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DIRECTIONS OF AUTOMORPHISMS OF LIE GROUPS OVER LOCAL FIELDS COMPARED TO THE DIRECTIONS OF LIE ALGEBRA AUTOMORPHISMS

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Abstract. To each totally disconnected, locally compact topological group G and each group A of automorphisms of G, a pseudo-metric space ∂A of "directions" has been associated by U. Baumgartner and the second author. Given a Lie group G over a local field, it is a natural idea to try to define a map

$$\Phi\colon \partial\,\mathrm{Aut}_{C^\omega}(G) \to \partial\,\mathrm{Aut}(L(G)), \quad \partial\alpha \mapsto \partial L(\alpha)$$

which takes the direction of an analytic automorphism of G to the direction of the associated Lie algebra automorphism. We show that, in general, Φ is not well-defined. Also, it may happen that $\partial L(\alpha) = \partial L(\beta)$ although $\partial\alpha \neq \partial\beta$. However, such pathologies are absent for a large class of groups: we show that $\Phi\colon \partial\,\mathrm{Inn}(G) \to \partial\,\mathrm{Aut}(L(G))$ is a well-defined isometric embedding for each generalized Cayley group G. Some counterexamples concerning the existence of small jointly tidy subgroups for flat groups of automorphisms are also provided.

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Introduction

In a recent article [2], U. Baumgartner and the second author associated a pseudo-metric space ∂A of "directions" to each totally disconnected, locally compact group G and group A of (bicontinuous) automorphisms of G. The completion $\overline{\partial G}$ of the metric space associated with the space $\partial G := \partial \operatorname{Inn}(G)$ of directions of inner automorphisms generalizes familiar objects. For example, $\overline{\partial G}$ is homeomorphic to the spherical building at infinity if G is a semisimple group over a local field [2]. We recall from [2] that an automorphism α of G is said to *move to infinity* if, for all compact open subgroups $V \subseteq W$ of G, there exists a positive integer n such that $\alpha^n(V) \not\subseteq W$. Each automorphism $\alpha \in A$ which moves to infinity can be assigned a "direction" $\partial\alpha \in \partial A$, and every element of ∂A arises in this way. The first part of the article is devoted to the relations between the space of directions $\partial \operatorname{Aut}_{C^\omega}(G)$ of analytic automorphisms of a Lie group G over a local field $\mathbb{K}$ and the space of directions $\partial \operatorname{Aut}(L(G))$ of automorphisms of (the additive group of) its Lie algebra. It is a natural idea to try to define a map

$$\Phi\colon \partial \operatorname{Aut}_{C^\omega}(G) \to \partial \operatorname{Aut}(L(G)), \quad \partial\alpha \mapsto \partial L(\alpha)$$

which takes the direction of an analytic automorphism of G to the direction of the associated linear automorphism. Section 2 compiles negative results: We show that, in general, Φ is not well-defined,[1] at least if $\operatorname{char}(\mathbb{K}) > 0$ (Example 2.3). Furthermore, it may happen that $\partial L(\alpha) = \partial L(\beta)$ although $\partial\alpha \neq \partial\beta$ (see Examples 2.4 and 2.5).

Section 3 is devoted to positive results: we describe additional conditions ensuring that $\Phi\colon \partial G \to \partial\operatorname{Aut}(L(G))$ is well-defined, respectively, an isometric embedding (Proposition 3.3). These conditions are satisfied by a large class of linear algebraic groups (called "generalized Cayley groups" here), for which a well-behaved analogue of the Cayley transform is available. These groups are variants of the Cayley groups investigated recently in [14]. Notably, we shall see in Corollary 3.5 that $\Phi\colon \partial G \to \partial \operatorname{Aut}(L(G))$ is a well-defined isometric embedding if G is a general linear group, a special linear group (if $\operatorname{char}(\mathbb{K}) = 0$), an orthogonal group (if $\operatorname{char}(\mathbb{K}) \neq 2$) or the group of all invertible upper triangular $n\times n$-matrices.

[1] Not even as a map into $\overline{\partial\operatorname{Aut}(L(G))}$.

We recall two concepts from the second author's structure theory of totally disconnected groups (see [20]–[22]): Given $\alpha \in \mathrm{Aut}(G)$, its *scale* is defined as the minimum index

$$s_G(\alpha) \;:=\; \min_U \,[U : U \cap \alpha^{-1}(U)] \in \mathbb{N}\,,$$

where U ranges through the set of compact open subgroups of G. If the minimum is attained at U, then the compact open subgroup U is called *tidy for* α (see [21]; cf. [20] for equivalent earlier definitions).

These concepts play an important role in the construction of the space of directions. Notably, the scale is needed to define the pseudo-metric on $\partial\,\mathrm{Aut}(G)$. The scale also facilitates a convenient characterization: An automorphism $\alpha \in \mathrm{Aut}(G)$ moves to infinity if and only if $s_G(\alpha) > 1$ (see [2, Lemma 3]).

Following [8], an automorphism α of G is called *tidy* if G has arbitrarily small compact open subgroups which are tidy for α. Tidiness of automorphisms is a useful regularity property, which rules out many pathologies (see [8], [12]). Also for our present purposes, tidiness is of interest: it ensures that α moves to infinity if and only if so does $L(\alpha)$ (see Lemma 1.3 (c) and Lemma 1.4 (b)).

We recall from [22] that a group $\mathcal{H} \leq \mathrm{Aut}(G)$ of automorphisms is called *flat* if G has a compact, open subgroup which is tidy for each $\alpha \in \mathcal{H}$. For some purposes, flat groups in totally disconnected groups can be used as substitutes for tori in algebraic groups. They are also used in the proof that the space of directions of a semisimple algebraic group G over a local field is homeomorphic to the spherical building of G (see [2]), as mentioned above.

The second part of this article (Section 4) provides new results concerning flat groups of automorphisms of a totally disconnected group G. In particular, we describe examples of flat groups $\mathcal{H}$ of tidy automorphisms such that G does not have small subgroups which are tidy for all $\alpha \in \mathcal{H}$ simultaneously.

1. Notation and auxiliary results

In this section, we recall some definitions and basic facts which are necessary to define and discuss the space of directions $\partial\,\mathrm{Aut}(G)$ of a totally disconnected, locally compact group G. We also compile some facts concerning the scale of automorphisms of Lie groups over local fields.

1.1. On the set of compact open subgroups of a totally disconnected, locally compact group G, a metric can be defined using the formula $d(V,W) := d_+(V,W) + d_+(W,V)$, where

$$d_+(V,W) \;:=\; \log[V : V \cap W]$$

(see [2, p. 395]). Two automorphisms α and β of G are said to be *asymptotic* if the sequence $(d(\alpha^{nk}(V), \beta^{n\ell}(W)))_{n\in\mathbb{N}}$ is bounded for some (hence any) compact open subgroups $V, W \subseteq G$ and certain $k, \ell \in \mathbb{N}$ (cf. Definition 6 and Lemma 7 in [2]). In this case, we write $\alpha \asymp \beta$. Given a group $A \leq \mathrm{Aut}(G)$ of automorphisms, being asymptotic is an equivalence relation on the set $A_>$ of automorphisms moving towards infinity [2, Lemma 10]. The equivalence class of $\alpha \in A_>$ is called its *direction*. We write $\partial A := \{\partial\alpha \colon \alpha \in A_>\}$ and abbreviate $\partial G := \partial \mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the group of all inner automorphisms of G. To construct a pseudo-metric on ∂A, as an auxiliary notion define

$$\begin{aligned} &\delta_n^{V,W}(\alpha,\beta) \\ &\quad := \;\; \min\left\{ \frac{d_+(\alpha^n(V), \beta^k(W))}{n \log(s_G(\alpha))} \colon k \in \mathbb{N} \;\; \text{s.t.} \;\; s_G(\beta)^k \leq s_G(\alpha)^n \right\} \end{aligned}$$

for $\alpha, \beta \in A_>$, compact open subgroups $V, W \subseteq G$ and $n \in \mathbb{N}$. It can be shown that

$$\delta_+(\alpha,\beta) \;:=\; \limsup_{n\to\infty} \delta_n^{V,W}(\alpha,\beta) \;\in\; [0,1]\,,$$

and that $\delta_+(\alpha,\beta)$ is independent of the choice of V and W (see [2, p. 406]). By [2, Corollary 16],

$$\delta(\alpha,\beta) \;:=\; \delta_+(\alpha,\beta) + \delta_+(\beta,\alpha)$$

defines a pseudo-metric δ on $A_>$. Furthermore, a well-defined pseudo-metric on ∂A (also denoted δ) is obtained via $\delta(\partial\alpha, \partial\beta) := \delta(\alpha,\beta)$ (see [2, Lemma 17]). The completion of the metric space $\partial A/\delta^{-1}(0)$ associated with ∂A is denoted by $\overline{\partial A}$. Finally, we let $\Delta_G(\alpha)$ be the module of $\alpha \in \mathrm{Aut}(G)$, defined as $\Delta_G(\alpha) := \frac{\mu(\alpha(U))}{\mu(U)}$, where μ is a Haar measure on G and $U \subseteq G$ a non-empty, relatively compact, open set.

