathcal{A}$. Letter $\mathcal{F}$ will denote always a non-principal filter on ω. A *basis* for $\mathcal{F}$ is a family $\mathcal{B} \subseteq \mathcal{F}$ such that for every $F \in \mathcal{F}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq F$. We say that $\mathcal{F}$ is *κ-generated* provided that κ is the minimum cardinality of a basis, and it is *$< \kappa$-generated* provided it is λ-generated for some $\lambda < \kappa$. A filter $\mathcal{F}$ on ω is a *Q-filter* provided that for every finite-to-one $f\colon \omega \to \omega$ there exists an $X \in \mathcal{F}$ such that $f \restriction X$ is one-to-one. A Q-filter which is an ultrafilter is called a *Q-point*. On the other hand, we say that $\mathcal{F}$ is *rapid* provided that for every $f\colon \omega \to \omega$ there is an $X \in \mathcal{F}$ such that $|X \cap f(n)| \leq n$ for every $n < \omega$. A rapid ultrafilter is called a *semi-Q-point*. Every Q-point is rapid but not every rapid ultrafilter is a Q-point. An ultrafilter $\mathcal{U}$ on a countably infinite set is a *P-point* provided that for every sequence $\langle U_n \in \mathcal{U}\colon n < \omega\rangle$ such that $U_{n+1} \subseteq^* U_n$ for every $n < \omega$, there exists a $U \in \mathcal{U}$ such that $U \subseteq^* U_n$ for every $n < \omega$. An ultrafilter which is both a P-point and a Q-point is called *selective* or *Ramsey* and an ultrafilter which is both a P-point and a semi-Q-point is called *semiselective*.

A family $\mathcal{G} \subseteq \omega^\omega$ is *dominating* provided that for every $h \in \omega^\omega$ there is a $g \in \mathcal{G}$ such that $h(n) < g(n)$ for every $n < \omega$. The number $\mathfrak{d}$ is the minimum cardinality of a dominating family in ω^ω.

The number $\mathsf{cov}(\mathcal{M})$ is the minimum cardinality of a family of meager sets whose union covers the real line. It is well known that both cardinals are uncountable and that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$.

The *Covering Property Axiom* of Krzysztof Ciesielski and Janusz Pawlikowski will be denoted CPA. Although we are not going to make explicit use of CPA in this note, we will consider the following theorem.

Proposition 1.1 (Ciesielski and Pawlikowski [7, Theorem 7.21])**.** *Let M be a countable transitive model of* ZFC+CH *and let $\mathbb{P}$ be the partial order in M to add ω_2 many Sacks reals with countable supports. Then* CPA *holds in M. In particular,* CPA *is consistent with* ZFC *set theory. Moreover, the value of 2^{ω_1} is preserved and it can be equal to ω_2 or bigger.*

2. The generic existence of ultrafilters

Definition 2.1 (Canjar [6, Definition 1])**.** We say that selective (respectively, semiselective, P-point) ultrafilters *generically exist* iff every $< \mathfrak{c}$-generated *filter* can be extended to a selective (respectively, semiselective, P-point) *ultrafilter*.

The next two theorems characterize the generic existence of selective and semiselective ultrafilters and P-points in terms of $\mathsf{cov}(\mathcal{M})$ and $\mathfrak{d}$.

Proposition 2.2 (Canjar [6, Theorem 2])**.** *The following three statements are equivalent:*

(1) $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$*;*
(2) *selective ultrafilters generically exist; and*
(3) *semiselective ultrafilters generically exist.*

Proposition 2.3 (Ketonen [8])**.** *The following are equivalent:*

(1) $\mathfrak{d} = \mathfrak{c}$*; and*
(2) P*-points generically exist.*

In [6, p. 240], R. Michael Canjar asked, assuming that $\mathfrak{c}$ is regular, is the existence of $2^{\mathfrak{c}}$-many selective ultrafilters equivalent to the generic existence of selectives? We will answer this negatively by constructing a model of ZFC where $\mathfrak{c} = \omega_2$ and there are $2^{\mathfrak{c}}$-many selective ultrafilters, but $\mathsf{cov}(\mathcal{M}) < \mathfrak{c}$. The same question for $\mathfrak{c}$ singular was answered in the negative by James E. Baumgartner who noticed that in the Bell-Kunen model described in [3], $\mathfrak{c} = \omega_{\omega_1}$ and $\mathsf{cov}(\mathcal{M}) = \omega_1$, and there are $2^{\mathfrak{c}}$-many selective ultrafilters on ω.

