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SETS AND POINTS OF COUNTABLE WEAK CHARACTER IN COMPACTA

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Abstract. A closed subset C of a compactum X is said to have countable weak character in X if there exists a countable family $\mathcal{F}$ of closed sets in X such that $\cap\mathcal{F} = C$ and every non-compact set S that is closed in $X \setminus C$ meets all elements of $\mathcal{F}$. We show that if X is a sequential compactum and the diagonal Δ_X has countable weak character in X^2 then X is metrizable. We also prove that a weakly perfect sequential compactum is perfect.

1. Introduction

In this paper we will consider only Tychonoff spaces. In notation and terminology we will follow [ENG]. In [AR1], the first author introduced the notion of weak first countability. Recall that a space X is *weakly first-countable* if for every point $x \in X$ there exists a countable family $\mathcal{B}_x$ of closed subsets containing x such that $U \subset X$ is open whenever for every $x \in U$ there exists $B_x \in \mathcal{B}_x$ such that $B_x \subset U$. Yakovlev [YAK] distinguished weak first countability from first countability in the class of compact spaces (consistently). The definition of first countability suggests a quest for an acceptable definition of "countable weak character" at a point. In [AR2], the first author gives a natural definition for this notion. However, it is not clear yet if having countable weak character at every point

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is equivalent to weak first countability for compact spaces. Let us give the definition from [AR2], which is designed especially for compacta.

Given a compact space X and its closed subset $A \subset X$, we say that F is *stationary* at A if F is closed and meets every non-compact S that is closed in $X \setminus A$. Given a compact space X and its closed subset $C \subset X$, the *weak character* of C in X is the smallest cardinal number τ such that there exists a τ-sized family $\mathcal{F}$ of sets stationary at C such that $\bigcap \mathcal{F} = C$. Such a family $\mathcal{F}$ will be called a *weak base* at C. The weak character of a point x in X is the weak character of $\{x\}$ in X. A compactum X is *weakly perfect* if every closed subset of X has countable weak character in X.

As one can see, elements of a weak base at a point $x \in X$ share a very strong property with open sets, namely, they block all approaches to x from outside. As we mentioned earlier we do not know if for compact spaces the definition of countable weak character naturally agrees with that of weak first countability. The following two results speak in favor of "yes". For the first result recall that a space X is *sequential* if for every non-closed subset A of X there exists a point $x \in \overline{A} \setminus A$ and a sequence $\{x_n\}_n$ of elements of A that converges to x.

Theorem 1.1. *A compactum X is weakly first-countable iff X is sequential and every x in X has countable weak character.*

Proof. ($\Rightarrow$) Sequentiality is proved in [AR1]. For each $x \in X$ fix countable $\mathcal{B}_x$ as in the definition of weak first countability. Let us show that $\mathcal{B}_x$ is a weak base at x. It is clear that $\bigcap \mathcal{B}_x = \{x\}$. We need to show that every $B \in \mathcal{B}_x$ is stationary at x. Assume the contrary. Then there exist a non-compact set $A \subset X \setminus \{x\}$ that is closed in $X \setminus \{x\}$ and an element $B_x \in \mathcal{B}_x$ that does not meet A. Put $U = X \setminus A$. The set $W = X \setminus (A \cup \{x\})$ is open and is equal to $U \setminus \{x\}$. Therefore, for every $y \in W$ there exist $B_y \in \mathcal{B}_y$ such that $B_y \subset W \subset U$. Since $B_x \cap A = \emptyset$ we have $B_x \subset U$. By the definition of weak first countability, U is open. This contradicts the fact that x is a limit point for A.

($\Leftarrow$) For each $x \in X$ fix a countable weak base $\mathcal{B}_x$ at x.

Fix an open set U and $x \in U$. Since X is compact, $\bigcap \mathcal{B}_x = \{x\}$ and each element of $\mathcal{B}_x$ is closed there exists an element $B_x \in \mathcal{B}_x$ such that $B_x \subset U$.

