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WHITNEY ARCS WITH GRAPH–DIRECTED STRUCTURE

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ABSTRACT. This paper proves that if arcs with graph-directed structure have Hausdorff dimension greater than 1, then they are Whitney's critical sets.

1. Introduction

In 1935, Whitney [11] published his celebrated example of a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ critical but not constant along a fractal arc. In this example, the image of the critical set of f contains an interval which has positive Lebesgue measure. The above Whitney phenomenon seems to contradict the Morse-Sard Theorem, which says the critical image has zero measure. This is due to the fact that the arc γ is a fractal and f has lower smoothness.

Definition 1. A connected set $A \subset \mathbb{R}^n$ is said to be a Whitney's critical set (Whitney set in brief), if there is a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ such that f is critical but not constant on A, i.e., the grads $\nabla f\mid_A \equiv 0$ and $f\mid_A$ is not a constant.

Remark 1. By an application of the Morse-Sard Theorem, the function f in Definition 1 cannot be sufficiently differentiable (Sard [7]).

The following two kinds of sets are not Whitney sets: (1) The set F holding the condition: Every pair of points in F is connected by a rectifiable arc lying in F (Whyburn [9]); (2) The graph G of any continuous function $g : \mathbb{R} \to \mathbb{R}$ (Choquet [1]).

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How can one characterize the Whitney set geometrically? The question was vaguely posed in Whitney's original paper, and can be stated as follows: Given a function f, how far from rectifiable must a closed connected set be in order to be a critical set for f on which f is not constant?

In [6], Norton proved that if γ is a *t-quasi-arc* such that $\dim_H \gamma > t$, then γ is a Whitney set. It is proved in [4] that although the Sierpinski gasket is not a Whitney set, but it contains a Whitney subset of full dimension, and the Koch curve is a Whitney set. In [14], Wu and Xi constructed a Whitney set which is an arc such that there does not exist a non-constant function critical and *monotone* along the arc.

Furthermore, self-similar arcs of Hausdorff dimension greater than 1, including the Koch curve, are Whitney sets ([13]).

In this paper, we want to prove that graph-directed arcs of dimension greater than 1 are Whitney sets.

A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a contractive similitude if there is a ratio ρ with $0 < \rho < 1$ such that for any $x, y \in \mathbb{R}^n$,

$$\mid S(x) - S(y) \mid = \rho \mid x - y \mid .$$

In order to prove the main result, we shall recall the notion of graph-directed sets. Suppose $(V, \mathcal{E})$ is a directed graph, where $V = \{1, 2, \cdots, m\}$ is a set of vertices and $\mathcal{E}$ is the set of edges of the graph. Write $\mathcal{E}_{i,j}$ for the set of edges from vertex i to vertex j. We suppose that for any $e \in \mathcal{E}$, there is a contracting similitude $S_e : \mathbb{R}^n \to \mathbb{R}^n$ with ratio $r_e \in (0, 1)$. Then by [5] there exists a unique family of non-empty compact sets $\Gamma_1, \Gamma_2 \cdots, \Gamma_m$ such that for any i,

$$\Gamma_i = \bigcup_{j=1}^{m} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(\Gamma_j). \tag{1.1}$$

Here $\{S_e : e \in \mathcal{E}\}$ is called a graph-directed iterated function system on $(V, \mathcal{E})$, and the sets $\{\Gamma_1, \Gamma_2 \cdots, \Gamma_m\}$ are called graph-directed sets with respect to the IFS.

Definition 2. Suppose $\{S_e : e \in \mathcal{E}\}$ is a graph-directed IFS on $(V, \mathcal{E})$, and $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_m\}$ are graph-directed sets satisfying (1.1). Then $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_m\}$ is said to be graph-directed arcs, if

(1) Γ_i is an arc for any i;
(2) if $e \in \mathcal{E}_{i,j}$, $e' \in \mathcal{E}_{i,j'}$ and $e \neq e'$, then $S_e(\Gamma_j) \cap S_{e'}(\Gamma_{j'})\ (\subset \Gamma_i)$ is either a singleton or an empty set.

In this paper, we will prove the following result.

Theorem 1. *Suppose* $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_m\}$ *are graph-directed arcs with* $\dim_H \Gamma_i > 1$ *for any* i. *Then* $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_m\}$ *are Whitney sets.*

Remark 2. In the above Theorem, we do not need the transitivity condition on directed graph.