Theorem 2.4. *There is a model N of* ZFC *so that $N \models$ "$\mathfrak{c} = \omega_2$," and in N, there are $2^{\mathfrak{c}}$-many selective ultrafilters* (P*-points*)*, but selective ultrafilters* (P*-points*) *do not generically exist.*

Proof: Let Φ stand for "there are $2^{\mathfrak{c}}$-many selective ultrafilters" and let M be such that $M \models$ "$\mathrm{ZFC} + \mathrm{CH} + 2^{\omega_1} = 2^{\omega_2} = \omega_3$." If $\mathbb{P} \in M$ is the partial order to add ω_2-many Sacks reals iteratively with countable supports and G is $\mathbb{P}$-generic over M, then

$$M[G] \models \text{“ZFC} + \text{CPA} + 2^{\omega_1} = 2^{\omega_2} = \omega_3.\text{”}$$

Now, CPA implies that $\mathfrak{c} = \omega_2$, $\mathsf{cov}(\mathcal{M}) = \mathfrak{d} = \omega_1$ ([7, p. 11]). That Φ holds in $M[G]$ can be seen either by obtaining 2^{ω_1}-many selective ultrafilters from CPA ([7, Proposition 6.1.2]) or by using a theorem by Baumgartner and Richard Laver [2, Theorem 4.5] to extend 2^{ω_1}-many selective ultrafilters in M to 2^{ω_1}-many selective ultrafilters in $M[G]$. Therefore,

$$M[G] \models \text{“}\mathfrak{c} \text{ is regular} + \Phi + \mathsf{cov}(\mathcal{M}) < \mathfrak{c}.\text{”}$$

Hence, $N = M[G]$ works. $\square$

3. Large q-points

In a remark in [6, p. 237], Canjar claimed that it is possible to construct $2^{\mathfrak{d}}$-many rapid ultrafilters from the hypothesis $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$. Actually, this already follows from the existence of rapid ultrafilters since it is well known (see [10, Theorem 4]) that $\mathcal{U} \otimes \mathcal{V}$ ($\mathcal{U} \otimes \mathcal{V} = \{A \subseteq \omega \times \omega \colon \{m < \omega \colon (A)_m \in \mathcal{V}\} \in \mathcal{U}\}$) is rapid provided $\mathcal{V}$ is. Therefore, if we let $\mathcal{U}$ vary on the set of non-principal ultrafilters on ω, we will obtain $2^{\mathfrak{c}}$-many different rapid ultrafilters on $\omega \times \omega$. However, we can improve considerably this result.

Theorem 3.1. *The identity* $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$ *implies that every filter on* ω *which is* $< \mathfrak{d}$*-generated can be extended to* $2^{\mathfrak{c}}$*-many different* $\mathfrak{c}$*-generated* Q*-points.*

The theorem will follow as a consequence of the three lemmas below. To see how this improves Canjar's claim as well as the argument above, notice first that ultrafilters of the form $\mathcal{U} \otimes \mathcal{V}$ with $\mathcal{V}$ rapid are rapid but not Q-points. On the other hand, Proposition 1.1 guarantees that there exists a countable transitive model M for the theory ZFC+ CPA + $\mathfrak{c} = \omega_2 = 2^{\omega_1}$. Let Ψ stand for "there are $2^{\mathfrak{c}}$-many Q-points." Since CPA implies that all cardinals in Cichon's diagram are equal to ω_1 (see [7, p. 11]), it follows from Proposition 1.1 that

$$M \models \text{“}\mathsf{cov}(\mathcal{M}) = \mathfrak{d} + \Psi + 2^{\mathfrak{d}} < 2^{\mathfrak{c}}.\text{”}$$

It is interesting to note that all the Q-points obtained in the proof of Theorem 3.1 are non-selective Q-points. This has to be the case since it is known that in the iterated perfect set model the identity $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$ holds and every selective ultrafilter is ω_1-generated (see [7, Corollary 1.5.4]).