1.2. All Lie groups considered in this article are finite-dimensional. We use "C^ω" as a shorthand for "analytic." Beyond C^ω-Lie groups over a local field $\mathbb{K}$ (as in [18]), it is possible to define C^k-Lie groups over $\mathbb{K}$ for each $k \in \mathbb{N} \cup \{\infty\}$, based on a notion of C^k-map between open subsets of finite-dimensional (or topological) vector spaces introduced in [3].[2] Every C^ω-Lie group is also a C^k-Lie group, but the converse is valid only in zero characteristic (see [9], [10]). For the sake of added generality, we therefore formulate our results for C^1-automorphisms of C^1-Lie groups. However, all of our concrete examples will be C^ω-Lie groups, and readers unfamiliar with C^k-maps over local fields are invited to replace "C^1" by "C^ω" throughout the article.

In the next two lemmas, $\mathbb{K}$ is a local field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$ and $|.|\colon \overline{\mathbb{K}} \to [0,\infty[$ an absolute value whose restriction to $\mathbb{K}^\times$ is $\mathrm{mod}_{\mathbb{K}}$ (cf. [19, Chapter I] and [17, § 14]). "Tidiness" and "s_G" are as in the Introduction.

Lemma 1.3. *Let α be a C^1-automorphism of a C^1-Lie group G over a local field $\mathbb{K}$, and $L(\alpha) := T_1(\alpha)$ be the associated linear automorphism of the tangent space $L(G) := T_1(G)$. Then the following holds:*

(a) *Let $\lambda_1, \ldots, \lambda_{\dim L(G)} \in \overline{\mathbb{K}}$ be the eigenvalues of $\alpha \otimes_{\mathbb{K}} \overline{\mathbb{K}} \in \mathrm{GL}(L(G) \otimes_{\mathbb{K}} \overline{\mathbb{K}})$, with repetitions according to the algebraic multiplicities. Then*

$$s_{L(G)}(L(\alpha)) \;=\; \prod_{j \text{ such that } |\lambda_j|>1} |\lambda_j|\,. \tag{1.1}$$

(b) *$s_G(\alpha)$ divides $s_{L(G)}(L(\alpha))$, whence $s_G(\alpha) \le s_{L(G)}(L(\alpha))$ in particular.*

(c) *$s_G(\alpha) = s_{L(G)}(L(\alpha))$ holds if and only if α is tidy.*

(d) *If α is an inner automorphism and there exists an injective, continuous homomorphism $f\colon G \to \mathrm{GL}_n(\mathbb{K})$ for some n, then $s_G(\alpha) = s_{L(G)}(L(\alpha))$.*

(e) *If $\mathrm{char}(\mathbb{K}) = 0$ or if G is one-dimensional, then $s_G(\alpha) = s_{L(G)}(L(\alpha))$.*

Proof. See [12] for (a)–(d) and the first half of (e). To prove the second half of (e), assume that $\dim_{\mathbb{K}}(L(G)) = 1$. If $s_{L(G)}(L(\alpha)) = 1$, then also $s_G(\alpha) = 1$, by (b). Now assume that $s_{L(G)}(L(\alpha)) > 1$.

[2] By [11, Lemmas 3.2 and 4.4], a map is C^1 if and only if it is strictly differentiable at each point in the sense of [4, 1.2.2]. For C^k-maps on subsets of $\mathbb{K}$, see already [17].

Since $L(G)$ is 1-dimensional, we have $L(\alpha) = \lambda\,\mathrm{id}_{L(G)}$ for some $\lambda \in \mathbb{K}^\times$. Then $|\lambda| = s_{L(G)}(L(\alpha)) > 1$, by (1.1). Hence $L(\alpha^{-1}) = \lambda^{-1}\,\mathrm{id}_{L(G)}$ with $|\lambda^{-1}|<1$ and thus $s_{L(G)}(L(\alpha^{-1}))=1$, entailing that also $s_G(\alpha^{-1}) = 1$. Therefore

$$\begin{aligned} s_G(\alpha) &= \frac{s_G(\alpha)}{s_G(\alpha^{-1})} = \Delta_G(\alpha) = \Delta_{L(G)}(L(\alpha)) \\ &= \frac{s_{L(G)}(L(\alpha))}{s_{L(G)}(L(\alpha^{-1}))} = s_{L(G)}(L(\alpha)), \end{aligned}$$

using [20, p. 354, Corollary 1] as well as Proposition 55 from [5, Chapter III, § 3.16]. □

We remark that if $\mathrm{char}(\mathbb{K}) = 0$ or G is a Zariski-connected reductive algebraic group over $\mathbb{K}$ and α an inner automorphism, then $s_G(\alpha)$ has been expressed by the right hand side of (1.1) already in [7] and [1], respectively.

Since α moves to infinity if and only if $s_G(\alpha) > 1$, Lemma 1.3 implies:

Lemma 1.4. *Let α be a C^1-automorphism of a C^1-Lie group G over a local field $\mathbb{K}$. Then the following holds:*

(a) *If α moves to infinity, then $L(\alpha)$ moves to infinity.*
(b) *If $s_G(\alpha) = s_{L(G)}(L(\alpha))$, then α moves to infinity if and only if $L(\alpha)$ moves to infinity.*
(c) *$L(\alpha)$ moves to infinity if and only if $L(\alpha)$ has an eigenvalue λ in $\overline{\mathbb{K}}$ of absolute value $|\lambda| > 1$.*

Proof. Part (a) is an immediate consequence of Lemma 1.3 (b). Part (b) is obvious, and (c) follows from Lemma 1.3 (a). □

2. Counterexamples concerning directions

In this section, we provide examples of Lie groups over local fields with pathological properties, as announced in the introduction.

The following simple observation is useful for our discussions.

Lemma 2.1. *Let G be a totally disconnected, locally compact group, V and W be compact, open subgroups of G and $\alpha, \beta \in \mathrm{Aut}(G)_>$. If $s_G(\alpha) = s_G(\beta)$ and $\beta(W) \supseteq W$, then*

$$\delta_n^{V,W}(\alpha,\beta) \;=\; \frac{d_+(\alpha^n(V), \beta^n(W))}{n\log(s_G(\alpha))} \quad \text{for each } n \in \mathbb{N}.$$

Proof. This is clear from the definitions. □

2.2. In our first and second example, we consider Lie groups over the field $\mathbb{K} := \mathbb{F}(\!(X)\!)$ of formal Laurent series over a finite field $\mathbb{F}$ of q elements. The corresponding ring of formal power series will be abbreviated $\mathbb{O} := \mathbb{F}[\![X]\!]$. The following notations are useful: Given $z = \sum_{k=-\infty}^{\infty} a_k X^k \in \mathbb{K}$, where $(a_k)_{k\in\mathbb{Z}} \in \mathbb{F}^{(-\mathbb{N})} \times \mathbb{F}^{\mathbb{N}_0}$, we define

$$z^{(1)} := \sum_{k=1}^{\infty} a_k X^k\,; \qquad z^{(2)} := \sum_{k=1}^{\infty} a_{-k} X^{-k}\,; \qquad z^{(3)} := a_0 X^0\,;$$

$$z^{(4)} := \sum_{k=0}^{\infty} a_k X^k;\; z^{(5)} := \sum_{k=0}^{\infty} a_{-(2k+1)} X^{-(2k+1)};\; z^{(6)} := \sum_{k=1}^{\infty} a_{-2k} X^{-2k}.$$

Thus $z = z^{(1)} + z^{(2)} + z^{(3)} = z^{(4)} + z^{(5)} + z^{(6)}$.

