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A NEW SELECTION PRINCIPLE

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ABSTRACT. Motivated by a recent result of Masami Sakai, we define a new selection operator for covers of topological spaces, inducing new selection hypotheses, and initiate a systematic study of the new hypotheses. Some intriguing problems remain open.

1. Subcovers with strong covering properties

We say that $\mathcal{U}$ is a *cover* of a set X if $X \notin \mathcal{U}$ and $X = \bigcup \mathcal{U}$.

Definition 1. For a family $\mathscr{A}$ of covers of a set X, $\mathscr{A}_\infty$ is the family of all $\mathcal{U}$ such that there exist infinite sets $\mathcal{U}_n \subseteq \mathcal{U}$, $n \in \mathbb{N}$, with $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\} \in \mathscr{A}$.

For topological spaces X, various special families of covers have been extensively studied in the literature, in a framework called *selection principles*; see the surveys [10], [4], and [13]. The main types of covers are defined as follows. Let $\mathcal{U}$ be a cover of X. $\mathcal{U}$ is an ω-*cover* of X if each finite $F \subseteq X$ is contained in some $U \in \mathcal{U}$. $\mathcal{U}$ is a γ-*cover* of X if $\mathcal{U}$ is infinite and each $x \in X$ belongs to all but finitely many $U \in \mathcal{U}$.

Let the boldfaced symbols $\boldsymbol{\mathcal{O}}$, $\boldsymbol{\Omega}$, $\boldsymbol{\Gamma}$ denote the families of all covers, ω-covers, and γ-covers, respectively. Then

$$\boldsymbol{\Gamma} \subseteq \boldsymbol{\Omega} \subseteq \boldsymbol{\mathcal{O}}.$$

Also, let $\mathcal{O}$, Ω, Γ denote the corresponding families of *open* covers.

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For a space X and collections $\mathscr{A}, \mathscr{B}$ of covers of X, the following property may or may not hold:

$\binom{\mathscr{A}}{\mathscr{B}}$: Every member of $\mathscr{A}$ has a subset which is a member of $\mathscr{B}$.

Masami Sakai [7] proved that for a Tychonoff space X, a local property called the *Pytkeev property* holds in the function space $C_p(X)$ if and only if X satisfies $\binom{\check{\Omega}}{\mathbf{\Omega}_\infty}$, where $\check{\Omega}$ is a certain subclass of Ω. It is open whether $\check{\Omega}$ can be replaced by Ω [7]. Motivated by this, Petr Simon and the present author proved that for Lindelöf spaces X, $C_p(X)$ satisfies the Pytkeev property if and only if X is zero-dimensional and satisfies $\binom{C_\Omega}{\mathbf{\Omega}_\infty}$, where C_Ω is the collection of all *clopen* ω-covers of X [12]. This motivates the study of additional properties of the form $\binom{\mathscr{A}}{\mathscr{B}_\infty}$.

Definition 2. A family $\mathscr{B}$ of open covers of X is *surjectively derefinable* if $\{f(U) : U \in \mathcal{U}\} \in \mathscr{B}$ for each $\mathcal{U} \in \mathscr{B}$ and each $f : \mathcal{U} \to P(X) \setminus \{X\}$ such that for each $U \in \mathcal{U}$ $f(U)$ is open and contains U. A similar definition applies to families of Borel covers, clopen covers, etc.

Example 3. $\mathbf{\mathcal{O}}$, $\mathbf{\Omega}$, and $\mathbf{\Gamma}$ are surjectively derefinable. For the latter we must explain why a (surjective) derefinement of a γ-cover is infinite, and this follows from the fact that it is an ω-cover.

Lemma 4. *Assume that $\mathscr{B}$ is a surjectively derefinable family of covers of X. Then*

$$\binom{\mathscr{A}}{\mathscr{B}_\infty} \Rightarrow \binom{\mathscr{A}}{\mathscr{B}}.$$

Proof: Assume that $\mathcal{U} \in \mathscr{A}$. By the assumption, there are infinite $\mathcal{U}_1, \mathcal{U}_2, \ldots \subseteq \mathcal{U}$ such that $\mathcal{V} = \{\bigcap \mathcal{U}_n : n \in \mathbb{N}\} \in \mathscr{B}$. For each n, choose $f(\bigcap \mathcal{U}_n) \in \mathcal{U}_n$. As $\mathscr{B}$ is surjectively derefinable, $\mathcal{W} = \{f(\bigcap \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathscr{B}$. Clearly, $\mathcal{W} \subseteq \mathcal{U}$. □

The converse need not hold. For example, $\binom{C_\Omega}{\mathbf{\Omega}}$ always holds, whereas $\binom{C_\Omega}{\mathbf{\Omega}_\infty}$ need not, as explained above. More examples will follow in the sequel.