In order to prove Theorem 3.1 we will work on $\omega \times \omega$ instead of ω. Consider the pair $(\omega \times \omega, \prec)$ where $\prec$ is defined as follows. Pick any well-order $\prec_k$ for the finite set A_k of all pairs in $\omega \times \omega$ with largest coordinate equal to k. If $(m_1, n_1), (m_2, n_2) \in A_k$, then

$(m_1, n_1) \prec (m_2, n_2)$ iff $(m_1, n_1) \prec_k (m_2, n_2)$; otherwise, $(m_1, n_1) \prec (m_2, n_2)$ iff $\max\{m_1, n_1\} < \max\{m_2, n_2\}$. This induces an order on $(\omega \times \omega)^{\omega\times\omega}$ by

$$h \prec g \Leftrightarrow h(m,n) \prec g(m,n) \qquad \forall (m,n) \in \omega \times \omega.$$

A set $\mathcal{G} \subseteq (\omega \times \omega)^{\omega\times\omega}$ is $\prec$-dominating iff for every $h \in (\omega \times \omega)^{\omega\times\omega}$ there exists a $g \in \mathcal{G}$ such that $h \prec g$. Let

$$\mathfrak{d}_{\prec} = \min \{|\mathcal{G}| \colon \mathcal{G} \subseteq (\omega \times \omega)^{\omega\times\omega} \text{ and } \mathcal{G} \text{ is } \prec\text{-dominating } \}.$$

Lemma 3.2. $\mathfrak{d} = \mathfrak{d}_{\prec}$.

Proof: This is immediate because $(\omega \times \omega, \prec)$ and $(\omega, <)$ are isomorphic. $\square$

Lemma 3.3. *Let $\mathcal{F}$ be a $< \mathsf{cov}(\mathcal{M})$-generated filter on $\omega\times\omega$. There exists a partition $\mathcal{P} = \{P_m \colon m < \omega\}$ of $\omega\times\omega$ such that $|A\cap P_m| = \omega$ for every $A \in \mathcal{F}$ and $m < \omega$.*

Proof: Let $\kappa < \mathsf{cov}(\mathcal{M})$ and let $\mathcal{B} = \{B_\xi \colon \xi < \kappa\}$ be a basis of $\mathcal{F}$. We identify the set of all partitions $\mathcal{P} = \{P_m \colon m < \omega\}$ of ω into infinite pieces with the set Y of all functions $f \colon \omega \times \omega \to \omega$ such that $f^{-1}\{m\}$ is infinite for every $m < \omega$. It is easy to check that Y is a G_δ-subset of the product space $X = \omega^{\omega\times\omega}$ where ω has the discrete topology. Hence, Y is a Polish space with the relative topology inherited from X. For every $\xi < \kappa$ the set

$$Y_\xi = \{f \in Y \colon \exists m < \omega \; |f^{-1}\{m\} \cap B_\xi| < \omega\}$$

is meager. Since $\kappa < \mathsf{cov}(\mathcal{M})$ there exists an $f \in Y \setminus \bigcup_{\xi<\kappa} Y_\xi$. Hence, the partition $\mathcal{P}_f = \{f^{-1}\{m\} \colon m < \omega\}$ satisfies the conclusion of the lemma. $\square$

If $F \colon \omega \times \omega \to \omega \times \omega$ is arbitrary, we say that $C \subseteq \omega \times \omega$ is F-rare provided that $F(a, b) \prec (c, d)$ for every $(a, b), (c, d) \in C$ with $(a, b) \prec (c, d)$. The next lemma follows the scheme presented in the proof of Lemma 9 from [6].

