Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 31, 2007
Pages 403-418
http://topology.auburn.edu/tp/

ALGEBRAIC STRUCTURE CLOSE TO THE SMALLEST IDEAL OF $\beta\mathbb{N}$

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Abstract. If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ with the property that for each n, $x_{n+1} > \Sigma_{t=1}^{n} x_t$, then we have shown that several notions of largeness for the set of finite sums, including *syndetic*, *piecewise syndetic*, and *central*, are equivalent to the set $\{x_{n+1} - \Sigma_{t=1}^{n} x_t : n \in \mathbb{N}\}$ being bounded. We show here that there exist such sequences with $\{x_{n+1} - \Sigma_{t=1}^{n} x_t : n \in \mathbb{N}\}$ unbounded but with density of the set of finite sums as close to 1 as we please. As a consequence we see that there are copies of the semigroup $\mathbb{H}$ close to, but missing, the smallest ideal of $\beta\mathbb{N}$. This semigroup is known to contain much of the algebraic structure of $\beta\mathbb{N}$, including all of its idempotents.

1. Introduction

Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, we let $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$, where $\mathcal{P}_f(\mathbb{N})$ is the set of finite nonempty subsets of $\mathbb{N}$. Sets of this form have been of interest ever since the Finite Sums Theorem was proved in 1974.

Theorem 1.1 (Finite Sums Theorem). *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$.*

2000 *Mathematics Subject Classification.* Primary 22A15; Secondary 54D35, 20M10.

Key words and phrases. Syndetic, piecewise syndetic, central, subsemigroups of $\beta\mathbb{N}$, missing smallest ideal.

*This article is a portion of the author's doctoral dissertation written at Howard University under the direction of Neil Hindman.

Proof. [4, Theorem 3.1]. □

The proof of Theorem 1.1, while elementary, was very complicated. Subsequently a much simpler proof was obtained by Fred Galvin and Steven Glazer who used the fact that the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ has an operation extending ordinary addition on $\mathbb{N}$ making $(\beta\mathbb{N}, +)$ a compact right topological semigroup. Any compact right topological semigroup must have idempotents and a subset A of $\mathbb{N}$ contains $FS(\langle x_n \rangle_{n=1}^{\infty})$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$ if and only if there is an idempotent p of $\beta\mathbb{N}$ in the closure of A.

As a compact right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal, $K(\beta\mathbb{N})$ which is the union of all of the minimal left ideals of $\beta\mathbb{N}$ and is also the union of all of the minimal right ideals of $\beta\mathbb{N}$. This ideal is known to have very rich algebraic structure. (For example, there are $2^{\mathfrak{c}}$ pairwise isomorphic groups in $K(\beta\mathbb{N})$, each of which contains a free group on $2^{\mathfrak{c}}$ generators.)

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying a point of $\mathbb{N}$ with the principal ultrafilter consisting of all subsets of $\mathbb{N}$ with that element as a member. Given $A \subseteq \mathbb{N}$, the closure of A, $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$. The set $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a basis for the open sets of $\beta\mathbb{N}$, as well as a basis for the closed sets of $\beta\mathbb{N}$. See [6] for an introduction to the algebraic structure of $K(\beta\mathbb{N})$.

Of special interest to us is the subsemigroup $\mathbb{H}$ of $\beta\mathbb{N}$.

Definition 1.2. $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$.

Since $\mathbb{H}$ is a compact right topological semigroup, it has a smallest ideal $K(\mathbb{H})$.

Theorem 1.3. *All of the idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}$ and each maximal group in the $K(\mathbb{H})$ contains a free group on $2^{\mathfrak{c}}$ generators. Also, $K(\mathbb{H})$ contains a copy of the $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ rectangular semigroup.*

Proof. [6, Lemma 6.8 and Theorem 7.35] and [7, Corollary 3.15]. □

Several notions of size have originated in topological dynamics, and they all have characterizations in terms of the algebraic structure of $\beta\mathbb{N}$. One of them does not have a simple elementary description, so we define it first.

Definition 1.4. A subset A of $\mathbb{N}$ is *central* if and only if there is some idempotent in $\overline{A} \cap K(\beta\mathbb{N})$.

The other notions with which we will be concerned have simple elementary definitions. Given $A \subseteq \mathbb{N}$ and $x \in \mathbb{N}$, $-x + A = \{y \in \mathbb{N} : x + y \in A\}$.

Definition 1.5. Let $A \subseteq \mathbb{N}$.

(a) The set A is *thick* if and only if for all $F \in \mathcal{P}_f(\mathbb{N})$ there exists $x \in \mathbb{N}$ such that $F + x \subseteq A$.

(b) The set A is *syndetic* if and only if there exists $G \in \mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N} = \bigcup_{t \in G} -t + A$.

(c) The set A is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{t \in G} -t + A$ is thick.

Notice that a set A is thick precisely when it contains arbitrarily long blocks, syndetic precisely when there is a bound on the gaps of A, and piecewise syndetic precisely when there is a bound b and arbitrarily long blocks of $\mathbb{N}$ in which A has no gaps longer than b.

All of these notions have simple algebraic characterizations in terms of $\beta\mathbb{N}$.