Proposition 5. *Every space satisfies $\binom{\Gamma}{\mathbf{\Gamma}_\infty}$.*

Proof: Assume that $\mathcal{U} \in \Gamma$. We may assume that $\mathcal{U}$ is countable (since an infinite subset of a γ-cover is again a γ-cover). Enumerate

$\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ bijectively, and take $\mathcal{U}_n = \{U_k : k \geq n\}$ for each n. Then $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\} \in \Gamma$. $\square$

$\binom{\Omega}{\mathbf{\Gamma}}$ is the classical γ-property [2].

Corollary 6. $\binom{\Omega}{\mathbf{\Gamma}} = \binom{\Omega}{\mathbf{\Gamma}_\infty}$.

Proof: By Proposition 5 and Lemma 4,

$$\binom{\Omega}{\mathbf{\Gamma}} = \binom{\Omega}{\mathbf{\Gamma}} \cap \binom{\Gamma}{\mathbf{\Gamma}_\infty} = \binom{\Omega}{\mathbf{\Gamma}_\infty},$$

the last equation being self-evident. $\square$

What about the other properties? $\binom{\mathcal{O}}{\boldsymbol{\mathcal{O}}_\infty}$ never holds, since any T_1 space with more than one element has a finite open cover. Taking the above results into account, only $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ and $\binom{\Omega}{\mathbf{\Omega}_\infty}$ are potentially new. It turns out that even the formally weaker property $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ is quite restrictive. According to Borel, a set $X \subseteq \mathbb{R}$ has *strong measure zero* if for each sequence of positive reals $\{\epsilon_n\}_{n\in\mathbb{N}}$, there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that for each n, the diameter of I_n is smaller than ϵ_n. It was established by Richard Laver that consistently, all strong measure zero sets of reals are countable. The following theorem of Miller is essentially proved in [12].

Theorem 7 (Miller, see [12])**.** *If $X \subseteq \mathbb{R}$ and X satisfies $\binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty}$, then X has strong measure zero.*

Proof: By standard arguments [15], we may assume that $X \subseteq \{0,1\}^{\mathbb{N}}$. It suffices to prove that for each increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers, there are for each n elements $s^n_m \in \{0,1\}^{k_n}$, $m \leq n$, such that $X = \bigcup_n([s^n_1] \cup \cdots \cup [s^n_n])$. (One can allow n sets of diameter ϵ_n in the original definition of strong measure zero by moving to an appropriate subsequence of the original sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$.)

For each n, let

$$\mathcal{U}_n = \{[s_1] \cup \cdots \cup [s_n] : s_1, \ldots, s_n \in \{0,1\}^{k_n}\},$$

and take $\mathcal{U} = \bigcup_n \mathcal{U}_n$. $\mathcal{U}$ is a clopen ω-cover of X. By $\binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty}$, there are infinite subsets $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of $\mathcal{U}$, such that $\{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ is a cover of X. As each $\mathcal{V}_n$ is infinite and each $\mathcal{U}_n$ is finite, we can find m_1 and $V_1 \in \mathcal{V}_1 \cap \mathcal{U}_{m_1}$, $m_2 > m_1$ and $V_2 \in \mathcal{V}_2 \cap \mathcal{U}_{m_2}$, etc. Then

$\{V_n : n \in \mathbb{N}\}$ is a cover of X, and the sets V_n are as required in the first paragraph of this proof. □

However, we have the following.