Example 2.3. We describe analytic automorphisms $\alpha, \beta \in \mathrm{Aut}(G)_>$ of the 2-dimensional Lie group $G := (\mathbb{K}^2, +)$ such that $\partial\alpha = \partial\beta$ but $\delta(\partial L(\alpha), \partial L(\beta)) > 0$, whence $\partial L(\alpha) \neq \partial L(\beta)$ in particular.

Let α be the (linear) automorphism $\alpha\colon G \to G$, $\alpha(v,w) := (X^{-1}v, X^{-1}w)$ for $v, w \in \mathbb{K}$, and define $\beta\colon G \to G$ via

$$\beta(v,w) := \big(v^{(1)} + X^{-1}(v^{(2)} + v^{(3)}) + w^{(3)},\, X^{-2}w^{(1)} + X^{-1}w^{(2)}\big)\,.$$

It is clear that β is bijective and a homomorphism of groups; since β coincides with the linear map $\mathbb{K}^2 \to \mathbb{K}^2$, $(v,w) \mapsto (v, X^{-2}w)$ on the open zero-neighbourhood $(X\mathbb{O}) \times (X\mathbb{O})$, we deduce that β is a $\mathbb{K}$-analytic automorphism of G and $L(\beta)(v,w) = (v, X^{-2}w)$.

To see that $\delta(\partial L(\alpha), \partial L(\beta)) > 0$, note first that $s_{L(G)}(L(\alpha)) = |X^{-1}|^2 > 0$ and $s_{L(G)}(L(\beta)) = |X^{-2}| > 1$ by Lemma 1.3 (a), whence both $L(\alpha)$ and $L(\beta)$ move to infinity. For each $n \in \mathbb{N}$, we have $L(\alpha)^n(\mathbb{O}\times\mathbb{O}) = (X^{-n}\mathbb{O})\times(X^{-n}\mathbb{O})$ and $L(\beta)^n(\mathbb{O}\times\mathbb{O}) = \mathbb{O}\times X^{-2n}\mathbb{O}$ with intersection $\mathbb{O} \times X^{-n}\mathbb{O}$, whence

$$\begin{aligned} d_+(L(\alpha)^n(\mathbb{O}^2), L(\beta)^n(\mathbb{O}^2)) &= \log\,[X^{-n}\mathbb{O} : \mathbb{O}] \\ &= \log(q^n) = n\log(q). \end{aligned} \tag{2.1}$$

Since $s_{L(G)}(L(\alpha)) = s_{L(G)}(L(\beta))$ and $L(\beta)(\mathbb{O}\times\mathbb{O}) = \mathbb{O}\times X^{-2}\mathbb{O} \supseteq \mathbb{O}\times\mathbb{O}$, Lemma 2.1 can be applied. Combined with (2.1), it shows that

$$\delta_n^{\mathbb{O}^2,\mathbb{O}^2}(L(\alpha),L(\beta)) = \frac{d_+(L(\alpha)^n(\mathbb{O}^2), L(\beta)^n(\mathbb{O}^2))}{n\log s_{L(G)}(L(\alpha))} = \frac{\log(q)}{\log s_{L(G)}(L(\alpha))}.$$

Letting $n\to\infty$, we infer that $\delta_+(L(\alpha),L(\beta)) = \frac{\log(q)}{\log s_{L(G)}(L(\alpha))} > 0$. Hence $\delta(\partial L(\alpha),\partial L(\beta)) = \delta_+(L(\alpha),L(\beta)) + \delta_+(L(\beta),L(\alpha)) > 0$.

To see that α moves to infinity, let $V\subseteq W$ be compact open subgroups of G. There exist $k,\ell\in\mathbb{N}_0$ such that $X^k\mathbb{O}\times X^k\mathbb{O}\subseteq V$ and $W\subseteq X^{-\ell}\mathbb{O}\times X^{-\ell}\mathbb{O}$. Set $n := k+\ell+1$. Then $v := (0,X^k)\in V$ but $\alpha^n(v) = (0, X^{-\ell-1})\notin W$, whence $\alpha^n(V)\not\subseteq W$. An analogous argument shows that β moves to infinity. To complete our discussion, note that

$$\alpha^n(\mathbb{O}\times\mathbb{O}) = (X^{-n}\mathbb{O})\times(X^{-n}\mathbb{O}) = \beta^n(\mathbb{O}\times\mathbb{O})$$

for all $n\in\mathbb{N}$, whence $d(\alpha^n(\mathbb{O}^2),\beta^n(\mathbb{O}^2)) = 0$ for all $n\in\mathbb{N}$ and thus $\partial\alpha = \partial\beta$.

Example 2.4. We describe analytic automorphisms $\alpha,\beta\in\mathrm{Aut}(G)_>$ of the 1-dimensional Lie group $G := (\mathbb{K},+)$ such that $\delta(\partial\alpha,\partial\beta)>0$ (and hence $\partial\alpha\neq\partial\beta$), but $L(\alpha) = L(\beta)$ and thus $\partial L(\alpha) = \partial L(\beta)$.

Consider the linear automorphism $\alpha\colon G\to G$, $\alpha(z) := X^{-1}z$ and the map

$$\beta\colon G\to G\,,\quad \beta(z) := X^{-1}z^{(4)} + X^{-2}z^{(5)} + z^{(6)}\,.$$

Then β is bijective and a homomorphism, and from $\beta|_{\mathbb{O}} = \alpha|_{\mathbb{O}}$ we now deduce that β is a $\mathbb{K}$-analytic automorphism with $L(\beta) = L(\alpha)$. By Lemma 1.4 (c), $L(\beta) = L(\alpha)\colon z\mapsto X^{-1}z$ moves to infinity and hence so do α and β, by Lemma 1.3 (e) and Lemma 1.4 (b). To see that $\delta(\partial\alpha,\partial\beta)>0$, note that $\alpha^n(\mathbb{O}) = X^{-n}\mathbb{O}$ and $\beta^n(\mathbb{O}) = \sum_{k=0}^{n-1}\mathbb{F}X^{-(2k+1)} + \mathbb{O}$ for $n\in\mathbb{N}$. For n odd, say $n = 2\ell+1$, this entails that

$$\alpha^n(\mathbb{O})\cap\beta^n(\mathbb{O}) \;=\; \sum_{k=0}^{\ell}\mathbb{F}X^{-(2k+1)} + \mathbb{O}\,,$$

whence $[\alpha^n(\mathbb{O}) : \alpha^n(\mathbb{O})\cap\beta^n(\mathbb{O})] = q^\ell$ and hence

$$d_+(\alpha^n(\mathbb{O}),\beta^n(\mathbb{O})) \;=\; \log(q^\ell) \;=\; \ell\,\log(q)\,. \tag{2.2}$$

Since $s_G(\alpha) = s_{L(G)}(L(\alpha)) = s_{L(G)}(L(\beta)) = s_G(\beta)$ by Lemma 1.3 (e) and $\beta(\mathbb{O}) = \mathbb{F}X^{-1} + \mathbb{O} \supseteq \mathbb{O}$, we can apply Lemma 2.1 to see that

$$\begin{aligned}\delta_+(\alpha,\beta) &= \limsup_{n\to\infty} \frac{d_+(\alpha^n(\mathbb{O}), \beta^n(\mathbb{O}))}{n\log(s_G(\alpha))}\\ &\geq \limsup_{\ell\to\infty} \frac{d_+(\alpha^{2\ell+1}(\mathbb{O}), \beta^{2\ell+1}(\mathbb{O}))}{(2\ell+1)\log(s_G(\alpha))}\\ &= \lim_{\ell\to\infty} \frac{\ell\log(q)}{(2\ell+1)\log(s_G(\alpha))} = \frac{\log(q)}{2\log(s_G(\alpha))} > 0\,,\end{aligned}$$

where we used (2.2) to pass to the third line. Hence $\delta(\partial\alpha, \partial\beta) > 0$.