Lemma 1.6. *Let $A \subseteq \mathbb{N}$.*

(a) The set A is syndetic if and only if for every left ideal L of $\beta\mathbb{N}$, $L \cap \overline{A} \neq \emptyset$.
(b) The set A is thick if and only if there is some left ideal L of $\beta\mathbb{N}$ with $L \subseteq \overline{A}$.
(c) The set A is piecewise syndetic if and only if $K(\beta\mathbb{N}) \cap \overline{A} \neq \emptyset$.

Proof. [2, Lemma 1.9]. □

It turns out that for well behaved sequences, these notions of size are all equivalent for $FS(\langle x_n \rangle_{n=1}^{\infty})$.

Theorem 1.7. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each n, $x_{n+1} > \sum_{t=1}^{n} x_t$. The following statements are equivalent.*

(a) $FS(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic.
(b) For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is piecewise syndetic.
(c) $FS(\langle x_n \rangle_{n=1}^{\infty})$ is syndetic.
(d) For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is syndetic.
(e) $FS(\langle x_n \rangle_{n=1}^{\infty})$ is central.
(f) For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is central.
(g) The set $\{x_{n+1} - \sum_{t=1}^{n} x_t : n \in \mathbb{N}\}$ is bounded.

Proof. [1, Corollary 4.2]. □

There are other notions of size which do not join the above list of equivalences. These involve various notions of density. Of course the ordinary density of a set $A \subseteq \mathbb{N}$ is simply $d(A) = \lim_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n}$, provided that limit exists. The ordinary upper and lower density are defined respectively by

$$\overline{d}(A) = \lim\sup_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n} \underline{d}(A) = \lim\inf_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n}.$$

Then $d(A)$ exists if and only if $\overline{d}(A)$ and $\underline{d}(A)$ exist and are equal.

Closely related to the notion of piecewise syndeticity is that of upper Banach density. (The terminology is due to Furstenberg. The notion was actually introduced by Polya in [8].)

Definition 1.8. Let $A \subseteq \mathbb{N}$. Then

$$d^*(A) = \sup\Big\{\alpha : \text{for all } n \in \mathbb{N} \text{ there exist } m, x \in \mathbb{N} \text{ such that } m \geq n \text{ and } \frac{|A \cap \{x+1, x+2, \ldots, x+m\}|}{m} \geq \alpha\Big\}.$$

The notion of Banach density determines an ideal of $\beta\mathbb{N}$.

Definition 1.9. $\Delta^* = \{p \in \beta\mathbb{N} : \text{for all } A \in p,\ d^*(A) > 0\}$.

It is easy to see that Δ^* is a closed two sided ideal of $(\beta\mathbb{N}, +)$, and consequently contains $c\ell K(\beta\mathbb{N})$. The content of the following theorem is that, in one sense Δ^* is not much bigger than $c\ell K(\beta\mathbb{N})$, since $p \in c\ell K(\beta\mathbb{N})$ if and only if each member of p is piecewise syndetic.

Theorem 1.10. *Let $A \subseteq \beta\mathbb{N}$.*

(a) If A is piecewise syndetic, then there is some $b \in \mathbb{N}$ such that $d^*\big(\bigcup_{t=1}^{b}(-t + A)\big) = 1$.
(b) If $d^*(A) > 0$, then for each $\epsilon > 0$, there is some $b \in \mathbb{N}$ such that $d^*\big(\bigcup_{t=1}^{b}(-t + A)\big) > 1 - \epsilon$.

Proof. (a) Pick b such that $\bigcup_{t=1}^{b}(-t + A)$ is thick.
(b) [5, Theorem 3.8]. □

In this paper we show that for each $\epsilon > 0$, there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that

(1) for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$,
(2) $\{x_{n+1} - \sum_{t=1}^n x_t : n \in \mathbb{N}\}$ is unbounded, and
(3) $1 > d^*\big(\bigcup_{t=1}^b \big(-t + FS(\langle x_n\rangle_{n=1}^\infty)\big)\big) > 1 - \epsilon$.

Then we show that as a consequence, there exist copies of $K(\mathbb{H})$ contained in $\Delta^* \setminus c\ell K(\beta\mathbb{N})$. As we have seen in Theorem 1.3, this says that there is substantial algebraic structure contained in $\Delta^* \setminus c\ell K(\beta\mathbb{N})$.

2. Subsemigroups contained in $\Delta^* \setminus c\ell K(\beta\mathbb{N})$

It follows immediately from [6, Lemma 6.27], that if $\langle x_n\rangle_{n=1}^\infty$ is a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$, then $T = \bigcap_{m=1}^\infty \overline{FS(\langle x_n\rangle_{n=m}^\infty)}$ is algebraically and topologically isomorphic to $\mathbb{H}$.

Theorem 2.1. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$, and let $T = \bigcap_{m=1}^\infty \overline{FS(\langle x_n\rangle_{n=m}^\infty)}$. The following statements are equivalent.*

(a) $T \cap K(\beta\mathbb{N}) \neq \emptyset$.
(b) $T \cap c\ell K(\beta\mathbb{N}) \neq \emptyset$.
(c) $\{x_{n+1} - \sum_{t=1}^n x_t : n \in \mathbb{N}\}$ is bounded.