Conjecture 8. *The Continuum Hypothesis implies*

(1) *there is a set of reals X satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ but not $\binom{\Omega}{\mathbf{\Gamma}}$, and*
(2) *there is a set of reals X satisfying $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ but not $\binom{\Omega}{\mathbf{\Omega}_\infty}$.*

Conjecture 8(1) implies, if true, a negative answer to Sakai's Question 4.7 in [8]. We will show that critical cardinalities (defined below) cannot be used to prove the consistency of items (1) and (2) of Conjecture 8.

X is an Ω-*Lindelöf* space if each open ω-cover of X contains a countable ω-cover of X. For Tychonoff spaces this is equivalent to: All finite powers of X are Lindelöf [2]. Separable zero-dimensional metrizable spaces are homeomorphic to subsets of $\mathbb{R}$ and are thus Ω-Lindelöf. Recall that a family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is *centered* if the intersection of each finite subset of $\mathcal{F}$ is infinite. $\mathcal{F}$ is *free* if $\bigcap \mathcal{F} = \emptyset$. $A \subseteq \mathbb{N}$ is a *pseudo-intersection* of $\mathcal{F}$ if A is infinite and for each $B \in \mathcal{F}$, $A \subseteq^* B$ (that is, $A \setminus B$ is finite). $[\mathbb{N}]^{\aleph_0}$ inherits its topology from $P(\mathbb{N})$, whose topology is defined by identifying $P(\mathbb{N})$ with $\{0,1\}^{\mathbb{N}}$.

Theorem 9. *For Ω-Lindelöf spaces X, the following are equivalent:*

(1) *X satisfies $\binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty}$;*
(2) *For each continuous free centered image $\mathcal{F}$ of X in $[\mathbb{N}]^{\aleph_0}$, $\mathcal{F} = \bigcup_n \mathcal{F}_n$ where each $\mathcal{F}_n$ has a pseudo-intersection.*

Proof: $(1 \Rightarrow 2)$ Assume that $\Psi : X \to [\mathbb{N}]^{\aleph_0}$ is continuous and that its image $\mathcal{F}$ is free and centered. For each n, let $U_n = \{x : n \in \Psi(x)\}$. $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a clopen ω-cover of X. Choose infinite $\mathcal{U}_n \subseteq \mathcal{U}$, $n \in \mathbb{N}$, such that $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is a cover of X, and set $A_n = \{m : U_m \in \mathcal{U}_n\}$, and $\mathcal{F}_n = \{I \in \mathcal{F} : A_n \subseteq I\}$. For each $I \in \mathcal{F}$, let $x \in X$ be such that $I = \Psi(x)$. Choose n such that $x \in \bigcap \mathcal{U}_n$. Then for each $m \in A_n$, $x \in U_m$ and therefore $m \in \Psi(x) = I$, that is, $I \in \mathcal{F}_n$.

$(2 \Rightarrow 1)$ Assume that $\mathcal{U}$ is a clopen ω-cover of X. Since X is Ω-Lindelöf, we may assume that $\mathcal{U}$ is countable. Fix a bijective enumeration $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$. As the sets U_n are clopen, the Marczewski function $\mu : X \to P(\mathbb{N})$, defined by $\mu(x) = \{n : x \in$

$U_n\}$, is continuous. Since $\mathcal{U}$ is an ω-cover of X, the image $\mathcal{F}$ of μ is a free centered subset of $[\mathbb{N}]^{\aleph_0}$ [13]. Let $\mathcal{F} = \bigcup_n \mathcal{F}_n$ be as in (2). For each n, let A_n be a pseudo-intersection of $\mathcal{F}_n$. Take $\mathcal{U}_{n,m} = \{U_k : m \leq k \in A_n\} \subseteq \mathcal{U}$. Then $\{\bigcap \mathcal{U}_{n,m} : m, n \in \mathbb{N}\}$ is a cover of X. □

The minimal cardinality of a centered $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ such that there is *no* partition $\mathcal{F} = \bigcup_n \mathcal{F}_n$ where each $\mathcal{F}_n$ has a pseudo-intersection is equal to $\mathfrak{p}$ [12].

The *critical cardinality* of a nontrivial family $\mathcal{J}$ of sets of reals is

$$\mathsf{non}(\mathcal{J}) = \min\{|X| : X \subseteq \mathbb{R} \text{ and } X \notin \mathcal{J}\}.$$

Corollary 10. $\mathsf{non}(\binom{\Omega}{\mathbf{\Omega}_\infty}) = \mathsf{non}(\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}) = \mathfrak{p}$.