2. Preliminaries

Suppose $(V, \mathcal{E})$ is a directed graph. Let $\mathcal{E}^*$ be the set of paths on $(V, \mathcal{E})$. A path in the graph is a finite string $\alpha = e_1 e_2 \cdots e_k$ of edges, such that the terminal vertex of each edge e_i is the initial vertex of the next edge e_{i+1}. The initial vertex of α is the initial vertex of e_1 and the terminal vertex of α is the terminal vertex of e_k. For any $k \in \mathbb{N}$, write $\mathcal{E}_{i,j}^k$ for the set of directed path $e^* = e_1 e_2 \cdots e_k$ ($e_i \in \mathcal{E}$) of length k starting at vertex i and ending at vertex j. Write $S_{e^*} = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$ and $r_{e^*} = r_{e_1} \cdot r_{e_2} \cdot \cdots \cdot r_{e_k}$. By iterating (1.1), we see that for $i = 1, \cdots, m$,

$$\Gamma_i = \bigcup_{j=1}^{m} \bigcup_{e^* \in \mathcal{E}_{i,j}^k} S_{e^*}(\Gamma_j). \tag{2.1}$$

Let $A^{(s)}$ be the *associated matrix* $A^{(s)}$ with (i,j)-th entry given by

$$A_{i,j}^{(s)} = \sum_{e \in \mathcal{E}_{i,j}} r_e^{(s)}. \tag{2.2}$$

Denote by $\rho\left(A^{(s)}\right)$ the spectrum radius of $A^{(s)}$. We say that $(V, \mathcal{E})$ is transitive, if for any $i, j \in V$, $[\cup_{k \geq 1} \mathcal{E}_{i,j}^k] \neq \varnothing$, i.e., $(V, \mathcal{E})$ is path connected. There is a related result as follows ([2]): Suppose $(V, \mathcal{E})$ is transitive. Let $\{K_1, \cdots K_m\}$ be graph-directed sets on $(V, \mathcal{E})$ such that (1.1) is a disjoint union for each $i \in V$. Then there is a number s such that $\dim_H K_i = \dim_B K_i = s$, and $0 < \mathcal{H}^s(K_i) < \infty$ for any $i \in V$, where s is the unique positive number satisfying $\rho\left(A^{(s)}\right) = 1$. Here $\dim_H K_i$ and $\dim_B K_i$ denote Hausdorff dimension and Box dimension of K_i, respectively. $\mathcal{H}^s(K_i)$ denote Hausdorff measure of K_i.

In this paper, we need a special case of the Whitney Extension Theorem which is stated as follows ([12]).

Lemma 1. *Suppose $A \subset \mathbb{R}^n$ is compact and $f : A \to \mathbb{R}$ is a real-valued function. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x,\ y \in A$ with $0 < |x - y| < \delta$,*

$$\frac{|f(x) - f(y)|}{|x - y|} < \varepsilon, \tag{2.3}$$

then there is a C^1 extension $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ of f such that $\overline{f}|_A = f$ and $\nabla\overline{f}|_A \equiv 0$.

Remark 3. Inequality (2.3) means $|f(x) - f(y)| = o(|x - y|)$.

We introduce the notion of sub-IFS of graph-directed IFS.

Definition 3. Let Ξ be a finite subset of $\mathcal{E}^*$. The family $\{S_{e^*} : e \in \Xi\}$ is called a sub-IFS on $(V, \mathcal{E})$ with respect to $\{S_e : e \in \mathcal{E}\}$.

In fact, $\{S_{e^*} : e \in \Xi\}$ is an IFS on (V, Ξ) when Ξ is considered as a set of edges. If $\{K_1, \cdots, K_m\}$ are graph-directed sets on $(V, \mathcal{E})$, and $\{K'_1, \cdots, K'_m\}$ are graph-directed sets on (V, Ξ), then $K'_i \subset K_i$ for any i. Let $\Xi_{i,j}$ be the set of edges of Ξ starting at vertex i and ending at vertex j.