Proof. Trivially (a) implies (b). To see that (b) implies (c), assume that $T \cap K(\beta\mathbb{N}) \neq \emptyset$. Then, in particular $\overline{FS(\langle x_n\rangle_{n=1}^\infty)} \cap c\ell K(\beta\mathbb{N}) \neq \emptyset$. So, since $\overline{FS(\langle x_n\rangle_{n=1}^\infty)}$ is open, $\overline{FS(\langle x_n\rangle_{n=1}^\infty)} \cap K(\beta\mathbb{N}) \neq \emptyset$. So $FS(\langle x_n\rangle_{n=1}^\infty)$ is piecewise syndetic so that Theorem 1.7 applies.

(c) implies (a). By Theorem 1.7, $FS(\langle x_n\rangle_{n=1}^\infty)$ is piecewise syndetic, so pick a minimal left ideal L of $\beta\mathbb{N}$ such that $\overline{FS(\langle x_n\rangle_{n=1}^\infty)} \cap L \neq \emptyset$. We shall show by induction that for all $m \in \mathbb{N}$, $\overline{FS(\langle x_n\rangle_{n=m}^\infty)} \cap L \neq \emptyset$. This holds for $m = 1$ by the choice of L. So let $m \in \mathbb{N}$, assume that $\overline{FS(\langle x_n\rangle_{n=m}^\infty)} \cap L \neq \emptyset$, and pick $p \in \overline{FS(\langle x_n\rangle_{n=m}^\infty)} \cap L$. Now $FS(\langle x_n\rangle_{n=m}^\infty) = \{x_m\} \cup FS(\langle x_n\rangle_{n=m+1}^\infty) \cup \big(x_m + FS(\langle x_n\rangle_{n=m+1}^\infty)\big)$. Since $p \in \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, $\{x_m\} \notin p$. So either $FS(\langle x_n\rangle_{n=m+1}^\infty) \in p$ of $x_m + FS(\langle x_n\rangle_{n=m+1}^\infty) \in p$. In the first case we are done, so assume that $x_m + FS(\langle x_n\rangle_{n=m+1}^\infty) \in p$. Then $FS(\langle x_n\rangle_{n=m+1}^\infty) \in -x_m + p$. Since $\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$ by [6, Exercise 4.3.5] we have by

[6, Lemma 1.43(c)] that L is a left ideal of $\beta\mathbb{Z}$, and consequently, $-x_m + p \in \overline{FS(\langle x_n\rangle_{n=m+1}^\infty)} \cap L$.

We thus have that $\{\overline{FS(\langle x_n\rangle_{n=m}^\infty)} \cap L : m \in \mathbb{N}\}$ is a collection of closed subsets of $\beta\mathbb{N}$ with the finite intersection property, which therefore has nonempty intersection. $\square$

As a consequence of Theorem 2.1 we have that if $\{x_{n+1} - \sum_{t=1}^n x_t : n \in \mathbb{N}\}$ is unbounded, then $T \cap c\ell K(\beta\mathbb{N}) = \emptyset$ and so T is a copy of $\mathbb{H}$ which misses $c\ell K(\beta\mathbb{N})$. We shall show in this section that much of this structure is contained in Δ^*, and is thus close to $K(\beta\mathbb{N})$.

While we will continue to discuss sequences in $\mathbb{N}$ where each term is greater than the sum of it's predecessors, we seek to control the largeness of a particular $FS(\langle x_n\rangle_{n=1}^\infty)$ when the set $\{x_{n+1} - \sum_{t=1}^n x_t : n \in \mathbb{N}\}$ is not bounded. We let $d \in \mathbb{N}$ and consider sequences $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^n x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. For $d \in \mathbb{N}$, the function $f(n) = \lfloor \frac{n+(d-1)}{d} \rfloor$, though unbounded, grows slowly.

We shall need the following simple lemmas, whose proofs we omit.

Lemma 2.2. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$ and let $F, G \in \mathcal{P}_f(\mathbb{N})$. Then $\sum_{t\in F} x_t < \sum_{t\in G} x_t$ if and only if $\max(F \triangle G) \in G$.*

Lemma 2.3. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$ and let $F, G \in \mathcal{P}_f(\mathbb{N})$. If $\sum_{t\in G} x_t = \min\{\sum_{t\in H} x_t : H \in \mathcal{P}_f(\mathbb{N})$ and $\sum_{t\in F} x_t < \sum_{t\in H} x_t\}$, and $r = \min \mathbb{N} \setminus F$, then $G = \{r\} \cup \{t \in F : t > r\}$.*

In Lemma 2.4 we provide, without proof, formulas for computing the n^{th} term and n^{th} partial sum of the given sequence.

Lemma 2.4. *Let $d \in \mathbb{N}$ and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^n x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. For $n \geq d+2$, if $n = dk + j$, where $k \in \mathbb{N}$ and $j \in \{2, 3, \ldots, d+1\}$, then*

$$x_n = 2^{j-2}\left(x_1 2^{dk} + 2^d\left(\frac{2^{dk}-1}{2^d-1}\right) + 1\right)$$

and

$$\sum_{i=1}^n x_i = 2^{j-1}\left(x_1 2^{dk} + 2^d\left(\frac{2^{dk}-1}{2^d-1}\right) + 1\right) - k - 1\,.$$

Given $F \in \mathcal{P}_f(\mathbb{N})$ such that $\min F \geq r$, we wish to know the number of times $FS(\langle x_n\rangle_{n=r}^{\infty})$ hits $\{1, 2, \ldots, \sum_{t\in F} x_t\}$. The next lemma provides the answer.