Proof: $\mathsf{non}(\binom{\Omega}{\mathbf{\Gamma}}) = \mathfrak{p}$ [1], and by the implications among the properties, $\mathfrak{p} \leq \mathsf{non}(\binom{\Omega}{\mathbf{\Omega}_\infty}) \leq \mathsf{non}(\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}) \leq \mathsf{non}(\binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty})$. By Theorem 9 and the above-mentioned result of [12], $\mathsf{non}(\binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty}) \leq \mathfrak{p}$. □

Proposition 11. *If all finite powers of X satisfy $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$, then X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$.*

Proof: If $\mathcal{U}$ is an open ω-cover of X, then for each k, $\mathcal{U}^k := \{U^k : U \in \mathcal{U}\}$ is an open ω-cover of X^k. Take infinite $\mathcal{V}_{k,n} \subseteq \mathcal{U}$ such that $\{\bigcap \mathcal{V}_{k,n}^k : n \in \mathbb{N}\}$ is a cover of X^k. Then each k-element subset of X is contained in some member of $\{\bigcap \mathcal{V}_{k,n} : n \in \mathbb{N}\}$, and therefore $\{\bigcap \mathcal{V}_{k,n} : n, k \in \mathbb{N}\}$ is an ω-cover of X. □

A subtle technical problem prevents us from using the methods of [3] to obtain the converse implication.

Problem 12. Is the converse implication in Proposition 11 provable?

Additional results concerning $\binom{\Omega}{\mathbf{\Omega}_\infty}$ can be found in [8].

2. A new selection principle

Fix a topological space X, and let $\mathscr{A}$ and $\mathscr{B}$ each be a collection of covers of X. The following selection principles, which X may or may not satisfy, were introduced in [9] to generalize a variety of classical properties, and were extensively studied in the literature (see the surveys [10], [4], [13]).

$\mathsf{S}_1(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathscr{A}$, there exist members $U_n \in \mathcal{U}_n$, $n\in\mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\} \in \mathscr{B}$.

$\mathsf{S}_{fin}(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathscr{A}$, there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n\in\mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n \in \mathscr{B}$.

$\mathsf{U}_{fin}(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathscr{A}$ which do not contain a finite subcover, there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n\in\mathbb{N}$, such that $\{\cup\mathcal{F}_n : n\in\mathbb{N}\} \in \mathscr{B}$.

We introduce the following new selection principle, which is a selective version of $\binom{\mathscr{A}}{\mathscr{B}_\infty}$.

$\bigcap_\infty(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of elements of $\mathscr{A}$, there is for each n an infinite set $\mathcal{V}_n \subseteq \mathcal{U}_n$, such that $\{\bigcap\mathcal{V}_n : n\in\mathbb{N}\}\in\mathscr{B}$.

Note that if $\mathscr{A}$ contains a finite element, then $\bigcap_\infty(\mathscr{A},\mathscr{B})$ automatically fails.

Since the sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is allowed to be constant, the following holds.

Lemma 13. $\bigcap_\infty(\mathscr{A},\mathscr{B}) \Rightarrow \binom{\mathscr{A}}{\mathscr{B}_\infty}$.

The following is easy to verify.

Proposition 14. *Assume that $\mathscr{B}$ is a surjectively derefinable family of covers of X. Then*

$$\bigcap_\infty(\mathscr{A},\mathscr{B}) \Rightarrow \mathsf{S}_1(\mathscr{A},\mathscr{B}).$$

As $\bigcap_\infty(\Gamma,\mathbf{\Gamma}) \Rightarrow \mathsf{S}_1(\Gamma,\mathbf{\Gamma})$, and $\mathsf{S}_1(\Gamma,\mathbf{\Gamma})$ is rather restrictive (e.g., every set of reals satisfying it is perfectly meager), it follows that $\bigcap_\infty(\Gamma,\mathbf{\Gamma})$ is strictly stronger than $\binom{\Gamma}{\mathbf{\Gamma}_\infty}$.

Theorem 15. $\bigcap_\infty(\Gamma,\mathbf{\Gamma}) = \mathsf{S}_1(\Gamma,\mathbf{\Gamma})$.