Lemma 2. *If $\min_i \dim_H(K_i) > 1$, then there is a sub-IFS $\{S_{e^*} : e \in \Xi\}$ on $(V, \mathcal{E})$ with $\Xi \subset \mathcal{E}^*$, such that the followings hold:*
(1) *For any $e_1^* \in \Xi_{i,j_1}$, $e_2^* \in \Xi_{i,j_2}$ with $e_1^* \neq e_2^*$,*

$$S_{e_1^*}(K_{j_1}) \cap S_{e_2^*}(K_{j_2}) = \varnothing;$$

(2) *There is a number $\tau > 1$ such that*

$$\rho(B^{(\tau)}) = 1,$$

where $\{B^{(s)}\}_{s \in \mathbb{R}}$ are associated matrices w.r.t. the IFS
$$\{S_{e^*} : e \in \Xi\};$$
(3) *If furthermore $\{K_i\}_i$ are arcs, then for any $e^* \in \Xi_{i,j}$, $S_{e^*}(K_j)$ does not contain any endpoint of K_i.*

Proof. Write $t = \min_i(\dim_H K_i)$. Fix $\overline{s}, s_0$ with $1 < \overline{s} < s_0 < t$. Let $c_1 = (\min_{e\in\mathcal{E}} r_e)/[\max_j(diam(K_j))]$. Fix $h \in V$, we denote by $\mathcal{N}_k(h)$ the collection of k-level 2-adic cubes intersecting K_h. Here the k-level 2-adic cubes are the cubes with the length of 2^{-k}.

The following fact is well-known (e.g. see [8] or [10]): For any collection $\mathcal{M}$ of k-level 2-adic cubes of $\mathbb{R}^n$, we can select $\mathcal{M}_1 \subset \mathcal{M}$ such that the elements of $\mathcal{M}_1$ are pairwise disjoint and $\#\mathcal{M}_1 \geq (\#\mathcal{M})/c$, where c is a constant depending on $\mathbb{R}^n$.

Since $\dim_H K_h \geq t > 1$, then $s^* = \overline{\dim}_B K_h \geq \dim_H K_h \geq t > 1$. By the definition of upper Box dimension, we have

$$s^* = \overline{\lim_{k\to\infty}} \frac{\log \#\mathcal{N}_k(h)}{k \log 2},$$

which implies that there is a subsequence $\{k_i\}_i$ such that

$$s^* = \lim_{i\to\infty} \frac{\log \#\mathcal{N}_{k_i}(h)}{k_i \log 2}.$$

Furthermore, there exists an integer $\mathbf{i}$ large enough such that

$$\#\mathcal{N}_{k_\mathbf{i}}(h) \geq 2^{k_\mathbf{i} s_0} \geq 4c \text{ and } 2^{k_\mathbf{i}(s_0-\overline{s})} > 2c/c_1^{\overline{s}}.$$

As we know, we can always select a subset $\mathcal{N}^*_{k_\mathbf{i}}(h)$ of $\mathcal{N}_{k_\mathbf{i}}(h)$, such that the elements of $\mathcal{N}^*_{k_\mathbf{i}}(h)$ are pairwise disjoint and

$$\#\mathcal{N}^*_{k_\mathbf{i}}(h) \geq \#\mathcal{N}_{k_\mathbf{i}}(h)/c \geq 2^{k_\mathbf{i} s_0}/c. \tag{2.4}$$

For any 2-adic cube $I \in \mathcal{N}^*_{k_\mathbf{i}}(h)$, since $I \cap K_h \neq \varnothing$, we take point $x \in I \cap K_h (\subseteq K_h)$. Then there exists a infinite directed path $e_1 e_2 \cdots e_k \cdots$ with e_k ending at i_k for each k such that

$$\{x\} = \lim_{k\to\infty} S_{e_k^*}(K_{i_k}),$$

where $e_k^* = e_1 e_2 \cdots e_k$ for each k. Here

$$diam[S_{e_k^*}(K_{i_k})] = (r_{e_1} \cdot r_{e_2} \cdot \cdots \cdot r_{e_k})[diam(K_{i_k})].$$

Then there exists a positive integer $N(I)$ depending on I such that

$$(\min_e r_e) \cdot 2^{-k_\mathbf{i}} \leq (r_{e_1} \cdot r_{e_2} \cdot \cdots \cdot r_{e_{N(I)}})[diam(K_{i_{N(I)}})] < 2^{-k_\mathbf{i}}. \tag{2.5}$$

For $I \in \mathcal{N}^*_{k_\mathbf{i}}(h)$, let

$$S^{(I)} = S_{e^*_{N(I)}},\ e^{(I)} = e^*_{N(I)} \text{ and } K^{(I)} = K_{i_{N(I)}}.$$

Then

$$S^{(I)}(K^{(I)}) \cap S^{(J)}(K^{(J)}) = \varnothing \text{ for distinct } I, J \in \mathcal{N}^*_{k_{\mathbf{i}}}(h), \tag{2.6}$$

due to (2.5) and $d(I, J) \geq 2^{-k_{\mathbf{i}}}$.