Lemma 2.5. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^{n} x_t$. Let $r \in \mathbb{N}$ and let $F \in \mathcal{P}_f(\mathbb{N})$ such that $\min F \geq r$. Then*

$$\left| FS(\langle x_n\rangle_{n=r}^{\infty}) \cap \left\{ 1, 2, \ldots, \sum_{t\in F} x_t \right\} \right| = \sum_{t\in F} 2^{t-r}\,.$$

Proof. We proceed by induction on $|F|$. Assume first that $F = \{l\}$, then

$$\begin{aligned} &FS(\langle x_n\rangle_{n=r}^{\infty}) \cap \big\{1, 2, \ldots, \textstyle\sum_{t\in F} x_t\big\} = \\ &\big\{ \textstyle\sum_{t\in G} x_t : G = F \text{ or } \emptyset \neq G \subseteq \{r, r+1, \ldots, l-1\}\big\} \end{aligned}$$

so $\big|FS(\langle x_n\rangle_{n=r}^{\infty}) \cap \big\{1, 2, \ldots, \sum_{t\in F} x_t\big\}\big| = 2^{l-r}$.

Let $m \in \mathbb{N}$ and assume that the result is true for all F with $|F| = m$. Let $F \in \mathcal{P}_f(\mathbb{N})$ such that $\min F \geq r$ and $|F| = m+1$. Let $l = \min F$ and let $H = F \setminus \{l\}$. Let $A = \{G \in \mathcal{P}_f(\mathbb{N}) : \min G \geq r$ and $\sum_{t\in H} x_t < \sum_{t\in G} x_t \leq \sum_{t\in F} x_t\}$ and let $B =$

$$\{F\} \cup \big\{H \cup K : \emptyset \neq K \subseteq \{r, r+1, \ldots, l-1\}\big\}.$$

We claim that $A = B$.

To see that $B \subseteq A$, note that directly $\sum_{t\in H} x_t < \sum_{t\in F} x_t \leq \sum_{t\in F} x_t$, so let $\emptyset \neq K \subseteq \{r, r+1, \ldots, l-1\}$ and let $G = H \cup K$. Then $\max(G \triangle F) = l \in F$ and $\max(G \triangle H) \in K \subseteq G$ so by Lemma 2.2, $\sum_{t\in H} x_t < \sum_{t\in G} x_t < \sum_{t\in F} x_t$.

To see that $A \subseteq B$, let $G \in A$. If $\sum_{t\in G} x_t = \sum_{t\in F} x_t$, then $G = F$, so assume that $\sum_{t\in H} x_t < \sum_{t\in G} x_t < \sum_{t\in F} x_t$. Let $k = \max(H \triangle G)$ and let $m = \max(F \triangle G)$. Then $k \in G$ and $m \in F$. We claim that $m \leq l$. Suppose instead that $m > l$. Then $m \in H \setminus G$ so $m < k$. Thus $k > l$ and $k \notin H$ so $k \notin F$ so $k \in G \setminus F$. This is a contradiction because $k > m = \max(F \triangle G)$.

Since $m \leq l$ and $m \in F$, we have $m = l$. Thus $H \subseteq G$, for if $t \in H$, then $t \in F$ and $t > l$ so $t \notin F \setminus G$ so $t \in G$. Also, given $t \in G$, if $t > l$, then $t \notin G \setminus F$ so $t \in H$. Thus $\emptyset \neq G \setminus H \subseteq \{r, r+1, \ldots, l-1\}$. Let $K = G \setminus H$.

Having established that $A = B$, we have that

$$\begin{aligned}\left|FS(\langle x_n\rangle_{n=r}^{\infty})\cap\left\{1,2,\ldots,\sum_{t\in F}x_t\right\}\right| &= \left|FS(\langle x_n\rangle_{n=r}^{\infty})\cap\left\{1,2,\ldots,\sum_{t\in H}x_t\right\}\right| \\ &\quad +|B| \\ &= \sum_{t\in H}2^{t-r}+2^{l-r} \\ &= \sum_{t\in F}2^{t-r}. \qquad \square\end{aligned}$$

We are now able to easily establish the following theorem.

Theorem 2.6. *Let $d \in \mathbb{N}$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \left\lfloor \frac{n+(d-1)}{d}\right\rfloor$. Then*

$$\lim_{n\to\infty}\frac{|FS(\langle x_n\rangle_{n=1}^{\infty})\cap\{1,2,\ldots,x_n\}|}{x_n}=\lim_{n\to\infty}\frac{2^{n-1}}{x_n}=\frac{2(2^d-1)}{(2^d-1)x_1+2^d}.$$

In particular

$$\begin{aligned}\overline{d}(FS(\langle x_n\rangle_{n=1}^{\infty})) &\geq \frac{2(2^d-1)}{(2^d-1)x_1+2^d} \text{ and} \\ \underline{d}(FS(\langle x_n\rangle_{n=1}^{\infty})) &\leq \frac{2(2^d-1)}{(2^d-1)x_1+2^d}.\end{aligned}$$

Proof. The first equality holds by Lemma 2.5, where $r = 1$.