Proof: Assume that X satisfies $\mathsf{S}_1(\Gamma,\mathbf{\Gamma})$. We will prove that X satisfies $\bigcap_\infty(\Gamma,\mathbf{\Gamma})$. The trick we use comes from the context of local properties, which we learned from Marion Scheepers.

Assume that $\mathcal{U}_n$, $n\in\mathbb{N}$, are open γ-covers of X. We may assume that they are all countable and that the sets $\mathcal{U}_n$, $n\in\mathbb{N}$, are pairwise disjoint.

Fix a surjection $f : \mathbb{N} \to \mathbb{N}$ such that for each n, $f^{-1}(n)$ is infinite. For a countable bijectively enumerated set $\mathcal{F} = \{U_n : n \in \mathbb{N}\}$ and $m \in \mathbb{N}$, define $\mathcal{F}(m) = \{U_n : n \geq m\}$. Fix a bijective enumeration for each of the covers $\mathcal{U}_n$, and apply $\mathsf{S}_1(\Gamma, \mathbf{\Gamma})$ to the sequence $\mathcal{U}_{f(n)}(n)$, $n \in \mathbb{N}$, to obtain sets $U_n \in \mathcal{U}_{f(n)}(n)$ such that $\{U_n : n \in \mathbb{N}\}$ is a γ-cover of X.

For each n, take $\mathcal{V}_n = \{U_m : f(m) = n\} \subseteq \mathcal{U}_n$. Each $U_m \in \mathcal{V}_n$ can belong to only finitely many $\mathcal{U}_{f(k)}(k)$ with $f(k) = n$ and cannot belong to any $\mathcal{U}_{f(k)}(k)$ with $f(k) \neq n$ (because $\mathcal{U}_n \cap \mathcal{U}_{f(k)} = \emptyset$). In particular, $\mathcal{V}_n$ is infinite.

$\mathcal{V} = \{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ is a γ-cover of X: For each $x \in X$, $x \in U_m$ for all large enough m, and as $n \to \infty$, $\min f^{-1}(n) \to \infty$ too. This also shows that $\mathcal{V}$ is an ω-cover of X, and thus $\mathcal{V}$ is infinite. □

$\mathsf{S}_1(\Omega, \mathbf{\Gamma}) = \binom{\Omega}{\mathbf{\Gamma}}$ [2].

Corollary 16. $\bigcap_\infty(\Omega, \mathbf{\Gamma}) = \mathsf{S}_1(\Omega, \mathbf{\Gamma})$.

Proof: By Theorem 15 and easy reasoning,
$\bigcap_\infty(\Omega, \mathbf{\Gamma}) = \binom{\Omega}{\mathbf{\Gamma}} \cap \bigcap_\infty(\Gamma, \mathbf{\Gamma}) = \binom{\Omega}{\mathbf{\Gamma}} \cap \mathsf{S}_1(\Gamma, \mathbf{\Gamma}) = \mathsf{S}_1(\Omega, \mathbf{\Gamma})$. □

Exactly the properties in Figure 1 remain to be explored.

$$\begin{array}{ccc} \bigcap_\infty(\Gamma, \mathbf{\Omega}) & \longrightarrow & \bigcap_\infty(\Gamma, \boldsymbol{\mathcal{O}}) \\ \uparrow & & \uparrow \\ \bigcap_\infty(\Omega, \mathbf{\Omega}) & \longrightarrow & \bigcap_\infty(\Omega, \boldsymbol{\mathcal{O}}) \end{array}$$

FIGURE 1. The surviving properties

Problem 17. Are any of the properties in Figure 1 equivalent to a classical selection hypothesis?

Problem 18. Can any implication be added to Figure 1?

For $\bigcap_\infty(\Omega, \mathbf{\Omega})$ and $\bigcap_\infty(\Omega, \boldsymbol{\mathcal{O}})$, Problem 17 is closely related to Conjecture 8, because these properties are sandwiched between $\binom{\Omega}{\mathbf{\Gamma}}$ and $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$.

For the remaining two properties, we have a partial answer for Problem 17. Let $\mathcal{B}_\Gamma$ denote the family of *countable Borel* γ-covers of X.