Example 2.5. We now describe a 1-dimensional p-adic Lie group G and C^ω-automorphisms $\alpha, \beta \in \mathrm{Aut}(G)_>$ such that $\delta(\partial\alpha, \partial\beta) > 0$ (and hence $\partial\alpha \neq \partial\beta$), but $L(\alpha) = L(\beta)$ and thus $\partial L(\alpha) = \partial L(\beta)$.

A suitable p-adic Lie group is $G := \mathbb{Q}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)$; the desired automorphisms are $\alpha\colon G \to G$, $\alpha(x,y) := (p^{-1}x, y)$ and

$$\beta\colon G \to G\,, \quad \beta(x,y) := \big(p^{-1}x,\, y + q(p^{-1}x)\big)\,,$$

where $q\colon \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ is the quotient map. It is clear that α and β are automorphisms; their inverses are given by $\alpha^{-1}(x,y) = (px, y)$ and $\beta^{-1}(x,y) = (px, y - q(x))$, respectively. Since α and β coincide on the open zero-neighbourhood $p\mathbb{Z}_p \times \{0\}$, we have $L(\beta) = L(\alpha)\colon \mathbb{Q}_p \to \mathbb{Q}_p$, $x \mapsto p^{-1}x$. Using Lemma 1.4 (c), we deduce that $L(\alpha) = L(\beta)$ moves to infinity. By Lemma 1.3 (e) and Lemma 1.4 (b), also α and β move to infinity.

We claim that $\alpha^n(V) \cap \beta^n(V) = V$ for each $n \in \mathbb{N}$, where $V := \mathbb{Z}_p \times \{0\}$. If this is the case, then $[\alpha^n(V) : \alpha^n(V) \cap \beta^n(V)] = [p^{-n}\mathbb{Z}_p \times \{0\} : \mathbb{Z}_p \times \{0\}] = [p^{-n}\mathbb{Z}_p : \mathbb{Z}_p] = p^n$ and thus

$$d_+(\alpha^n(V), \beta^n(V)) = n\log(p)\,.$$

Since $s_G(\alpha) = s_{L(G)}(L(\alpha)) = s_{L(G)}(L(\beta)) = s_G(\beta)$ by Lemma 1.3 (e) and $\beta(V) \supseteq \beta(p\mathbb{Z}_p \times \{0\}) = \mathbb{Z}_p \times \{0\} = V$, we can apply Lemma 2.1 to see that

$$\delta_n^{V,V}(\alpha,\beta) = \frac{d_+(\alpha^n(V), \beta^n(V))}{n\log(s_G(\alpha))} = \frac{\log(p)}{\log(s_G(\alpha))}$$

for each $n \in \mathbb{N}$. Hence $\delta(\partial\alpha, \partial\beta) \geq \delta_+(\alpha,\beta) = \frac{\log(p)}{\log(s_G(\alpha))} > 0$. It only remains to prove the claim. If $x = \sum_{k=0}^\infty a_k p^k \in \mathbb{Z}_p$ with $a_k \in \{0, 1, \ldots, p-1\}$, then

$$\beta^n(x,0) = \Big(p^{-n}x,\ \sum_{k=1}^{n}\big(\sum_{j=0}^{n-k} a_j\big)p^{-k} + \mathbb{Z}_p\Big) \tag{2.3}$$

for each $n \in \mathbb{N}$, by a simple induction. If this element is in $p^{-n}\mathbb{Z}_p \times \{0\}$, then $\sum_{k=1}^{n}\big(\sum_{j=0}^{n-k} a_j\big)p^{-k} \in \mathbb{Z}_p$. Hence $a_0 p^{-n} \equiv \sum_{k=1}^{n}\big(\sum_{j=0}^{n-k} a_j\big)p^{-k} \equiv 0$ modulo $p^{-(n-1)}\mathbb{Z}_p$ and thus $a_0 = 0$. Therefore $\sum_{k=1}^{n-1}\big(\sum_{j=1}^{n-k} a_j\big)p^{-k} \in \mathbb{Z}_p$. Repeating the argument, we find that $a_0 = a_1 = \cdots = a_{n-1} = 0$ and thus $\beta^n(x,0) \in \mathbb{Z}_p \times \{0\}$. We have shown that $\alpha^n(\mathbb{Z}_p \times \{0\}) \cap \beta^n(\mathbb{Z}_p \times \{0\}) \subseteq \mathbb{Z}_p \times \{0\}$. On the other hand, $\beta^n(\mathbb{Z}_p \times \{0\}) \supseteq \beta^n((p^n\mathbb{Z}_p) \times \{0\}) = \mathbb{Z}_p \times \{0\}$. Hence $\beta^n(\mathbb{Z}_p \times \{0\}) \cap \alpha^n(\mathbb{Z}_p \times \{0\}) = \mathbb{Z}_p \times \{0\}$, as claimed.

It is unknown whether the pathology described in Example 2.3 can occur also if $\mathrm{char}(\mathbb{K}) = 0$. If not, we could define a map $\partial\,\mathrm{Aut}(G) \to \partial\,\mathrm{Aut}(L(G)), \partial\alpha \mapsto \partial L(\alpha)$, for each p-adic Lie group G. By Example 2.5 above, this map would not always be injective.

3. Conditions ensuring that Φ is well-behaved

In this section, we describe situations where the pathologies just encountered can be ruled out. In particular, we shall see that $\Phi\colon \partial G \to \mathrm{Aut}(L(G))$ is a well-defined isometric embedding for G in a large class of linear algebraic groups (the "generalized Cayley groups"). Recall that if G is a totally disconnected, locally compact group, $\alpha \in \mathrm{Aut}(G)$ and $V \subseteq G$ a compact open subgroup, then $(\alpha^n(V))_{n\in\mathbb{N}_0}$ is called the *ray generated by α based at V* (see [2, p. 394]). The heart of this section is a technical result (the "Intertwining Lemma"), which enables us to compare rays in two Lie groups (which need not even be locally isomorphic).

As the basis for our considerations, we need some basic facts concerning the local structure of Lie groups over local fields and Haar measure on them. The following notation will be used: If $(E, \|.\|)$ is a normed space, $r > 0$ and $x \in E$, we write $B_r^E(x)$ (or simply $B_r(x)$) for the ball $\{y \in E\colon \|y - x\| < r\}$.

Lemma 3.1. *Let G be a C^1-Lie group over a local field $\mathbb{K}$ and $\phi\colon P \to Q$ be a C^1-diffeomorphism from an open identity neighbourhood $P \subseteq G$ onto an open 0-neighbourhood Q in a finite-dimensional $\mathbb{K}$-vector space E, such that $\phi(1) = 0$.*

Let $\|.\|$ be an ultrametric norm on E. Then there exists $r > 0$ such that $B_r := B_r^E(0)$ is contained in Q, and the following holds:

(a) *$W_s := \phi^{-1}(B_s)$ is a compact open subgroup of G, for each $s \in \,]0, r]$. In particular, $x * y := \phi(\phi^{-1}(x)\phi^{-1}(y))$ defines a group multiplication on B_r which makes $\phi|_{W_r}^{B_r}$ an isomorphism of C^1-Lie groups.*

(b) *B_s is a normal subgroup of $(B_r, *)$, for each $s \in \,]0, r]$.*

(c) *$x * B_s = B_s * x = x + B_s = B_s(x)$, for each $x \in B_r$ and $s \in \,]0, r]$.*

(d) *The Haar measure on $(B_r, +)$ coincides with Haar measure on $(B_r, *)$.*

Proof. See [10, Proposition 2.1 (a), (b)] for the proof of (a)–(c).[3]

(d)[4] Let μ be a Haar measure on $(B_r, +)$. Given $x \in B_r$, we consider the left translation map $\lambda_x\colon B_r \to B_r$, $\lambda_x(y) := x * y$. Then $\lambda_x(B_s(y)) = x*(y*B_s) = x*y+B_s$ and thus $\mu(\lambda_x(B_s(y))) = \mu(x * y + B_s) = \mu(y + B_s) = \mu(B_s(y))$ for all $y \in B_r$ and all $s \in \,]0, r]$. If $U \subseteq B_r$ is an open subset, then $U = \bigcup_{j\in J} B_{s_j}(y_j)$ for a countable family $(B_{s_j}(y_j))_{j\in J}$ of mutually disjoint balls (e.g., we can take the set of all balls $B_{s(y)}(y)$ for $y \in U$ with $s(y) := \max\{t \in \,]0, 1]\colon B_t(y) \subseteq U\}$; cf. also Theorem 1 in Appendix 2 of [18, Part II, Chapter III]). Hence $\mu(\lambda_x(U)) = \sum_{j\in J} \mu(\lambda_x(B_{s_j}(y_j))) = \sum_{j\in J} \mu(B_{s_j}(y_j)) = \mu(U)$. Since μ is outer regular, we deduce that $\mu(\lambda_x(A)) = \mu(A)$ for each Borel set $A \subseteq B_r$. Hence μ is a Haar measure on $(B_r, *)$. □

The next lemma is the key to our positive results.