If $n = dk + j$ where $n \in \mathbb{N}$ and $j \in \{2, 3, \ldots, d + 1\}$, then by Lemma 2.4,

$$\begin{aligned}\frac{|FS(\langle x_n\rangle_{n=1}^{\infty})\cap\{1,2,\ldots,x_n\}|}{x_n} &= \frac{2^{dk+j-1}}{2^{(j-2)}\left(x_1 2^{dk}+2^d\left(\frac{2^{dk}-1}{2^d-1}\right)+1\right)} \\ &= \frac{2(2^d-1)}{x_1(2^d-1)+2^d-2^{-dk}}.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\frac{|FS(\langle x_n\rangle_{n=1}^{\infty})\cap\{1,2,\ldots,x_n\}|}{x_n} &= \lim_{k\to\infty}\frac{2(2^d-1)}{x_1(2^d-1)+2^d-2^{-dk}} \\ &= \frac{2(2^d-1)}{(2^d-1)x_1+2^d}.\end{aligned}$$

The "in particular" follows immediately from the fact that for any $A \in \mathbb{N}$ and subsequence $\langle x_n \rangle_{n=1}^{\infty}$ of $\mathbb{N}$, if $\lim_{n\to\infty} \frac{|A\cap\{1,2,\ldots,x_n\}|}{x_n}$ exists, then

$$\underline{d}(A) \leq \lim_{n\to\infty} \frac{|A \cap \{1,2,\ldots,x_n\}|}{x_n} \leq \overline{d}(A)\,. \qquad \square$$

We now begin a sequence of lemmas designed to establish that $FS(\langle x_n \rangle_{n=1}^{\infty})$ has an actual density – that is that

$$\lim_{m\to\infty} \frac{|FS(\langle x_n \rangle_{n=1}^{\infty}) \cap \{1,2,\ldots,m\}|}{m}$$

exists.

Lemma 2.7. *Let $d \in \mathbb{N}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. If $n \in \mathbb{N}$ and $n \geq 2$, then $x_{n+1} \geq 2x_n$.*

Proof. If $n \in \{2,3,\ldots,d\}$, then $x_n = 2^{n-2}x_1 + 2^{n-2}$ and $x_{n+1} = 2^{n-1}x_1 + 2^{n-1}$. Also $x_{d+1} = 2^{d-1}x_1 + 2^{d-1}$ and $x_{d+2} = 2^d x_1 + 2^d + 1$.

If $k \in \mathbb{N}$, $j \in \{2,3,\ldots,d\}$ and $n = kd + j$ one has directly by Lemma 2.4 that $x_{n+1} = 2x_n$. Finally, if $n = kd + d + 1$, then by Lemma 2.4,

$$\begin{aligned} x_n &= x_1 2^{kd+d-1} + \frac{2^{dk+2d-1} - 2^{d-1}}{2^d - 1} \text{ and} \\ x_{n+1} &= x_1 2^{kd+d} + \frac{2^{dk+2d} - 2^d}{2^d - 1} + 1 \\ &= 2x_n + 1\,. \end{aligned}$$

$\square$

Lemma 2.8. *Let $d \in \mathbb{N}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. If $k, r \in \mathbb{N}$ and $k \geq r$, then $\frac{k+1}{x_k} \leq \frac{r+1}{x_r}$.*

Proof. For this it suffices to observe that for any $n \in \mathbb{N}$, $\frac{n+2}{x_{n+1}} \leq \frac{n+2}{2x_n} < \frac{n+1}{x_n}$. $\square$

Lemma 2.9. *Let $d \in \mathbb{N}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. If $m \in \mathbb{N}$, $m \geq 2$, $F \in \mathcal{P}_f(\mathbb{N})$, and $\min F \geq m$, then $x_m \sum_{t\in F} 2^{t-1} \leq 2^{m-1} \sum_{t\in F} x_t$.*

Proof. We proceed by induction on the cardinality of F. Let $m \in \mathbb{N}$ with $m \geq 2$ and let $F = \{s\}$. We prove by induction on s that if $s \geq m$, then $x_m 2^{s-1} \leq 2^{m-1} x_s$. Grounding the proof by induction, for $s = m$, $x_m 2^{m-1} \leq 2^{m-1} x_m$. Assume for $s \geq m$ that $x_m 2^{s-1} \leq 2^{m-1} x_s$. By Lemma 2.7, $x_s \geq 2x_s$. Therefore,

$$\begin{aligned} 2^{m-1} x_{s+1} &\geq 2 \cdot 2^{m-1} x_s \\ &\geq 2 x_m 2^{s-1} \\ &= x_m 2^s \,. \end{aligned}$$

Returning to the inductive proof on the cardinality of F, assume that $r \in \mathbb{N}$ and whenever $|F| = r$ and $\min F \geq m$, one has $x_m \sum_{t \in F} 2^{t-1} \leq 2^{m-1} \sum_{t \in F} x_t$. Let $|F| = r + 1$ such that $\min F \geq m$, let $n = \min F$, and let $G = F \setminus \{n\}$. Then $|G| = r$. Since $\min F \geq m$, $\min G > m$. From the inductive hypothesis, $x_m \sum_{t \in G} 2^{t-1} \leq 2^{m-1} \sum_{t \in G} x_t$. Then

$$\begin{aligned} x_m \left(\sum_{t \in F} 2^{t-1} \right) &= x_m 2^{n-1} + x_m \sum_{t \in G} 2^{t-1} \\ &\leq x_n 2^{m-1} + 2^{m-1} \sum_{t \in G} x_t \\ &= 2^{m-1} \left(\sum_{t \in G} x_t + x_n \right) \\ &= 2^{m-1} \sum_{t \in F} x_t \,. \end{aligned}$$