Now fix any $W \subset X$. Assume that for each $x \in W$ there exists $B_x \in \mathcal{B}_x$ such that $B_x \subset W$. We need to show that W is open. Assume the contrary. By sequentiality, there exists a sequence $\{a_n\}_n$ of elements of $X \setminus W$ that converges to a point $x \in W$. Then the set $\{a_n : n \in \omega\}$ of terms of the sequence forms a non-compact set in $X \setminus \{x\}$ that does not meet B_x. This contradicts stationarity of B_x at x. □

According to Theorem 1.1, to show that the definition of countable weak character agrees with the definition of weak first countability we only need to prove that countable weak character at each point implies sequentiality. Our next observation is a convincing step in this directions. First, recall that $S = \{x_\alpha : \alpha < \omega_1\}$ is an ω_1-long free sequence in X if $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha > \beta\}} = \emptyset$ for every $\beta < \omega_1$.

Theorem 1.2. *Let X be a compactum. Suppose that every $x \in X$ has countable weak character. Then X is countably tight.*

Proof. Assume the contrary. Then by Juhász-Szentmiklóssy theorem [J&S] there exists a free sequence $S = \{x_\alpha : \alpha < \omega_1\}$ that has a single complete accumulation point x in X. The set $A = \overline{S} \setminus \{x\}$ is non-compact and closed in $X \setminus \{x\}$. Let $\mathcal{B}_x$ be a fixed countable weak base at x. Since each B in $\mathcal{B}_x$ is stationary at x, $B \cap A \neq \emptyset$. Fix $a_B \in A \cap B$ for every $B \in \mathcal{B}_x$. Then x is a limit point for $\{a_B : B \in \mathcal{B}_x\}$. However, each a_n is a limit point for some countable subset of S. Therefore, x is a limit point for a countable subset of S, a contradiction with S being a free sequence. □

Since there are models of ZFC in which every countably tight compactum is sequential (see [BAL] and [DO1]) we have the following result.

Corollary 1.3. *It is consistent with the axioms of ZFC that a compactum X is weakly first-countable iff every x in X has countable weak character.*

Question 1.4. *Let X be a compactum that has countable weak character at every point. Is X weakly first-countable?*

As we mentioned earlier, to answer Question 1.4 in affirmative we only need to show that countable weak character at every point in a compact space X implies sequentiality. In connection with this observation it might be worth to stress the importance of closeness (or smallness) of elements of a weak base. This is demonstrated by the next example.

Example 1.5. *There exists a compactum X such that every point x of X has uncountable weak character, uncountable tightness, and a countable family $\mathcal{B}_x$ with the following properties:*

(1) $\bigcap \mathcal{B}_x = \{x\}$;
(2) *Every $B \in \mathcal{B}_x$ meets every non-compact S that is closed in $X \setminus \{x\}$.*

Proof. Let $\omega^* = \beta\omega \setminus \omega$. It is clear that no point of ω^* has countable weak character or countable tightness. Let $\{A_n\}_n$ be a collection of disjoint subsets of ω^* such that $\bigcup_n A_n = \omega^*$ and every infinite closed subset of ω^* meets every A_n. For each $x \in \omega^*$, let $B_n(x) = \bigcup_{i>n}[A_i \cup \{x\}]$. Put $\mathcal{B}_x = \{B_n(x)\}_n$. The properties 1 and 2 are clearly met. □

Even if one finds a model of ZFC in which Question 1.4 has a negative answer, we still think that the definition of countable weak character deserves attention (under a different name maybe). Since we believe in the theorem we will keep the name of the notion we are about to study. In [AR2], the first author concentrated on weak character at special points. We will study countable weak character at special closed sets in compacta.

It is a classical theorem of Sneider [SNE] that if X is compact and the diagonal $\Delta_X = \{(x, x) : x \in X\}$ has countable character in $X \times X$ then X is metrizable. We will show that in Sneider theorem "countable character" can be replaced by "countable weak character" if we restrict the theorem to sequential compacta (Theorem 2.6). We also show that a weakly perfect sequential compactum is perfect (Theorem 2.2). Of course, sequentiality will be dropped from the hypothesis of each theorem if an answer to Question 1.4 is "yes". The proofs of these two theorems, which are the main results of this work, are based on a new fact (to be proved too) that every hereditarily ω_1-Lindelöf space is hereditarily Lindelöf. Recall that a space X is ω_1-*Lindelöf* if every ω_1-sized subset A of X has a *complete accumulation point* in X, that is, a point whose every open neighborhood contains ω_1 many elements of A. This property is equivalent to the property that every ω_1-sized open cover of X contains a countable subcover. Although the fact we mentioned is easy to prove it shows itself very handy in proving Lindelöf property.