We notice that the similar ratio of $S^{(I)}$, denoted by $r^{(I)}$, is greater than $c_1 2^{-k_{\mathbf{i}}}$, where c_1 is defined above.

Let $\Phi_h = \mathcal{N}^*_{k_{\mathbf{i}}}(h)$. However, when $\{K_i\}_i$ are arcs, let $\Phi_h = \{I \in \mathcal{N}^*_{k_{\mathbf{i}}}(h) : I \cap \{a, b\} = \varnothing\}$, where a and b are endpoints of K_h. Let $\Xi_h = \{e^* = e^{(I)} : I \in \Phi_h\}$. We have

$$\begin{aligned} \textstyle\sum_{I\in\Phi_h} (r^{(I)})^{\overline{s}} &\geqslant \left(c_1 2^{-k_{\mathbf{i}}}\right)^{\overline{s}} \left(\#\mathcal{N}^*_{k_{\mathbf{i}}}(h) - 2\right) \\ &\geqslant c_1^{\overline{s}} 2^{-k_{\mathbf{i}}\overline{s}} \left(2^{k_{\mathbf{i}} s_0}/c - 2\right) \geqslant c_1^{\overline{s}} 2^{-k_{\mathbf{i}}\overline{s}} (2^{k_{\mathbf{i}} s_0}/c)/2 \\ &= (c_1^{\overline{s}}/2c) 2^{k_{\mathbf{i}}(\overline{s}-s_0)} > 1, \end{aligned}$$

i.e.,

$$\sum_{e^*\in\Xi_h} (r_{e^*})^{\overline{s}} > 1. \tag{2.7}$$

Write $\Xi = \cup_{h\in V} \Xi_h$, we obtain a new directed graph (V, Ξ). For any $e^{(I)} \in \Xi$, there is a contracting similitude $S^{(I)}$ with similar ratio $r^{(I)}$.

Let $B^{(s)}$ denote the $m \times m$ *associated matrix* with respect to (V, Ξ), i.e., $(B^{(s)})_{i,j} = \sum_{e^{(I)}\in\Xi_{i,j}} (r^{(I)})^s$. It follows from (2.7) that each row sum of $B^{(\overline{s})}$ is greater than 1, consequently $\rho\left(B^{(\overline{s})}\right) > 1$ by the Perron-Frobenius Theorem. On the other hand, $\rho\left(B^{(s)}\right) \to 0$ as $s \to \infty$. Since $f(s) = \rho\left(B^{(s)}\right)$ is continuous and strictly decreasing with s (by the Perron-Frobenius Theorem), there is a unique positive number $\tau \in (\overline{s}, +\infty)$ satisfying

$$\rho\left(B^{(\tau)}\right) = 1,$$

where $\tau > \overline{s} > 1$. □

Remark 4. The idea of above proof came from p.128-131 of [10], where self-similar sets are discussed. In fact, we can state this lemma in general form.

3. Proof of Theorem

In this paper, we will identify an arc $\gamma(\subset \mathbb{R}^n)$ with its representation $\gamma : [0,\ 1] \longrightarrow \mathbb{R}^n$.

An arc γ is said to be a *monotone arc*, if there exist $f : \gamma \to [0,1]$, $d_1 \neq d_2$ and $t_1, t_2 \in (0,1)$ such that $|f(x) - f(y)| = o(|x-y|)$ for $x, y \in \gamma$, $(f \circ \gamma)$ is monotone on $[0,1]$ with $(f\circ\gamma)|_{[0,t_1]} \equiv d_1$, $(f\circ\gamma)|_{[t_2,1]} \equiv d_2$. Denote by Ω the set of monotone arcs.

Lemma 3. *If $\gamma \in \Omega$, then γ is a Whitney set.*

Proof. Since $|f(x) - f(y)| = o(|x-y|)$ and $d_1 \neq d_2$, the lemma follows from Lemma 1 and Definition 1. □

We will prove Theorem 1 by using the following propositions.