Theorem 19.

(1) $\mathsf{S}_1(\mathcal{B}_\Gamma, \boldsymbol{\mathcal{O}}) = \bigcap_\infty(\mathcal{B}_\Gamma, \boldsymbol{\mathcal{O}})$.
(2) $\mathsf{S}_1(\mathcal{B}_\Gamma, \boldsymbol{\Omega}) = \bigcap_\infty(\mathcal{B}_\Gamma, \boldsymbol{\Omega})$.
(3) $\mathsf{S}_1(\mathcal{B}_\Gamma, \boldsymbol{\mathcal{O}}) \Rightarrow \bigcap_\infty(\Gamma, \boldsymbol{\mathcal{O}}) \Rightarrow \mathsf{S}_1(\Gamma, \boldsymbol{\mathcal{O}})$.
(4) $\mathsf{S}_1(\mathcal{B}_\Gamma, \boldsymbol{\Omega}) \Rightarrow \bigcap_\infty(\Gamma, \boldsymbol{\Omega}) \Rightarrow \mathsf{S}_1(\Gamma, \boldsymbol{\Omega})$.

Proof: We only prove the implications which do not follow from Proposition 14.

(1) Assume that X satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \boldsymbol{\mathcal{O}})$ and $\mathcal{U}_n$, $n \in \mathbb{N}$, are countable Borel γ-covers of X. Enumerate bijectively, for each n, $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. Define $\Psi : X \to \mathbb{N}^\mathbb{N}$ by

$$\Psi(x)(n) = \min\{m : (\forall k \geq m)\ x \in U^n_k\}.$$

Since Ψ is Borel and X satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \boldsymbol{\mathcal{O}})$, $\Psi[X]$ is not dominating [11]. Let $g \in \mathbb{N}^\mathbb{N}$ be a witness for that. Take $\mathcal{V}_n = \{U^n_m : m \geq g(n)\}$. For each $x \in X$, there are infinitely many n such that $\Psi(x)(n) \leq g(n)$, and therefore $x \in \bigcap \mathcal{V}_n$.

(2) is similar. Here $\Psi[X]$ is not finitely dominating [11], and this is what we need.

(3) and (4) follow from (1) and (2), respectively, because γ-covers may be assumed to be countable. □

Corollary 20.

(1) $\mathsf{non}(\bigcap_\infty(\Omega, \boldsymbol{\Omega})) = \mathsf{non}(\bigcap_\infty(\Omega, \boldsymbol{\mathcal{O}})) = \mathfrak{p}$.
(2) $\mathsf{non}(\bigcap_\infty(\Gamma, \boldsymbol{\Omega})) = \mathsf{non}(\bigcap_\infty(\Gamma, \boldsymbol{\mathcal{O}})) = \mathfrak{d}$.

Proof: (1) follows from corollaries 16 and 10 and the implications among the properties, together with $\mathsf{non}(\mathsf{S}_1(\Omega, \boldsymbol{\Gamma})) = \mathfrak{p}$ [1].

(2) follows from Theorem 19 and the fact that the critical cardinalities of the Borel version of the classical principles are the same as in their open version and are both $\mathfrak{d}$ [3], [11]. □

There are some additional interesting connections between the new and the classical selection principles.

Theorem 21 (Sakai [8])**.** $\mathsf{S}_{fin}(\Omega, \boldsymbol{\Omega}) \cap \binom{\Omega}{\boldsymbol{\Omega}_\infty} \Rightarrow \mathsf{S}_1(\Omega, \boldsymbol{\Omega})$.

Proof: For completeness, we give a proof.

Clearly,

$$\mathsf{S}_{fin}(\Omega, \boldsymbol{\Omega}) \cap \binom{\Omega}{\boldsymbol{\Omega}_\infty} = \mathsf{S}_{fin}(\Omega, \boldsymbol{\Omega}_\infty).$$

It therefore suffices to show that $\mathsf{S}_{fin}(\Omega, \mathbf{\Omega}_\infty)$ implies $\mathsf{S}_1(\Omega, \mathbf{\Omega})$. Indeed, assume that $\mathcal{U}_n$, $n \in \mathbb{N}$, are open ω-covers of X. Choose finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{F}_n \in \mathbf{\Omega}_\infty$ for X.