2. Study

For our main results we need the following theorem.

Theorem 2.1. *Every hereditarily ω_1-Lindelöf space X is hereditarily Lindelöf.*

Proof. Assume the contrary. Then there exists a strictly increasing family $\{U_\alpha : \alpha < \omega_1\}$ of open sets in X. For each $\alpha < \omega_1$, fix $x_\alpha \in U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. Clearly, no point in U_α for $\alpha < \omega_1$ can be a complete accumulation point for $\{x_\alpha : \alpha < \omega_1\}$, contradicting ω_1-Lindelöfness of $\bigcup_\alpha U_\alpha$. □

Theorem 2.2. *Let X be a sequential compactum. If X is weakly perfect then X is perfect.*

Proof. We will reach the conclusion by showing that X is hereditarily Lindelöf. Fix F an ω_1-sized subset of X. Let C be the set of all complete accumulation points for F in X. By virtue of Theorem 2.1, it suffices to show that F meets C.

Assume the contrary. Then $F \setminus C$ is uncountable. Since C has countable weak character in X and $F \setminus C$ is uncountable, there exists $B_1 \subset X$ stationary at C such that $F \setminus B_1$ has cardinality ω_1. Inductively for each $n > 1$, define B_n with the following properties:

P1: B_n is stationary at B_{n-1};
P2: $F \setminus B_n$ has cardinality ω_1.

Let us make three remarks.

Remark 1. The set $B = \bigcup_n B_n$ is open. Indeed, fix $x \in B$. By Theorem 1.1 we need to show that B contains a set stationary at x. There exists n such that $x \in B_n$. Let us show that $B_{n+1} \subset B$ is stationary at x. Assume the contrary. Then there exists a non-compact $A \subset X \setminus \{x\}$ that is closed in $X \setminus \{x\}$ and does not meet B_{n+1}. Since $B_n \subset B_{n+1}$, A is closed in $X \setminus B_n$. This contradicts stationarity of B_{n+1} at B_n which is guaranteed by P1.

Remark 2. For infinitely many n's, the set $F_n = (B_n \setminus B_{n-1}) \cap F$ has cardinality ω_1. Indeed, B is open by Remark 1. Since $C \subset B$, the set $F \setminus B$ is countable. Now apply property P2.

Remark 3. If $x_n \in F_n = (B_n \setminus B_{n-1}) \cap F$ for each n. Then at least one limit point for $S = \{x_n : n \in \omega\}$ belongs to $X \setminus B$. Indeed, by sequentiality, elements of an infinite subset of S form a sequence $\{x_m\}_m$ converging to some x. Let us show that $x \in X \setminus B$. Assume the contrary. Then $x \in B_n$ for some n. We may assume that each x_m is in F_k for $k > n+1$. However, this contradicts stationarity of B_{n+1} at x, which is proved in Remark 1.

By Remark 2, there exists an ω_1-sized family $\mathcal{S}$ of countable sets with the following properties:

(1) $S_1 \cap S_2 = \emptyset$ for any distinct $S_1, S_2 \in \mathcal{S}$;
(2) Every $S \in \mathcal{S}$ meets F_n for infinitely many n's.

By Remark 3, for each $S \in \mathcal{S}$ there exists $x_S \in X \setminus B$ a limit point for S. If x_S is the same for uncountably many $S \in \mathcal{S}$, then by properties 1 and 2, x_S is a complete accumulation point for F lying off C, a contradiction. Now we may assume that $x_S \neq x_P$ for distinct S and P in $\mathcal{S}$. By compactness of $X \setminus B$, there exists $x \in X \setminus B$ a complete accumulation point for $\{x_S : S \in \mathcal{S}\}$. By properties 1 and 2, x is a complete accumulation point for F lying off C, a contradiction. □

For our next main result we need the following three lemmas. Since the proof of the first one is obvious it is omitted.