Lemma 3.2 (Intertwining Lemma). *Let G_1 and G_2 be C^1-Lie groups over a local field $\mathbb{K}$. Let α_j and β_j be C^1-automorphisms of G_j and $\Omega_j \subseteq G_j$ be an identity neighbourhood such that $\alpha_j(\Omega_j) = \beta_j(\Omega_j) = \Omega_j$, for $j \in \{1, 2\}$. Assume further that there exists a map $\kappa\colon \Omega_1 \to \Omega_2$ such that $\kappa(1) = 1$, $\alpha_2 \circ \kappa = \kappa \circ \alpha_1|_{\Omega_1}$, $\beta_2 \circ \kappa = \kappa \circ \beta_1|_{\Omega_1}$,*

[3] The hypotheses of *loc. cit.* that $E = L(G)$ and $d\phi(1) = \mathrm{id}_{L(G)}$ is not used in the proof.

[4] Cf. [18, Part II, Chapter IV, Exercise 5] for a different argument in the analytic case.

and such that $\eta := \kappa|_R^S \colon R \to S$ *is a* C^1*-diffeomorphism for some open identity neighbourhoods* $R \subseteq \Omega_1$ *and* $S \subseteq \Omega_2$*. Then the following can be achieved after shrinking* R *and* S *if necessary:*

(a) *R and S are compact open subgroups of G_1 and G_2, respectively.*
(b) *η takes Haar measure on R to Haar measure on S.*
(c) *Let $\mathcal{B}$ be the set of all compact open subgroups U of G_1 such that $U \subseteq R$ and $\kappa(U)$ is a compact open subgroup of G_2. Then $\mathcal{B}$ is a basis for the filter of identity neighbourhoods in G_1.*
(d) *Given $U \in \mathcal{B}$, let $U' := \kappa(U)$. Then $d_+(\alpha_1^n(U), \beta_1^k(U)) = d_+(\alpha_2^n(U'), \beta_2^k(U'))$ for all $n, k \in \mathbb{Z}$ and $d_+(\beta_1^n(U), \alpha_1^k(U)) = d_+(\beta_2^n(U'), \alpha_2^k(U'))$.*
(e) *If each of α_1, α_2, β_1 and β_2 moves to infinity, then $\partial\alpha_1 = \partial\beta_1$ if and only if $\partial\alpha_2 = \partial\beta_2$.*
(f) *If $s_{G_1}(\alpha_1) = s_{G_2}(\alpha_2) > 1$ and $s_{G_1}(\beta_1) = s_{G_2}(\beta_2) > 1$, then $\delta_+(\alpha_1, \beta_1) = \delta_+(\alpha_2, \beta_2)$, $\delta_+(\beta_1, \alpha_1) = \delta_+(\beta_2, \alpha_2)$ and $\delta(\alpha_1, \beta_1) = \delta(\alpha_2, \beta_2)$.*

Proof. Let $\|.\|$ be an ultrametric norm on $E := L(G_2)$. Pick a chart $\phi\colon P \to Q \subseteq E$ for G_2 around 1, such that $\phi(1) = 0$ and $P \subseteq S$. By Lemma 3.1, after shrinking P and Q we may assume the following: $Q = B_r \subseteq E$ for some $r > 0$; $W_s := \phi^{-1}(B_s)$ is a compact open subgroup of G_2, for each $s \in \,]0, r]$; and the image measure $\mu_2 := \phi^{-1}(\mu)$ is a Haar measure on $P = W_r$, where μ is a given Haar measure on $(B_r, +)$. Since also $\psi\colon \eta^{-1}(P) \to Q$, $\psi(x) := \phi(\eta(x))$ is a diffeomorphism with $\psi(1) = 0$, applying Lemma 3.1 again we see that after shrinking r, we may assume that furthermore $V_s := \psi^{-1}(B_s) = \eta^{-1}(W_s)$ is a compact open subgroup of G_1 for each $s \in \,]0, r]$ and $\mu_1 := \psi^{-1}(\mu) = \eta^{-1}(\mu_2)$ a Haar measure on V_r.

(a) After replacing R and S with V_r and W_r, respectively, (a) holds.

(b) By the preceding, indeed $\eta(\mu_1) = \eta(\eta^{-1}(\mu_2)) = \mu_2$.

(c) It is clear that the sets V_s provide a basis of identity neighbourhoods, and we have $\{V_s \colon s \in \,]0, r]\} \subseteq \mathcal{B}$ since $\kappa(V_s) = W_s$.

(d) Let $U \in \mathcal{B}$ and abbreviate $U' := \kappa(U)$. Then $\kappa(\alpha_1^{-n}(\beta_1^k(U))) = \alpha_2^{-n}(\beta_2^k(\kappa(U))) = \alpha_2^{-n}(\beta_2^k(U'))$ for all $n, k \in \mathbb{Z}$, entailing that

$$\begin{aligned}
&[\alpha_1^n(U) : \alpha_1^n(U) \cap \beta_1^k(U)] \\
&= [U : U \cap \alpha_1^{-n}\beta_1^k(U)] = \frac{\mu_1(U)}{\mu_1(U \cap \alpha_1^{-n}\beta_1^k(U))} \\
&= \frac{\mu_2(\eta(U))}{\mu_2(\eta(U \cap \alpha_1^{-n}\beta_1^k(U)))} = \frac{\mu_2(\kappa(U))}{\mu_2(\kappa(U) \cap \kappa(\alpha_1^{-n}\beta_1^k(U)))} \\
&= \frac{\mu_2(U')}{\mu_2(U' \cap \alpha_2^{-n}\beta_2^k(U'))} = [U' : U' \cap \alpha_2^{-n}\beta_2^k(U')] \\
&= [\alpha_2^n(U') : \alpha_2^n(U') \cap \beta_2^k(U')]
\end{aligned}$$

and thus $d_+(\alpha_1^n(U), \beta_1^k(U)) = d_+(\alpha_2^n(U'), \beta_2^k(U'))$. Interchanging the roles of α_j and β_j yields $d_+(\beta_1^n(U), \alpha_1^k(U)) = d_+(\beta_2^n(U'), \alpha_2^k(U'))$.

(e) Let $U \in \mathcal{B}$ and $U' := \kappa(U)$. By (d), we have $d(\alpha_1^n(U), \beta_1^k(U)) = d(\alpha_2^n(U'), \beta_2^k(U'))$ for all $k, n \in \mathbb{N}$. Therefore, if the sequence $(d(\alpha_1^{nk}(U), \beta_1^{n\ell}(U)))_{n\in\mathbb{N}}$ is bounded for certain $k, \ell \in \mathbb{N}$, then also $(d(\alpha_2^{nk}(U'), \beta_2^{n\ell}(U')))_{n\in\mathbb{N}}$ is bounded and conversely (as both sequences coincide). Thus $\partial\alpha_1 = \partial\beta_1$ if and only if $\partial\alpha_2 = \partial\beta_2$.