Take infinite $\mathcal{V}_n \subseteq \bigcup_n \mathcal{F}_n$, $n \in \mathbb{N}$, such that $\{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ is an ω-cover of X. To each n, assign m_n such that m_n is increasing with n and $\mathcal{V}_n \cap \mathcal{F}_{m_n} \neq \emptyset$, and choose any $U_{m_n} \in \mathcal{V}_n \cap \mathcal{F}_{m_n}$. For $k \notin \{m_n : n \in \mathbb{N}\}$, choose any $U_k \in \mathcal{U}_k$. As $\{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ refines $\{U_n : n \in \mathbb{N}\}$, $\{U_n : n \in \mathbb{N}\}$ is an ω-cover of X. □

Corollary 22. $\mathsf{S}_{fin}(\Omega, \mathbf{\Omega}_\infty) = \mathsf{S}_1(\Omega, \mathbf{\Omega}_\infty)$.

Proof: By Sakai's Theorem 21,

$$\mathsf{S}_{fin}(\Omega, \mathbf{\Omega}_\infty) = \mathsf{S}_{fin}(\Omega, \mathbf{\Omega}) \cap \binom{\Omega}{\mathbf{\Omega}_\infty} = \mathsf{S}_1(\Omega, \mathbf{\Omega}) \cap \binom{\Omega}{\mathbf{\Omega}_\infty} = \mathsf{S}_1(\Omega, \mathbf{\Omega}_\infty). \qquad \square$$

The following result is inspired by results from [17].

Theorem 23. *For Lindelöf zero-dimensional spaces,* $\mathsf{U}_{fin}(\mathcal{O}, \Gamma) = \mathsf{U}_{fin}(\mathcal{O}, \boldsymbol{\mathcal{O}}_\infty)$.

Proof: Note that $\Gamma \subseteq \boldsymbol{\mathcal{O}}_\infty$. We therefore prove that $\mathsf{U}_{fin}(\mathcal{O}, \boldsymbol{\mathcal{O}}_\infty)$ implies $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$. Assume that X is Lindelöf zero-dimensional, and satisfies $\mathsf{U}_{fin}(\mathcal{O}, \boldsymbol{\mathcal{O}}_\infty)$. It suffices to prove that every continuous image of X in $\mathbb{N}^\mathbb{N}$ is bounded [6].

Assume that Y is a continuous image of X in $\mathbb{N}^\mathbb{N}$. We may assume that all elements of Y are increasing functions. If there is an infinite $I \subseteq \mathbb{N}$ such that $\{f \restriction I : f \in Y\}$ is bounded, then Y is bounded. We therefore assume that there is k such that for each $n \geq k$, $\{f(n) : f \in Y\}$ is infinite.

For each $n \geq k$, let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$, where $U^n_m = \{f \in Y : f(n) \leq m\}$ for each m. $\mathcal{U}_n$ does not contain Y as an element. Thus, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \geq k$, such that $\mathcal{V} = \{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \boldsymbol{\mathcal{O}}_\infty$. We may assume that each $\mathcal{F}_n$ is nonempty. For each n, the sets U^n_m are increasing with m, and therefore, there is $g(n) \in \mathbb{N}$ such that $\bigcup \mathcal{F}_n = U^n_{g(n)}$.

Let $\mathcal{V}_m$, $m \in \mathbb{N}$, be infinite subsets of $\mathcal{V} = \{U^n_{g(n)} : n \in \mathbb{N}\}$ such that $\{\bigcap \mathcal{V}_m : m \in \mathbb{N}\}$ is a cover of Y. For each m, let $I_m = \{n \geq k : U^n_{g(n)} \in \mathcal{V}_m\}$. I_m is infinite, and $\{f \restriction I_m : f \in \bigcap \mathcal{V}_m\}$ is bounded by $g \restriction I_m$. Thus, $\bigcap \mathcal{V}_m$ is bounded. It follows that $Y = \bigcup_m \bigcap \mathcal{V}_m$ is bounded. □

Theorem 23 can be contrasted with the fact that $\mathsf{U}_{fin}(\mathcal{O},\Gamma) \neq \mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ [16].

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