Lemma 2.3. *Let X be compact. Let $\{F_n\}_n$ be a weak base at Δ_X in X^2. For $x \in X$, let $B_n(x) = \{y : (x,y) \in F_n\}$. Then $\{B_n(x)\}_n$ forms a weak base at x in X.*

Lemma 2.4. *Let X be compact and let Δ_X have countable weak character in X^2. Then there exists a countable weak base $\{D_n\}_n$ at Δ_X such that each D_n is symmetric with respect to the diagonal. That is, if $(x,y) \in D_n$ then $(y,x) \in D_n$.*

Proof. Let $\{F_n\}_n$ be an arbitrary countable weak base at Δ_X. For each n, let F_n' be the reflection of F_n in the diagonal. That is, $F_n' = \{(y,x) : (x,y) \in F_n\}$. Clearly, the reflection of a closed set is closed. Put $D_n = F_n \cup F_n'$. Each D_n is closed as the union of two closed sets. Now let S be a non-compact set that lies off the diagonal and is closed in $X^2 \setminus \Delta_X$. Since F_n is stationary at Δ_X, S meets every F_n, and therefore, every $D_n = F_n \cup F_n'$.

We only need to show that $\Delta_X = \bigcap_n D_n$. Assume the contrary and fix $(x, y) \in \bigcap_n D_n$, where $x \neq y$. Then, $(x, y) \in F'_n$ for all n. Therefore, $(y, x) \in \bigcap_n F_n$, a contradiction. □

Lemma 2.5. *Let X be a compact space and let Δ_X have countable weak character in X^2. If X is first-countable then X is metrizable.*

Proof. Let $\{D_n\}_n$ be a countable weak base at Δ_X in X^2. By Sneider's theorem [SNE], it suffices to show that the interior of each D_n contains Δ_X. Fix $x \in X$. Let $\{O_i\}_i$ be a base at x. We need to show that there exists open O_i such that $O_i \times O_i \subset D_n$. Assume no such i exists. Then for each i there exists $(x_i, y_i) \in O_i \times O_i \setminus D_n$. The set $S = \{(x_i, y_i) : i \in \omega\}$ is a non-compact subset that lies off the diagonal and is closed there. By our construction, S does not meet D_n, a contradiction with stationarity of D_n at Δ_X. □

Observe that in the proof of Lemma 2.5 we do not use closeness of elements of weak base at Δ_X. This little detail will be used later. We will prove now our second main result.

Theorem 2.6. *Let X be a sequential compact space. If Δ_X has countable weak character in X^2 then X is metrizable.*

Proof. Fix $\{D_n\}_n$ a countable weak base at Δ_X that consists of symmetric sets (Lemma 2.4). For each $x \in X$, let $B_n(x) = \{y : (x, y) \in D_n\}$.

By Lemma 2.5, it suffices to show that X is first-countable. By Theorem 2.1, it suffices to show that X is hereditarily ω_1-Lindelöf. Fix an arbitrary ω_1-sized collection $\{U_\alpha\}_{\alpha<\omega_1}$ of opens sets in X.

Let $W_\alpha = U_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}$. Let $S_{\alpha,n} = \{x \in W_\alpha : B_n(x) \subset U_\alpha\}$. Since $B_n(x)$'s form a weak base at x, the set $\bigcup_n S_{\alpha,n}$ cover W_α. Our goal is to show that there exists $\alpha^* < \omega_1$ such that $W_\alpha = \emptyset$ for all $\alpha > \alpha^*$. To reach the goal we need to find $\alpha^* < \omega_1$ such that $S_{\alpha,n} = \emptyset$ for all $\alpha > \alpha^*$ and all n.

Fix n. We claim that $S_{\alpha,n}$ is not empty only for finitely many indices α. Assume the contrary and let $\{\alpha_k\}_k$ be a strictly increasing sequence such that $S_{\alpha_k,n} \neq \emptyset$ for all k. For each k, fix $x_k \in S_{\alpha_k,n}$. By compactness and sequentiality of X, we may assume that the sequence $\{x_k\}_k$ converges to some x in X. Since x_{k+1} lies outside of $\bigcup\{U_\alpha : \alpha \leq \alpha_k\}$, the limit x belongs to $X \setminus \bigcup\{U_\alpha : \alpha < \sup\{\alpha_k\}_k\}$. Since the set $\{x_k : x \in \omega\}$ is non-compact and closed in $X \setminus \{x\}$, the set $B_n(x)$ contains almost all x_k's. Assume $x_5 \in B_n(x)$.