Lemma 3.4. *Let $\mathcal{F}$ be $< \mathsf{cov}(\mathcal{M})$-generated with $|(A)_m| = \omega$ for every $A \in \mathcal{F}$, $m < \omega$ and let $F \colon \omega \times \omega \to \omega \times \omega$ be arbitrary. There exists an F-rare set $C \subseteq \omega \times \omega$ such that $|(A \cap C)_m| = \omega$ for every $A \in \mathcal{F}$, $m < \omega$.*

Proof: Let $\kappa < \mathsf{cov}(\mathcal{M})$ and let $\mathcal{B} = \{B_\xi \colon \xi < \kappa\}$ be a basis of $\mathcal{F}$. Take the product topology on $X = 2^{\omega\times\omega}$ with $2 = \{0,1\}$ discrete. Then, X is a Polish space and $Y = \{\chi_A \in X \colon A \text{ is } F\text{-rare}\}$ is a closed subset of X. Therefore, it is also a Polish space with the relative topology inherited from X. For every $\xi < \kappa$, the set

$$Y_\xi = \{\chi_A \in X \colon \exists m < \omega \ |(A \cap B_\xi)_m| < \omega\}$$

is meager. Since $\kappa < \mathsf{cov}(\mathcal{M})$, there is a $C \in Y \setminus \bigcup_{\xi<\kappa} Y_\xi$. This C satisfies the conclusion of the lemma. □

Lemma 3.5. *Assume* $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$ *and let* $\mathcal{F}$ *be a* $< \mathfrak{d}$*-generated filter on* $\omega \times \omega$ *such that* $|(X)_m| = \omega$ *for every* $X \in \mathcal{F}$ *and* $m < \omega$*. Then, there are* $2^{\mathfrak{c}}$*-many* $\mathfrak{c}$*-generated Q-points on* $\omega \times \omega$ *extending* $\mathcal{F}$*.*

Proof: Let $\kappa < \mathsf{cov}(\mathcal{M})$ and $\mathcal{B} = \{B_\xi \colon \xi < \kappa\}$ be a basis of $\mathcal{F}$ and fix an independent family $\mathcal{K}$ on ω of cardinality $\mathfrak{c}$. The family $\mathcal{J} = \{X_A \colon A \in \mathcal{K}\}$, and $X_A = \bigcup\{\{m\} \times \omega \colon m \in A\}$ for $A \in \mathcal{K}$ is an independent family on $\omega\times\omega$ and $|\mathcal{J}| = \mathfrak{c}$. Also, for every $g \colon \mathcal{J} \to 2$, $\mathcal{J}_g = \{S \in \mathcal{J} \colon g(S) = 1\} \cup \{(\omega\times\omega) \setminus S \colon S \in \mathcal{J} \ \& \ g(S) = 0\}$ is an independent family and $|\mathcal{J}_g| = \mathfrak{c}$. Let $\langle F_\xi \in (\omega\times\omega)^{\omega\times\omega} \colon \xi < \mathfrak{d}\rangle$ be a $\prec$-dominating family. We use Lemma 3.3 to construct inductively two sequences $\langle C_\xi \subseteq \omega\times\omega \colon \xi < \mathfrak{d}\rangle$ and $\langle \mathcal{F}_\xi \colon \xi < \mathfrak{d}\rangle$ such that for every $\xi < \mathfrak{d}$

(a) $\mathcal{F}_\xi$ is $\leq \max\{\kappa, |\xi|\}$-generated and extends $\mathcal{F}$;
(b) $\xi < \eta < \mathfrak{d} \Longrightarrow \mathcal{F}_\xi \subseteq \mathcal{F}_\eta$;
(c) $\mathcal{F}_{\xi+1} = \langle \mathcal{F}_\xi \cup \{C_\xi\}\rangle$ and $\mathcal{F}_\xi = \bigcup\{\mathcal{F}_\eta \colon \eta < \xi\}$ for ξ limit;
(d) $|(X)_m| = \omega$ for every $X \in \mathcal{F}_\xi$, $m < \omega$; and
(e) C_ξ is F_ξ-rare.