(f) Let U and U' be as before. Then

$$\begin{aligned}
&\delta_n^{U,U}(\alpha_1, \beta_1) \\
&= \min\left\{ \frac{d_+(\alpha_1^n(U), \beta_1^k(U))}{n \log(s_{G_1}(\alpha_1))} : k \in \mathbb{N} \text{ s.t. } s_{G_1}(\beta_1)^k \le s_{G_1}(\alpha_1)^n \right\} \\
&= \min\left\{ \frac{d_+(\alpha_2^n(U'), \beta_2^k(U'))}{n \log(s_{G_2}(\alpha_2))} : k \in \mathbb{N} \text{ s.t. } s_{G_2}(\beta_2)^k \le s_{G_2}(\alpha_2)^n \right\} \\
&= \delta_n^{U',U'}(\alpha_2, \beta_2)
\end{aligned}$$

for each $n \in \mathbb{N}$, using (d) and the hypothesis that $s_{G_1}(\alpha_1) = s_{G_2}(\alpha_2)$ and $s_{G_1}(\beta_1) = s_{G_2}(\beta_2)$. As a consequence, $\delta_+(\alpha_1, \beta_1) = \delta_+(\alpha_2, \beta_2)$. The same argument gives $\delta_+(\beta_1, \alpha_1) = \delta_+(\beta_2, \alpha_2)$, whence also $\delta(\alpha_1, \beta_1) = \delta(\alpha_2, \beta_2)$. □

Given a Lie group G and $x \in G$, we use the notation $I_x \colon G \to G$, $y \mapsto xyx^{-1}$ for the inner automorphism associated with x and set $\mathrm{Ad}_x := L(I_x) := T_1(I_x)$. In the following, we consider G (and each conjugation-invariant subset) as a G-space via $x \,.\, y := I_x(y)$. We consider $L(G)$ as a G-space via $x.y := \mathrm{Ad}_x(y)$.

When speaking of a linear algebraic group G over a local field $\mathbb{K}$, more precisely we mean the Lie group of $\mathbb{K}$-rational points (cf. [15, Chapter I, Proposition 2.5.2]). We occasionally write $\mathfrak{g} := L(G)$ for the Lie algebra of a Lie group (or linear algebraic group) G.

Proposition 3.3. *Let G be a C^1-Lie group over a local field $\mathbb{K}$. Assume that there exists a map $\kappa\colon \Omega \to L(G)$ on a conjugation-invariant identity neighbourhood $\Omega \subseteq G$ such that $\kappa(1) = 0$, κ is G-equivariant (i.e. $\kappa \circ I_x|_\Omega = \mathrm{Ad}_x \circ \kappa$ for all $x \in G$), and $\kappa|_R^S$ is a C^1-diffeomorphism for some identity neighbourhood $R \subseteq \Omega$ and some 0-neighbourhood $S \subseteq L(G)$. Then the map*

$$\Phi\colon \partial G \to \partial \mathrm{Aut}(L(G)), \quad \Phi(\partial\alpha) := \partial L(\alpha)$$

is well-defined and injective. If, furthermore, $s_G(\alpha) = s_{L(G)}(L(\alpha))$ for each $\alpha \in \mathrm{Inn}(G)_>$, then Φ is an isometric embedding.

Remark 3.4. If G is a linear algebraic group over $\mathbb{K}$, then $s_G(\alpha) = s_{L(G)}(L(\alpha))$ for each $\alpha \in \mathrm{Inn}(G)$, as a special case of Lemma 1.3 (d).

Proof of Proposition 3.3. If $\alpha, \beta \in \mathrm{Inn}(G)$ move to infinity, then also $L(\alpha)$ and $L(\beta)$ move to infinity (see Lemma 1.4 (a)). Applying Lemma 3.2 (e) to the automorphisms α, β and $L(\alpha)$, $L(\beta)$ of the Lie groups G and $(\mathfrak{g}, +)$, respectively, we see that $\partial L(\alpha) = \partial L(\beta)$ if and only if $\partial\alpha = \partial\beta$. Hence Φ is well-defined and injective. If, furthermore, $s_G(\alpha) = s_{L(G)}(L(\alpha))$ and $s_G(\beta) = s_{L(G)}(L(\beta))$ for all $\alpha, \beta \in \mathrm{Inn}(G)_>$, then $\delta(\partial\alpha, \partial\beta) = \delta(\partial L(\alpha), \partial L(\beta))$ by Lemma 3.2 (f) and thus Φ is an isometry. $\square$

To illustrate Proposition 3.3, we now consider various classes of examples, which can be discussed by elementary means. Afterwards, we define a quite general class of linear algebraic groups with similar properties.

Corollary 3.5. *Let $\mathbb{K}$ be a local field and $n \in \mathbb{N}$. Assume that G is either*

(a) *the general linear group $\mathrm{GL}_n(\mathbb{K})$; or*
(b) *the special linear group $\mathrm{SL}_n(\mathbb{K}) = \{g \in \mathrm{GL}_n(\mathbb{K})\colon \det g = 1\}$, provided that $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > 0$ and $\mathrm{char}(\mathbb{K})$ does not divide n; or*

(c) *the orthogonal group* $\mathrm{O}_n(\mathbb{K}) = \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^T = g^{-1}\}$, *where* $\mathrm{char}(\mathbb{K}) \neq 2$*; or*
(d) *the group* $\mathrm{UT}_n(\mathbb{K}) = \{(a_{ij})_{i,j=1}^n \in \mathrm{GL}_n(\mathbb{K})\colon i > j \Rightarrow a_{ij} = 0\}$ *of all invertible upper triangular matrices.*

Then $\Phi\colon \partial G \to \partial\,\mathrm{Aut}(L(G))$, $\partial\alpha \mapsto \partial L(\alpha)$ *is well-defined and an isometric embedding.*

Proof. (a) If $G = \mathrm{GL}_n(\mathbb{K})$, let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{K})$ and consider the map $\kappa\colon G \to \mathfrak{g}$, $\kappa(x) := x - \mathbf{1}$. Then $\kappa(\mathbf{1}) = 0$ and κ is a C^ω-diffeomorphism onto the open subset $G - \mathbf{1}$ of $\mathfrak{g}$ (with inverse $x \mapsto x + \mathbf{1}$). Given $x, y \in G$, we have

$$\mathrm{Ad}_x(\kappa(y)) = x\kappa(y)x^{-1} = x(y - \mathbf{1})x^{-1} = xyx^{-1} - \mathbf{1} = \kappa(I_x(y))\,,$$

whence κ is G-equivariant. Now apply Proposition 3.3.

(b) The map $\kappa\colon \mathrm{SL}_n(\mathbb{K}) \to \mathfrak{sl}_n(\mathbb{K})$, $\kappa(g) := g - \frac{\mathrm{tr}(g)}{n}\mathbf{1}$ is G-equivariant, and $\kappa(\mathbf{1}) = 0$. Furthermore, $d\kappa(\mathbf{1}) = \mathrm{id} \in \mathrm{GL}(\mathfrak{g})$, since $d\kappa(\mathbf{1}).\gamma'(0) = (\kappa \circ \gamma)'(0) = \gamma'(0) + \frac{\mathrm{tr}\,\gamma'(0)}{n}\mathbf{1} = \gamma'(0)$ for each analytic map $\gamma\colon U \to \mathrm{SL}_n(\mathbb{K})$ on some 0-neighbourhood $U \subseteq \mathbb{K}$ with $\gamma(0) = \mathbf{1}$. The second summand vanishes because $\gamma'(0) \in \mathfrak{sl}_n(\mathbb{K})$. Hence κ is a local diffeomorphism at $\mathbf{1}$, and Proposition 3.3 applies.