Proposition 1. *If $\Gamma_j \in \Omega$ and $\mathcal{E}_{i,j} \neq \varnothing$, then $\Gamma_i \in \Omega$.*

Proposition 2. *Suppose $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_m\}$ are graph-directed arcs with $\min_i(\dim_H \Gamma_i) > 1$. Then $\Omega \cap \{\Gamma_1, \Gamma_2, \cdots, \Gamma_m\} \neq \varnothing$.*

Propositions 1 and 2$\Longrightarrow$Theorem 1

Suppose on the contrary that $\Omega^c \cap \{\Gamma_1, \cdots, \Gamma_m\} \neq \varnothing$. By Proposition 2, we assume $\{\Gamma_1, \cdots, \Gamma_k\} \subseteq \Omega$ and $\{\Gamma_{k+1}, \cdots \Gamma_m\} \subseteq \Omega^c$ with $1 \leq k < m$. Applying Proposition 1, we have $\mathcal{E}_{i,j} = \varnothing$ for any $i > k$ and $j \leq k$. As a result, $\{\Gamma_{k+1}, \cdots, \Gamma_m\}$ are graph-directed arcs on $\{V', \mathcal{E}'\}$, where $V' = \{(k+1), \cdots, m\}$ and $\mathcal{E}' = \{e \in \mathcal{E}_{i,j} : i, j \geq k+1\}$. Then it follows from Proposition 2 that $\{\Gamma_{k+1}, \cdots, \Gamma_m\} \cap \Omega \neq \varnothing$. This yields a contradiction. □

Proof of Propositions 1. Suppose $\Gamma_j \in \Omega$ and there is a contracting similitude S_e such that $S_e(\Gamma_j) \subset \Gamma_i$. We will show that $\Gamma_i \in \Omega$.

Let $S_e : \mathbb{R}^n \to \mathbb{R}^n$ be a contracting similitude of ratio r_e with $0 < r_e < 1$. Denote by a and b the two endpoints of Γ_i, and $\overline{a}$ and $\overline{b}$ the endpoints of Γ_j. Let $S_e(\overline{a}) = a_1$ and $S_e\left(\overline{b}\right) = b_1$.

For any arc γ, let $\gamma(x,y)$ denote the subarc of γ lying between x and y. Since $\Gamma_j \in \Omega$, there exist $f : \Gamma_j \to [0,1]$, which is monotone along Γ_j, and x_1, $y_1 \in \Gamma_j$ such that

$$|f(x) - f(y)| = o(|x-y|) \text{ for } x, y \in \Gamma_j,$$

and $f|_{\Gamma_j(\overline{a},x_1)} \equiv d_1$, $f|_{\Gamma_j(y_1,\bar{b})} \equiv d_2$ with $d_1 \neq d_2$. Let $S_e(x_1) = a_2$ and $S_e(y_1) = b_2$. Let $\gamma_1 = \Gamma_i(a, a_1)$, $\gamma_2 = \Gamma_i(a_1, a_2)$, $\gamma_3 = \Gamma_i(a_2, b_2)$, $\gamma_4 = \Gamma_i(b_2, b_1)$ and $\gamma_5 = \Gamma_i(b_1, b)$ (Figure 1). Let $\delta = \min_{|i-j|>1} d(\gamma_i, \gamma_j) > 0$.

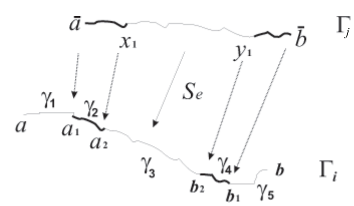

FIGURE 1

A function $g : \Gamma_i \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} d_1 & \text{if } x \in \Gamma_i(a, a_2) = \gamma_1 \cup \gamma_2, \\ f(S_e^{-1}(x)) & \text{if } x \in \Gamma_i(a_1, b_1) = \gamma_2 \cup \gamma_3 \cup \gamma_4, \\ d_2 & \text{if } x \in \Gamma_i(b_2, b) = \gamma_4 \cup \gamma_5. \end{cases}$$

To apply Lemma 1, we shall verify (2.3) for g, that is

(3.1) $\left|g(x') - g(y')\right| = o(\left|x' - y'\right|)$ for $x', y' \in \Gamma_i$ with $|x' - y'| < \delta$.

Suppose $x', y' \in \Gamma_i$ with $|x' - y'| < \delta$, and let $x' \in \gamma_p$ and $y' \in \gamma_q$. Since $\delta = \min_{|i-j|>1} d(\gamma_i, \gamma_j)$, we have

$$p = q \text{ or } |p - q| = 1.$$

We shall distinguish the following cases.