□

Lemma 2.10. *Let $d \in \mathbb{N}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \left\lfloor \frac{n+(d-1)}{d} \right\rfloor$. If $k \in \mathbb{N}$, $k \geq 2$, $F \in \mathcal{P}_f(\mathbb{N})$, and $\max F = k$, then $x_k \sum_{t \in F} 2^{t-1} \geq 2^{k-1} \sum_{t \in F} x_t$.*

Proof. We proceed by induction on $|F|$. To ground the induction we establish by induction on k the stronger statement that if $s \leq k$, then $x_k 2^{s-1} \geq 2^{k-1} x_s$. This is trivially true if $s = k$. If $x_k 2^{s-1} \geq 2^{k-1} x_s$, then $x_{k+1} 2^{s-1} \geq 2 x_k 2^{s-1} \geq 2^k x_s$, where the first inequality holds by Lemma 2.7.

Now let $r \in \mathbb{N}$ and assume the conclusion holds for all F with $|F| = r$. Let $F \in \mathcal{P}_f(\mathbb{N})$ with $|F| = r+1$ be given, let $l = \min F$, and let $G = F \setminus \{l\}$. Then by assumption $x_k \sum_{t\in G} 2^{t-1} \geq 2^{k-1} \sum_{t\in G} x_t$. Thus

$$\begin{aligned} x_k \sum_{t\in F} 2^{t-1} &= x_k \left(2^{l-1} + \sum_{t\in G} 2^{t-1} \right) \\ &\geq 2^{k-1} x_l + \sum_{t\in G} x_t \\ &= 2^{k-1} \sum_{t\in F} x_t \,. \end{aligned}$$

□

Given $m \in \mathbb{N}$ and $\sum_{n\in G} x_n \in FS(\langle x_n\rangle_{n=m}^\infty)$, the next two lemmas provide constructions for the immediate successor and predecessor, respectively, of $\sum_{n\in G} x_n$.

Lemma 2.11. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$. Let $m \in \mathbb{N}$ and let $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq m$. If $G \in \mathcal{P}_f(\mathbb{N})$, $\min G \geq m$,*

$$\sum_{n\in G} x_n = \min\left\{ \sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N}), \min F \geq m, \text{ and } \sum_{n\in F} x_n > \sum_{n\in H} x_n \right\},$$

and $s = \min\{t \in \mathbb{N} \setminus H : t \geq m\}$, then $G = \{s\} \cup \{t \in H : t > s\}$.

Proof. For each $n \in \mathbb{N}$, let $y_n = x_{m+n-1}$. Let $H' = H - m + 1$, where $H' = H - m + 1 = \{t - m + 1 : t \in H\}$. Let $G' = G - m + 1$. Then $\sum_{t\in H'} y_t = \sum_{t\in H} x_t$ and $\sum_{t\in G'} y_t = \min\{a \in FS(\langle y_n\rangle_{n=1}^\infty) : a > \sum_{t\in H'} y_t\}$. Let $r = \min(\mathbb{N}\setminus H')$. Then by Lemma 2.3, $G' = \{r\} \cup \{t \in H' : t > r\}$. Then $G = G' + m - 1 = \{s\} \cup \{t \in H : t > s\}$. □

Lemma 2.12. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$. Let $m \in \mathbb{N}$ and let $L \in \mathcal{P}_f(\mathbb{N})$ with $\min L \geq m$ and $L \neq \{m\}$. If $K \in \mathcal{P}_f(\mathbb{N})$, $\min K \geq m$,*

$$\sum_{n\in K} x_n = \max\left\{ \sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N}), \min F \geq m, \text{ and } \sum_{n\in F} x_n < \sum_{n\in L} x_n \right\},$$

and $l = \min L$, then $K = (L\setminus\{l\}) \cup \{t \in \mathbb{N} : m \leq t < l\}$.

Proof. Note that

$$\sum_{n\in L} x_n = \min\left\{\sum_{n\in F} x_n : F\in\mathcal{P}_f(\mathbb{N}), \min F\geq m, \text{ and } \sum_{n\in F} x_n > \sum_{n\in K} x_n\right\},$$

Let $s=\min\{t\in\mathbb{N}\setminus K : t\geq m\}$. Then by Lemma 2.11, $L=\{s\}\cup\{t\in K : t>s\}$. Therefore,

$$l=\min L = s = \min\{t\in\mathbb{N}\setminus K : t\geq m\})$$

and so it can easily be verified that

$$K=(L\backslash\{l\})\cup\{t\in\mathbb{N} : m\leq t<l\}. \qquad \square$$

The notations $F(m,y)$ and $G(m,y)$ defined below do not reflect the fact that they also depend on the sequence $\langle x_n\rangle_{n=1}^{\infty}$.

Definition 2.13. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n\in\mathbb{N}$, $x_{n+1}>\sum_{t=1}^{n} x_t$. Let $m\in\mathbb{N}$ and let $y\in\mathbb{N}$ such that $x_m<y$. Then $F(m,y)$ and $G(m,y)$ are finite nonempty subsets of $\mathbb{N}$ such that $\min F(m,y)\geq m$, $\min G(m,y)\geq m$, and

$$\sum_{n\in F(m,y)} x_n = \min\left\{\sum_{n\in F} x_n : F\in\mathcal{P}_f(\mathbb{N}), \min F\geq m, \text{ and } y\leq \sum_{n\in F} x_n\right\}$$

and

$$\sum_{n\in G(m,y)} x_n = \max\left\{\sum_{n\in F} x_n : F\in\mathcal{P}_f(\mathbb{N}), \min F\geq m, \text{ and } \sum_{n\in F} x_n < y\right\}.$$

We omit the routine proof of the following lemma.