(c) If $\mathrm{char}(\mathbb{K}) \neq 2$ and $G = \mathrm{O}_n(\mathbb{K})$, we let $\mathfrak{g} := L(G) = \mathfrak{o}_n(\mathbb{K}) = \{X \in \mathfrak{gl}_n(\mathbb{K})\colon X^T = -X\}$ be the orthogonal Lie algebra. Then

$$\Omega := \{g \in \mathrm{O}_n(\mathbb{K})\colon \mathbf{1} + g \in \mathrm{GL}_n(\mathbb{K})\} \tag{3.1}$$

is an open conjugation-invariant identity neighbourhood in G and the Cayley transform

$$\kappa\colon \Omega \to \mathfrak{g}\,, \qquad \kappa(g) := (\mathbf{1} - g)(\mathbf{1} + g)^{-1}$$

is a C^ω-diffeomorphism onto an open 0-neighbourhood in $\mathfrak{g}$ (as we recall in the appendix). We have $\kappa(\mathbf{1}) = 0$, and furthermore κ is G-equivariant, since $\mathrm{Ad}_x(\kappa(y)) = x\kappa(y)x^{-1} = x(\mathbf{1}-y)(\mathbf{1}+y)^{-1}x^{-1} = (\mathbf{1} - xyx^{-1})(\mathbf{1} + xyx^{-1})^{-1} = \kappa(xyx^{-1})$ for all $x \in G$ and $y \in \Omega$. Hence Proposition 3.3 applies.

(d) Let $\mathfrak{ut}_n(\mathbb{K}) := \{(a_{ij})_{i,j=1}^n \in M_n(\mathbb{K})\colon i > j \Rightarrow a_{ij} = 0\}$ be the Lie algebra of upper triangular matrices. Then Proposition 3.3 applies with $\kappa\colon \mathrm{UT}_n(\mathbb{K}) \to \mathfrak{ut}_n(\mathbb{K})$, $x \mapsto x - \mathbf{1}$. □

The following definition captures the essence of the arguments just used.

Definition 3.6. Let G be a Zariski-connected linear algebraic group over an infinite field $\mathbb{K}$, with Lie algebra $\mathfrak{g}$. We say that G is a *generalized Cayley group* if there exists a G-equivariant rational map $\kappa\colon G \rightharpoonup \mathfrak{g}$ defined over $\mathbb{K}$, such that $\kappa(1)$ is defined,[5] $\kappa(1) = 0$ and $d\kappa(1) \in \mathrm{GL}(\mathfrak{g})$.

Remark 3.7. Following [14], G is called a ($\mathbb{K}$-) *Cayley group* if there exists a G-equivariant birational isomorphism $\kappa\colon G \rightharpoonup \mathfrak{g}$ (defined over $\mathbb{K}$). Also some weakened versions of this concept were considered in [14]. Notably, for $\mathbb{K}$ an algebraically closed field of characteristic 0, they showed that each connected linear algebraic group G admits a G-equivariant and dominant morphism $G \to \mathfrak{g}$ of affine varieties [14, Theorem 10.2]. Another result is more relevant for us: For $\mathbb{K}$ as before, each connected reductive group G admits a G-equivariant birational isomorphism $\kappa\colon G \to \mathfrak{g}$ which is a morphism of algebraic varieties, takes 1 to 0, and is étale at 1 (see [14, Corollary to Lemma 10.3]; cf. also [13] for related earlier studies). Hence, every reductive group over an algebraically closed field of characteristic 0 is a generalized Cayley group in our sense. Unfortunately, no comparable results seem to be available yet for algebraic groups over ground fields which are not algebraically closed or have positive characteristic. But it is to be expected that also many of these will be generalized Cayley groups. The examples given in Corollary 3.5 (d) show that also some non-reductive groups are (generalized) Cayley groups (see also [14, Example 1.21]).

Corollary 3.8. *If G is a generalized Cayley group over a local field $\mathbb{K}$, then $\Phi\colon \partial G \to \partial\,\mathrm{GL}(\mathfrak{g})$, $\partial\alpha \mapsto \partial L(\alpha)$ is a well-defined isometric embedding.*

Proof. For κ as in Definition 3.6, its domain of definition Ω is conjugation-invariant. Also all other hypotheses of Proposition 3.3 are satisfied. □

Remark 3.9. Note that the isometry Φ in Corollary 3.8 factors to an isometry $\partial G/\delta^{-1}(0) \to \partial\,\mathrm{GL}(\mathfrak{g})/\delta^{-1}(0)$ between the corresponding metric spaces, which extends uniquely to an isometry $\overline{\partial G} \to \overline{\partial\,\mathrm{GL}(\mathfrak{g})}$ between the completions.

[5] The arrow $\rightharpoonup$ indicates that κ is only partially defined. For further information concerning rational maps, cf. [6].

4. Non-existence of small jointly tidy subgroups

In this section, we provide counterexamples showing that the existence of small tidy subgroups for each individual automorphism in a flat group $\mathcal{H}$ need not ensure the existence of small jointly tidy subgroups for $\mathcal{H}$, not even if $\mathcal{H}$ is finitely generated. We also show that a flat group may contain automorphisms which are not tidy, although it is generated by a set of tidy automorphisms. For general information concerning flat groups, see [22].

Throughout this section, J is an infinite set, F a non-trivial finite group and $G := F^J$, equipped with the (compact) product topology. The group $\mathrm{Sym}(J)$ of all bijective self-maps of J admits a permutation representation $\pi\colon \mathrm{Sym}(J) \to \mathrm{Aut}(G)$ on G by automorphisms via $\pi(\sigma)(f)(j) := f(\sigma^{-1}(j))$ for $\sigma \in \mathrm{Sym}(J)$, $f \in G$, $j \in J$. Furthermore, $\mathrm{Sym}(J)$ (and its subgroups) act on J in an obvious way.

Lemma 4.1. *Let $H \leq \mathrm{Sym}(J)$ be a subgroup and $\mathcal{H} := \pi(H) \leq \mathrm{Aut}(G)$.*

(a) *Then G is tidy for each $\alpha \in \mathcal{H}$, and hence $\mathcal{H}$ is flat. Furthermore, $s_G(\alpha) = 1$ for each $\alpha \in \mathcal{H}$.*

(b) *A compact open subgroup $U \subseteq G$ is tidy for $\alpha \in \mathcal{H}$ if and only if $\alpha(U) = U$.*

(c) *If every H-orbit in J is infinite (e.g., if H acts transitively on J), then G is the only jointly tidy subgroup for $\mathcal{H}$.*

(d) *If $\sigma \in H$ and all $\langle\sigma\rangle$-orbits in J are finite, then $\pi(\sigma)$ is a tidy automorphism of G.*

Proof. (a) For $\alpha \in \mathcal{H}$, we have $\alpha^{-1}(G) = G$ and therefore $[G : G \cap \alpha^{-1}(G)] = 1$, which is minimal. Hence G is tidy for each $\alpha \in \mathcal{H}$ and $s_G(\alpha) = 1$.

(b) Since $s_G(\alpha) = s_G(\alpha^{-1}) = 1$ by (a), it is well-known that U is tidy if and only if $\alpha(U) = U$ holds.[6]

(c) Assume that all orbits of H are infinite and $U \subseteq G$ is tidy for each $\alpha \in \mathcal{H}$. Then $\{(x_j)_{j\in J} \in G\colon j \in A \Rightarrow x_j = 1\} \subseteq U$ for some finite set $A \subseteq J$. Given $x \in F$ and $k \in J$, let $f_k(x)$ be the element of G with k-th component x and all other entries 1. The orbit $H.k$ being infinite, there exists $m \in H.k \setminus A$. Since $m \in H.k$,

[6] This is obvious from an alternative definition of tidy subgroups [21, Definition 2.1], which is equivalent to the one we use by [21, Theorem 3.1]. Cf. also [21, Corollary 3.11].

there exists $\sigma \in H$ such that $\sigma(m) = k$. Then $f_m(x) \in U$ and hence $f_k(x) = \pi(\sigma)(f_m(x)) \in \pi(\sigma)(U) = U$, using (b). Hence U contains the dense subgroup $\langle f_k(x)\colon k \in J, x \in F\rangle$ of G and thus $U = G$.