Then $(x, x_5) \in D_n$. By symmetry, $(x_5, x) \in D_n$. Therefore, $x \in B_n(x_5)$. Therefore, $x \in U_{\alpha_5}$, a contradiction. □

In the rest of the paper we will give a number of observations related to the main results. In Theorem 2.6, we use all aspects of stationarity including closeness. Following classical approaches it is natural to drop the requirements on closeness of elements of weak base of the diagonal. A compactum X has a *thin diagonal* if there exists a family $\{F_n\}_n$ of subsets of X^2 such that $\Delta_X = \bigcap_n F_n$ and any non-compact subset $S \subset X^2 \setminus \Delta_X$ closed in $X^2 \setminus \Delta_X$ meets all F_n's. Such a family of F_n's will be called a *thin diagonal sequence.*

Theorem 2.7. *Let X be a compactum with a thin diagonal. If X is Frechet-Uryson then X is metrizable.*

Proof. Fix a thin diagonal sequence $\{D_n\}_n$. Let us show that X is first-countable. Fix $x \in X$. Let $B_n(x) = \{y : (x, y) \in D_n\}$. It suffices to show that the interior of $B_n(x)$ contains x for all n. If the set $\overline{X \setminus B_n(x)}$ does not contain x for each n, we are done. Otherwise, by Frechet-Uryson property, there exists a sequence $\{y_k\}_k$ of elements of $X \setminus B_n(x)$ that converges to x for some n. Then the set $\{(x, y_n) : n \in \omega\}$ is a non-compact subset closed in $X^2 \setminus \Delta_X$ that does not meet D_n, a contradiction.

As we pointed out after Lemma 2.5, the conclusion of Lemma 2.5 holds even if elements of weak base of the diagonal are not closed. Therefore, we are done. □

Theorem 2.8. *It is consistent with the axioms of ZFC that every homogeneous compactum of countable tightness with a thin diagonal is metrizable.*

Proof. It is known that there are models of ZFC in which every countably tight compactum has a point of countable character (see [DO2]). Therefore, in such models every homogeneous compactum of countable tightness is Frechet-Uryson. Now, apply Theorem 2.7. □

It is important to point out that some convergence-type properties must be assumed when dealing with thin diagonals. This is witnessed by the following example.

Example 2.9. *There exists a compactum X which is not countably tight and has a thin diagonal.*

Proof. Let $\omega^* = \beta\omega\backslash\omega$. Let $\{A_n\}_n$ be a collection of disjoint subsets of $\omega^*\times\omega^*$ such that $\bigcup_n A_n = \omega^*\times\omega^*$ and every infinite closed subset of $\omega^* \times \omega^*$ meets every A_n. Put $D_n = \bigcup_{i>n}[A_i \cup \Delta_{\omega^*}]$. □

Question 2.10. *Is there a sequential (or countably tight) compactum with a thin diagonal which is not metrizable?*

One way to construct such an example is answering the following question in affirmative.

Question 2.11. *Is there a Yakovlev locally compact space with a G_δ-diagonal?*

By a Yakovlev locally compact space X we understand a first-countable locally compact space whose one-point compactification has countable weak character and uncountable character. To show a connection between the above two questions we need the following lemma.

Lemma 2.12. *Let X and Y be compact, $a \in X$ and $b \in Y$. If F_a and F_b are stationary at a and b, respectively, then $F_a \times F_b$ is stationary at (a, b) in $X \times Y$.*

Proof. Let S be non-compact and closed in $X \times Y \setminus \{(a, b)\}$. Then, $(a, b) \in \overline{S}$. If $\{y : (a, y) \in S\}$ is not compact then S meets $\{a\}\times F_b \subset F_a \times F_b$, and we are done.