Lemma 2.14. *Let $d\in\mathbb{N}$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n\in\mathbb{N}$, $x_{n+1}=\sum_{t=1}^{n} x_t+\left\lfloor\frac{n+(d-1)}{d}\right\rfloor$. Let $m\in\mathbb{N}$ with $m\geq 2$. Then*

$$\lim_{y\to\infty}\frac{\sum_{n\in G(m,y)} x_n}{y} = \lim_{y\to\infty}\frac{\sum_{n\in F(m,y)} x_n}{y} = 1\,.$$

We are now able to establish that the asymptotic densities of the finite sums from our sequences exist.

Theorem 2.15. *Let $d,r\in\mathbb{N}$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n\in\mathbb{N}$, $x_{n+1}=\sum_{t=1}^{n} x_t+\lfloor\frac{n+(d-1)}{d}\rfloor$. Then*

$$d\big(FS(\langle x_n\rangle_{n=r}^{\infty})\big)=\frac{(2^d-1)2^{2-r}}{(2^d-1)x_1+2^d}\,.$$

Proof. Let $\alpha = \dfrac{(2^d-1)2^{2-r}}{(2^d-1)x_1+2^d}$. Let $\epsilon > 0$ be given such that $\epsilon < \alpha$. Since $\lim\limits_{m\to\infty} \dfrac{2^{m-r}}{x_m} = \alpha$ by Theorem 2.6, pick $m \in \mathbb{N}$ such that $m > r$ and for all $k \geq m$, $\left|\dfrac{2^{k-r}}{x_k} - \alpha\right| < \dfrac{\epsilon}{2}$. By Lemma 2.14, since

$$\lim_{y\to\infty} \frac{\sum_{t\in G(m,y)} x_t}{y} = \lim_{y\to\infty} \frac{\sum_{t\in F(m,y)} x_t}{y} = 1\,,$$

pick $s \in \mathbb{N}$ such that if $y \geq s$, then

$$1 - \frac{\epsilon}{2\alpha-\epsilon} < \frac{\sum_{t\in G(m,y)} x_t}{y} < \frac{\sum_{t\in F(m,y)} x_t}{y} < 1 + \frac{\epsilon}{2\alpha+\epsilon}.$$

Let $y > \max\{x_m, s\}$. Then

$$\begin{aligned}
\frac{|FS(\langle x_n\rangle_{n=r}^{\infty}) \cap \{1,2,\ldots,y\}|}{y} &\leq \frac{\left|FS(\langle x_n\rangle_{n=r}^{\infty}) \cap \{1,2,\ldots,\sum_{t\in F(m,y)} x_t\}\right|}{y} \\
&= \frac{\sum_{t\in F(m,y)} 2^{t-r}}{y} \text{ by Lemma 2.5} \\
&= \frac{\sum_{t\in F(m,y)} 2^{t-r}}{\sum_{t\in F(m,y)} x_t} \cdot \frac{\sum_{t\in F(m,y)} x_t}{y} \\
&\leq \frac{2^{m-r}}{x_m} \cdot \frac{\sum_{t\in F(m,y)} x_t}{y} \text{ by Lemma 2.9} \\
&< \left(\alpha + \frac{\epsilon}{2}\right)\left(1 + \frac{\epsilon}{2\alpha+\epsilon}\right) \\
&= \alpha + \epsilon\,.
\end{aligned}$$

Let $k = \max G(m,y)$. Then

$$\begin{aligned}
\frac{|FS(\langle x_n\rangle_{n=r}^\infty)\cap\{1,2,\ldots,y\}|}{y} &\geq \frac{\left|FS(\langle x_n\rangle_{n=r}^\infty)\cap\{1,2,\ldots,\sum_{t\in G(m,y)} x_t\}\right|}{y}\\
&= \frac{\sum_{t\in G(m,y)} 2^{t-r}}{y} \text{ by Lemma 2.5}\\
&= \frac{\sum_{t\in G(m,y)} 2^{t-r}}{\sum_{t\in G(m,y)} x_t}\cdot\frac{\sum_{t\in G(m,y)} x_t}{y}\\
&\geq \frac{2^{k-r}}{x_k}\cdot\frac{\sum_{t\in G(m,y)} x_t}{y} \text{ by Lemma 2.10}\\
&> \left(\alpha-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{2\alpha-\epsilon}\right)\\
&= \alpha-\epsilon\,.
\end{aligned}$$

□

Corollary 2.16. *For each $\epsilon > 0$ there exists $d \in \mathbb{N}$ such that, if $x_1 = 1$ and for each $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^n x_t + \lfloor\frac{n+d-1}{d}\rfloor$, then $d(FS(\langle x_n\rangle_{n=1}^\infty)) > 1-\epsilon$.*

Proof. Pick $d \in \mathbb{N}$ such that $\frac{1}{2^{d+1}-1} < \epsilon$. Then by Theorem 2.15 $d\big(FS(\langle x_n\rangle_{n=1}^\infty)\big) = \frac{(2^d-1)2}{(2^d-1)+2^d} = 1-\frac{1}{2^{d+1}}$. □

Notice that, as a consequence of Theorem 2.1, if $\langle x_n\rangle_{n=1}^\infty$ is a sequence such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^n + \lfloor\frac{n+(d-1)}{d}\rfloor$ and $T = \bigcap_{m=1}^\infty \overline{FS(\langle x_n\rangle_{n=m}^\infty)}$, then $T \cap c\ell K(\beta\mathbb{N}) = \emptyset$.