(d) If all orbits of σ are finite and $V \subseteq G$ is an identity neighbourhood, there exists a finite subset $A \subseteq J$ such that $U := \{(x_j)_{j\in J} \in G\colon j \in A \Rightarrow x_j = 1\} \subseteq V$. Since σ has finite orbits, after increasing A we may assume that A is a union of σ-orbits. Set $\alpha := \pi(\sigma)$. Then $\alpha(U) = U$, and thus U is a subgroup of V which is tidy for α (by (b)). □

Example 4.2. In this example, we specialize to the case $J = \mathbb{Z}$. Thus $G = F^{\mathbb{Z}}$. We let $\mathcal{H} := \pi(\mathrm{Sym}_{\mathrm{fin}}(\mathbb{Z}))$, where

$$\mathrm{Sym}_{\mathrm{fin}}(\mathbb{Z}) \;:=\; \{\sigma \in \mathrm{Sym}(\mathbb{Z})\colon \sigma(n) = n \text{ for all but finitely many } n\}$$

is the group of all finite permutations of $\mathbb{Z}$. Since $\mathrm{Sym}_{\mathrm{fin}}(\mathbb{Z})$ acts transitively and each $\sigma \in \mathrm{Sym}_{\mathrm{fin}}(\mathbb{Z})$ has finite orbits, Lemma 4.1 shows that $\mathcal{H}$ is flat, each $\alpha \in \mathcal{H}$ is tidy, but G is the only compact open subgroup of G which is tidy for all $\alpha \in \mathcal{H}$ simultaneously.

There even is a finitely generated counterexample.

Example 4.3. Let J be a finitely generated, algebraically periodic, infinite group (for example, a "Tarski monster" as in [16]). Then J acts on itself via left translation. We let $H \leq \mathrm{Sym}(J)$ be the corresponding group of permutations and $\mathcal{H} := \pi(H) \leq \mathrm{Aut}(G)$. Then $\mathcal{H}$ is a finitely generated group. Since H acts transitively on J and each $\sigma \in H$ has finite orbits, Lemma 4.1 shows that $\mathcal{H}$ is flat and each $\alpha \in \mathcal{H}$ is tidy, but G is the only jointly tidy subgroup for $\mathcal{H}$.

If a flat group $\mathcal{H}$ of automorphisms is generated by a set of tidy automorphisms, this does not imply that each $\alpha \in \mathcal{H}$ is tidy.

Example 4.4. Take $J := \mathbb{Z}$; thus $G = F^{\mathbb{Z}}$. We define $\sigma, \tau \in \mathrm{Sym}(\mathbb{Z})$ via $\sigma(2k+1) := 2k$, $\sigma(2k) := 2k+1$, $\tau(2k-1) := 2k$ and $\tau(2k) := 2k-1$ for $k \in \mathbb{Z}$. We also set $H := \langle \sigma, \tau\rangle \leq \mathrm{Sym}(\mathbb{Z})$ and $\mathcal{H} := \pi(H)$. Then $\mathcal{H}$ is flat, by Lemma 4.1 (a). Furthermore, $\pi(\sigma)$ and $\sigma(\tau)$ are tidy, since all orbits of σ and τ have two elements (see Lemma 4.1 (d)). However, although both generators $\pi(\sigma)$ and $\pi(\tau)$

of $\mathcal{H}$ are tidy, the group $\mathcal{H}$ contains automorphisms which are not tidy. For example, consider the element $\alpha := \pi(\sigma \circ \tau) \in \mathcal{H}$. Since $(\sigma \circ \tau)(2k) = 2k - 2$ and $(\sigma \circ \tau)(2k - 1) = 2k + 1$ for each $k \in \mathbb{Z}$, we see that $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ are the orbits of $\sigma \circ \tau$. Since both of these are infinite, the only subgroup of G tidy for each $\beta \in \langle\alpha\rangle$ is all of G, by Lemma 4.1 (c). Since a subgroup is tidy for α if and only if it is tidy for each $\beta \in \langle\alpha\rangle$ (cf. Lemma 4.1 (b)), we deduce that G is the only subgroup of G tidy for α, and thus α is not tidy.

Appendix: Basic properties of the Cayley transform

We recall some properties of the Cayley transform which we used to prove Corollary 3.5 (c) (cf. [6, p. 123-125] for further information).

Lemma. *Let $\mathbb{K}$ be a local field of characteristic* $\mathrm{char}(\mathbb{K}) \neq 2$, *and* $n \in \mathbb{N}$. *Then* $\Omega := \{x \in M_n(\mathbb{K})\colon \mathbf{1} + x \in \mathrm{GL}_n(\mathbb{K})\}$ *is an open subset of* $M_n(\mathbb{K})$ *such that* $\{\mathbf{0}, \mathbf{1}\} \subseteq \Omega$, *and*

$$\theta\colon \Omega \to \Omega\,, \quad \theta(x) := (\mathbf{1} - x)(\mathbf{1} + x)^{-1}$$

is a bijective self-map of Ω with $\theta \circ \theta = \mathrm{id}_\Omega$. The following holds:

(a) $\mathbf{1} + \theta(x) \in \mathrm{GL}_n(\mathbb{K})$ *for* $x \in \Omega$, *and* $(\mathbf{1} + \theta(x))^{-1} = \frac{1}{2}(\mathbf{1} + x)$.

(b) $\Omega_1 := \Omega \cap \mathrm{O}_n(\mathbb{K})$ *is an open conjugation-invariant identity neighbourhood in* $\mathrm{O}_n(\mathbb{K})$ *and* $\Omega_2 := \Omega \cap \mathfrak{o}_n(\mathbb{K})$ *is an open* 0*-neighbourhood in* $\mathfrak{o}_n(\mathbb{K})$, *such that* $\theta(\Omega_1) = \Omega_2$.

(c) *The map* $\theta|_{\Omega_1}^{\Omega_2}\colon \Omega_1 \to \Omega_2$ *is a* C^ω*-diffeomorphism.*

Proof. (a) We have $(\mathbf{1}+\theta(x))(\mathbf{1}+x) = (\mathbf{1}+(\mathbf{1}-x)(\mathbf{1}+x)^{-1})(\mathbf{1}+x) = \mathbf{1} + x + \mathbf{1} - x = 2\mathbf{1}$ for each $x \in \Omega$, showing that indeed $\mathbf{1} + \theta(x) \in \mathrm{GL}_n(\mathbb{K})$ (and thus $\theta(x) \in \Omega$), with $(\mathbf{1} + \theta(x))^{-1} = \frac{1}{2}(\mathbf{1} + x)$.

θ is an involution as $\theta(\theta(x)) = (\mathbf{1}-(\mathbf{1}-x)(\mathbf{1}+x)^{-1})(\mathbf{1}+\theta(x))^{-1} = \frac{1}{2}(\mathbf{1} - (\mathbf{1} - x)(\mathbf{1} + x)^{-1})(\mathbf{1} + x) = \frac{1}{2}(\mathbf{1} + x - (\mathbf{1} - x)) = x$ for each $x \in \Omega$, using (a). As a consequence, $\theta\colon \Omega \to \Omega$ is a bijection.

(b) If $g \in \Omega_1$, then $\theta(g)^T = (\mathbf{1} + g^T)^{-1}(\mathbf{1} - g^T) = (\mathbf{1} + g^{-1})^{-1} (\mathbf{1} - g^{-1}) = (g^{-1}(g + \mathbf{1}))^{-1} g^{-1}(g - \mathbf{1}) = (g + \mathbf{1})^{-1}(g - \mathbf{1}) = -\theta(g)$. Thus $\theta(g) \in \mathfrak{o}_n(\mathbb{K}) \cap \Omega = \Omega_2$. Conversely, $\theta(x)^T = (\mathbf{1} + x^T)^{-1} (\mathbf{1} - x^T) = (\mathbf{1} - x)^{-1}(\mathbf{1} + x)$ for $x \in \Omega_2$ and thus $\theta(x)^T\theta(x) = (\mathbf{1} - x)^{-1}(\mathbf{1} + x)(\mathbf{1} - x)(\mathbf{1} + x)^{-1} = \mathbf{1}$, using that all matrices involved commute. Thus $\theta(x) \in \Omega \cap \mathrm{O}_n(\mathbb{K}) = \Omega_1$. Now (b) follows.

(c) Since θ is C^ω, also $\theta|_{\Omega_1}^{\Omega_2}$ and $\theta|_{\Omega_2}^{\Omega_1} = (\theta|_{\Omega_1}^{\Omega_2})^{-1}$ are C^ω. □

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