Now we may assume that no element of S shares a single coordinate with (a, b). We need to show that there exists $x \in F_a$ and $y \in F_b$ such that $(x, y) \in S$. Assume that no such pair exists. Let $B = \{y : (x, y) \in S \ for\ some\ x \in F_a\}$. Let us show that B is compact. Notice that $B \cup \{b\}$ is the image of the projection of $[F_a \times Y] \cap [S \cup \{(a, b\}]$ to the second coordinate axis. Due to compactness of $S \cup \{(a, b)\}$ and continuity of projections, $B \cup \{b\}$ is compact. Since B is closed in $Y \setminus \{b\}$ and, by our assumption, does not meet F_b, it has to be compact. Similarly, the set $A = \{x : (x, y) \in S \ for\ some\ y \in F_b\}$ is compact not containing a. Take open sets $U \ni a$ and $V \ni b$ whose closures do not meet A and B, respectively. Pick any $(x, y) \in S \cap (\overline{U \times V})$. Let us show that $x \notin F_a$. If x were in F_a then y would have been in B but $y \in \overline{V} \subset Y \setminus B$. Therefore, the projections of $S \cap (\overline{U \times V})$ to the first coordinate axis does not meet F_a, a contradiction with stationarity of F_a at a. □

The next lemma establishes the promised relation between Questions 2.10 and 2.11.

Lemma 2.13. *Let X be a Yakovlev locally compact space with a G_δ-diagonal. Then the one-point compactification $X \cup \{\infty\}$ of X has a thin diagonal and is sequential.*

Proof. Put $Y = X \cup \{\infty\}$. Let $\{F_n\}_n$ be a nested weak base at ∞ in Y. To prove sequentiality, fix any non-closed $A \subset Y$. If $A \cap X$ is not closed in X then apply first countability of X. Otherwise, we may assume that $A = X$. In this case, select $x_n \in F_n \setminus \{\infty\}$. Clearly, $x_n \to \infty$.

Now, let us prove that Y has a thin diagonal. Let $\{U_n\}_n$ be a nested family of open subsets of X^2 such that $\bigcap_n U_n = \Delta_X$. Let $D_n = U_n \cup (F_n \times F_n)$. Let us show that $\bigcap_n D_n = \Delta_Y$. Fix (x, y) off the diagonal. There exists n such that $(x, y) \notin U_n$. There exists m such that $x, y \notin F_m$. Let $k = \max\{n, m\}$. Then $(x, y) \notin D_k$.

Now let us show that every non-compact set S closed in $Y^2 \setminus \Delta_Y$ meets every D_n. If $(x, x) \in \Delta_Y \setminus \{(\infty, \infty)\}$ is limit for S then S meets each U_n since they are open neighborhoods of (x, x). Otherwise, (∞, ∞) is the only limit point for S on the diagonal. By Lemma 2.12, S meets every F_n. □

Our last observation in this work is an answer to a question posed in [AR2]. A locally compact space X is said to have *countable weak character at infinity* if the one point compactification $X \cup \{\infty\}$ of x has a countable weak base at ∞.

Lemma 2.14. *Let X and Y be locally compact spaces. If X and Y have countable weak character at infinity then so does $X \times Y$.*

Proof. Let $X' = X \cup \{\infty_X\}$ and $Y' = Y \cup \{\infty_Y\}$ be the one-point compactifications of X and Y, respectively. Let $Z = (\{\infty_X\} \times Y') \cup (X' \times \{\infty_Y\})$. We need to show that Z' has countable weak character in $X' \times Y'$.

Let $\{F_n\}_n$ and $\{G_n\}_n$ be weak bases at ∞_X and ∞_Y in X' and Y', respectively. Let $D_n = (F_n \times Y') \cup (X' \times G_n)$. Clearly, $Z = \bigcap_n D_n$. Let us show that D_n is stationary at Z. Since Y' is stationary at any $y \in Y'$, by Lemma 2.12, the set $F_n \times Y'$ is stationary at any (∞_X, y). □

Let us finish this paper with several questions related to our study.

Question 2.15. *Let C be a compactum in ω^* that has countable weak character in ω^*. Does C have a non-empty interior?*

Question 2.16. *Let X be locally compact space that has countable weak character at infinity. Let Y be a locally compact continuous image of X. Does Y have countable weak character at infinity?*

Question 2.17. *Let X be a weakly first-countable compactum. Is it true in ZFC that X has a point of first countability?*

Question 2.18. *Let X be a weakly first-countable compactum. Suppose X is hereditarily separable. Is X first-countable?*

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