Case 1. When $x', y' \in \gamma_1 \cup \gamma_2$ or $x', y' \in \gamma_4 \cup \gamma_5$,we have

$$\left|g(x') - g(y')\right| = 0.$$

Case 2. When $x', y' \in \gamma_2 \cup \gamma_3 \cup \gamma_4$,

$$\begin{aligned} \left|g(x') - g(y')\right| &= |f(S_e^{-1}x') - f(S_e^{-1}y')| \\ &= o(|S_e^{-1}x' - S_e^{-1}y'|) \\ &= o(|x' - y'|). \end{aligned}$$

Then (3.1) is proved and thus Proposition 1 follows. □

Proof of Propositions 2. By Lemma 2, there is a sub-IFS on (V, Ξ) with $\rho(B^{(\tau)}) = 1$, where $\tau > 1$ and $\{B^{(s)}\}_{s\in\mathbb{R}^+}$ are *associated matrices* w.r.t. this sub-IFS.

It follows from the Perron-Frobenius Theorem that there is a nonnegative and non-zero eigenvector $X = (x_1, \cdots, x_m)^T$ satisfying

$$B^{(\tau)}X = X. \tag{3.2}$$

Here $X \geq 0$ and $X \neq 0$. Without loss of generality, we may assume $x_h > 0$ for some $h \in V$. We will show $\Gamma_h \in \Omega$.

Step 1. Defining measures and functions on $\{\Gamma_i\}_i$.

Let Ξ^* be the set of admissible paths in the form $\alpha = e_1^* \cdots e_k^*$ with $e_i^* \in \Xi$. Denote by S_α and r_α the similitude and its ratio w.r.t. $\alpha \in \Xi^*$.

It follows from (3.2) and (1) of Lemma 2 that, there are Borel measures $\{\mu_i\}_{i\in V}$ on $\{\Gamma_i\}_i$ such that

$$\mu_i\left(\Gamma_i\right) = x_i,$$

and for any $\alpha \in \Xi^*$ starting at i and ending at j,

$$\mu_i\left(S_\alpha\left(\Gamma_j\right)\right) = (r_\alpha)^\tau x_j.$$

Notice that for this path α and any subarc $\gamma \subset \Gamma_j$,

$$\mu_i\left(S_\alpha\left(\gamma\right)\right) = (r_\alpha)^\tau \mu_j(\gamma). \tag{3.3}$$

Suppose $\{K_1', \cdots, K_m'\}$ are graph-directed sets w.r.t. the sub-IFS on (V, Ξ). Then μ_i is supported by K_i'.

For each $i \in V$, fix one endpoint a_i of Γ_i. We define a function $f_i : \Gamma_i \to \mathbb{R}$ by

$$f_i\left(x\right) = \mu_i\left[\Gamma_i\left(a_i, x\right)\right] \text{ for any } x \in \Gamma_i, \tag{3.4}$$

where $\Gamma_i\left(x, y\right)$ is the subarc of Γ_i lying between x and y. Here

$$|f_i\left(x\right) - f_i\left(y\right)| = \mu_i\left[\Gamma_i\left(x, y\right)\right] \tag{3.5}$$

f_i is monotone, and f_h is not constant since $\mu_h(\Gamma_h) = x_h > 0$. Since K_i' does not contain any endpoint of Γ_i, then f_i is constant on some neighborhood of endpoint. It follows from (3.3), (3.4) and (3.5) that

$$|f_i(S_\alpha(x)) - f_i(S_\alpha(y))| = (r_\alpha)^\tau |f_j(x) - f_j(y)|. \tag{3.6}$$

On the arc Γ_j, we get the following disjoint subarcs

$$\cup_q\{S_{e^*}(\Gamma_q) : e^* \in \Xi_{j,q}\} = \Pi_j,$$

which do not include the endpoints of Γ_j. Then we get a decomposition $\Gamma_j = \cup_{t=1}^{2(\#\Pi_j)+1} \gamma_t^{(j)}$ of subarcs satisfying
(1) $\gamma_t^{(j)} \in \Pi_j$ if and only if t is even;
(2) $\mu_j(\gamma_t^{(j)}) = 0$ if t is odd;
(3) $\gamma_t^{(j)} \cap \gamma_{t'}^{(j)} = \varnothing$ whenever $|t-t'|>1$;
(4) $\gamma_t^{(j)} \cap \gamma_{t+1}^{(j)}$ is a singleton for any t;
(5) Each endpoint of Γ_j is contained in $\gamma_1^{(j)}$ or $\gamma_{2(\#\Pi_j)+1}^{(j)}$.