Then, the family

$$\mathcal{H}_g = \bigcup\{\mathcal{F}_\xi \colon \xi < \mathfrak{d}\} \cup \mathcal{J}_g \cup \left\{(\omega\times\omega) \setminus \bigcap \mathcal{B} \colon \mathcal{B} \subseteq \mathcal{J}_g \ \& \ |\mathcal{B}| \geq \omega\right\}$$

has the SFIP. Pick for each $g \in 2^{\mathcal{J}}$ a nonprincipal ultrafilter $\mathcal{U}_g$ on $\omega\times\omega$ extending $\mathcal{H}_g$. The usual argument (see [4, Proposition 9.5]) shows that $\mathcal{U}_g$ is $\mathfrak{c}$-generated. Also, notice that if $g, h \in 2^{\mathcal{J}}$ and $g \neq h$, then $\mathcal{U}_g \neq \mathcal{U}_h$. To see that each $\mathcal{U}_g$ is a Q-point, pick any $f \colon \omega\times\omega \to \omega\times\omega$ finite-to-one. If $F \colon \omega\times\omega \to \omega\times\omega$ is defined by $F(m,n) = \prec\text{-}\max(f^{-1}\{f(m,n)\})$, then, since the family $\langle F_\xi \colon \xi < \mathfrak{d}\rangle$ is $\prec$-dominating, there exists a $\xi < \mathfrak{d}$ such that

$F(m,n) \prec F_\xi(m,n)$ for every $(m,n) \in C_\xi$. But condition (e) from above implies that $f(m_1,n_1) \neq f(m_2,n_2)$ for every distinct $(m_1,n_1),(m_2,n_2) \in C_\xi$. Thus, $f \restriction C_\xi$ is one-to-one.

To complete the proof of Theorem 3.1, assume that $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$ and start with a $< \mathfrak{d}$-generated filter $\mathcal{F}$ on $\omega \times \omega$. Then, use Lemma 3.3 to find a partition $\mathcal{P} = \{P_m \colon m < \omega\}$ of $\omega \times \omega$ such that $|X \cap P_m| = \omega$ for every $m < \omega$. Consider a bijection $b \colon \omega\times\omega \to \omega\times\omega$ such that $b[P_m] = \{m\} \times \omega$ for every $m < \omega$. Then, the filter $\mathcal{F}^* = \{b[X] \colon X \in \mathcal{F}\}$ satisfies the hypotheses of Lemma 3.4 and there are $2^{\mathfrak{c}}$-many $\mathfrak{c}$-generated Q-points extending $\mathcal{F}^*$. For each of these Q-points $\mathcal{U}^*$, the ultrafilter $\mathcal{U} = \{b^{-1}[U] \colon U \in \mathcal{U}^*\}$ is a Q-point on $\omega \times \omega$ extending $\mathcal{F}$. $\square$

Corollary 3.6. *The following four statements are equivalent:*

(1) $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$*;*
(2) *every* $< \mathfrak{d}$*-generated filter can be extended to a* Q*-point;*
(3) *every* $< \mathfrak{d}$*-generated filter can be extended to* $2^{\mathfrak{c}}$*-many* $\mathfrak{c}$*-generated* Q*-points; and*
(4) *there is a* $\mathsf{cov}(\mathcal{M})$*-generated rapid filter.*

Proof: The equivalence (1) $\Leftrightarrow$ (2) follows from [6, Theorem 3]. The implication (3) $\Rightarrow$ (2) is trivial, and (1) $\Rightarrow$ (3) is Theorem 3.1 above. That (1) $\Leftrightarrow$ (4) follows from [5, Lemma 4.6.3(b)] and the fact that the set formed by the increasing enumerations of members of a basis of a rapid filter constitutes a dominating family in ω^ω. $\square$

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