As a means to proving that $T \cap \Delta^* \neq \emptyset$, we introduce the definition of a partition regular family of sets and record a theorem which characterizes this notion.

Definition 2.17. Let $\mathcal{R}$ be a nonempty set of sets. We say that $\mathcal{R}$ is *partition regular* if and only if whenever $\mathcal{F}$ is a finite set of sets and $\bigcup\mathcal{F} \in \mathcal{R}$, there exists $A \in \mathcal{F}$ and $B \in \mathcal{R}$ such that $B \subseteq A$.

Theorem 2.18. *Let D be a set and let $\mathcal{R} \subseteq \mathcal{P}(D)$ be nonempty and assume that $\emptyset \notin \mathcal{R}$. Let $\mathcal{R}^\uparrow = \{B \in \mathcal{P}(D) : A \subseteq B$ for some $A \in \mathcal{R}\}$. Then $\mathcal{R}$ is partition regular if and only if whenever $\mathcal{A} \subseteq \mathcal{P}(D)$ has* *the property that every finite nonempty subfamily of $\mathcal{A}$ has an intersection which is in $\mathcal{R}^{\uparrow}$, there is an ultrafilter p on D such that $\mathcal{A} \subseteq p \subseteq \mathcal{R}^{\uparrow}$.*

Proof. [6, Theorem 3.11]. □

Lemma 2.19. *Let $\mathcal{R} = \{A \subseteq \mathbb{N} : d^*(A) > 0\}$. Then $\mathcal{R}$ is partition regular.*

Proof. Let $\mathcal{F}$ be a finite set of sets such that $\bigcup \mathcal{F} \in \mathcal{R}$. We note here that $\mathcal{R} = \mathcal{R}^{\uparrow}$. Since $\bigcup \mathcal{F} \in \mathcal{R}$, $d^*(\bigcup \mathcal{F}) > 0$. By [6, Lemma 20.2], $0 < d^*(\bigcup \mathcal{F}) \leq \sum_{A \in \mathcal{F}} d^*(A)$. Therefore for some $A \in \mathcal{F}$, $d^*(A) > 0$. □

Lemma 2.20. *Let $d \in \mathbb{N}$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$ and let $T = \bigcap_{m=1}^{\infty} \overline{FS(\langle x_n \rangle_{n=m}^{\infty})}$. Then $T \cap \Delta^* \neq \emptyset$.*

Proof. Let $\mathcal{A} = \{FS(\langle x_n \rangle_{n=m}^{\infty}) : m \in \mathbb{N}\}$. Let $F \in \mathcal{P}_f(\mathbb{N})$. Pick $r = \max F$. Then $\bigcap_{m \in F} FS(\langle x_n \rangle_{n=m}^{\infty}) = FS(\langle x_n \rangle_{n=r}^{\infty})$. By Theorem 2.15,

$$d\big(FS(\langle x_n \rangle_{n=r}^{\infty})\big) = \frac{(2^d - 1)2^{2-r}}{(2^d - 1)x_1 + 2^d} > 0\,.$$

Since $0 < d\big(FS(\langle x_n \rangle_{n=r}^{\infty})\big) \leq d^*\big(FS\langle x_n \rangle_{n=r}^{\infty}\big)$, $\bigcap_{m \in F} FS(\langle x_n \rangle_{n=m}^{\infty}) \in \mathcal{R}$. Since $\mathcal{R}$ is partition regular by Lemma 2.19, we have by Theorem 2.18 an ultrafilter p on $\mathbb{N}$ such that $\mathcal{A} \subseteq p \subseteq \mathcal{R}$. Therefore $p \in T$ and for all $A \in p$, $d^*(A) > 0$. □

Recall that the semigroup $\mathbb{H}$ contains all of the idempotents of $\beta\mathbb{N}$ and has substantial algebraic structure.

Theorem 2.21. *Let $d \in \mathbb{N}$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $x_{n+1} = \sum_{t=1}^{n} x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$ and let $T = \bigcap_{m=1}^{\infty} \overline{FS(\langle x_n \rangle_{n=m}^{\infty})}$. Then $T \cap \Delta^*$ is a subsemigroup of $\beta\mathbb{N}$ which misses $K(\beta\mathbb{N})$ and contains a topological and algebraic copy of $K(\mathbb{H})$.*

Proof. By [6, Theorem 6.27] T is a subsemigroup of $\beta\mathbb{N}$ and there exists $\tau : \mathbb{H} \to T$ which is an isomorphism and a homeomorphism. By [6, Theorems 20.5 and 20.6] Δ^* is an ideal of $\beta\mathbb{N}$. By Lemma 2.20 $T \cap \Delta^* \neq \emptyset$, so $T \cap \Delta^*$ is an ideal of T and thus contains $K(T) = \tau[K(\mathbb{H})]$. □

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