Step 2. Estimating $\dfrac{|f_i(x)-f_i(y)|}{|x-y|}$ **for** $x, y \in \Gamma_i$. Given distinct points $x, y \in \Gamma_i$, we assume that there is a largest path $\alpha \in \Xi^*$ such that both x and y lie in $S_\alpha(\Gamma_j)$ (α may be an empty word). Write $x' = S_\alpha^{-1}(x)$, $y' = S_\alpha^{-1}(y)(\in \Gamma_j)$, and suppose $x' \in \gamma_t^{(j)}$ and $y' \in \gamma_{t'}^{(j)}$. Under the assumption on α, the following case is *impossible*:

$$t = t' \text{ and } \gamma_t^{(j)} \in \Pi_j.$$

We shall distinguish three cases.

Case 1. $t = t'$ and $\gamma_t^{(j)} \notin \Pi_j$.

Since $\mu_j(\gamma_t^{(j)}) = 0$, we have

$$|f_i(x) - f_i(y)| = (r_\alpha)^\tau \left|f_j(x') - f_j(y')\right| \le (r_\alpha)^\tau \mu_j(\gamma_t^{(j)}) = 0.$$

Case 2. $|t-t'|>1$.

We have $|x'-y'| \ge \min_{|t_1-t_2|>1} d(\gamma_{t_1}^{(j)}, \gamma_{t_2}^{(j)})$, and thus

$$|x-y| = |S_\alpha(x') - S_\alpha(y')| = r_\alpha |x'-y'| \ge r_\alpha [\min_{|t_1-t_2|>1} d(\gamma_{t_1}^{(j)}, \gamma_{t_2}^{(j)})],$$

which implies that $r_\alpha \le |x-y| / [\min_{|t_1-t_2|>1} d(\gamma_{t_1}^{(j)}, \gamma_{t_2}^{(j)})]$. Hence

$$\begin{aligned}
\frac{|f_i(x)-f_i(y)|}{|x-y|} &= \frac{|f_i(S_\alpha(x')) - f_i(S_\alpha(y'))|}{|S_\alpha(x') - S_\alpha(y')|} \\
&= \frac{(r_\alpha)^\tau \left|f_j(x') - f_j(y')\right|}{r_\alpha |x'-y'|} \\
&\le (r_\alpha)^{\tau-1} \mu_j(\Gamma_j) [\min_{|t_1-t_2|>1} d(\gamma_{t_1}^{(j)}, \gamma_{t_2}^{(j)})]^{-1} \\
&\le |x-y|^{\tau-1} \mu_j(\Gamma_j) [\min_{|t_1-t_2|>1} d(\gamma_{t_1}^{(j)}, \gamma_{t_2}^{(j)})]^{-\tau}.
\end{aligned}$$

Therefore, in this case we have

$$|f_i(x) - f_i(y)| = o(|x - y|).$$

Case 3. $|t - t'| = 1$.

We assume that $t' = t + 1$ and t' is even. Suppose $\gamma_{t+1}^{(j)} = S_{e^*}(\Gamma_p) \subset \Gamma_j$ with $e^* \in \Xi$. Then $\Gamma_p = \cup_{u=1}^{2(\#\Pi_p)+1} \gamma_u^{(p)}$. Suppose $\gamma_t^{(j)} \cap \gamma_{t+1}^{(j)} = \{z'\}$. Without loss of generality, let $z' \in S_{e^*}(\gamma_1^{(p)})$.

If $y' \in S_{e^*}(\gamma_1^{(p)})$, then

$$\begin{aligned} \left|f_j(x') - f_j(y')\right| &= \mu_j(\Gamma_j(x', y')) \\ &= \mu_j(\Gamma_j(x', z')) + \mu_j(\Gamma_j(z', y')) \\ &= 0 + 0 = 0. \end{aligned}$$

As a result, we have

$$|f_i(x) - f_i(y)| = (r_\alpha)^\tau \left|f_j(x') - f_j(y')\right| = 0.$$

If $y' \notin S_{e^*}(\gamma_1^{(p)})$, then

$$\left|f_j(x') - f_j(y')\right| = \mu_j(\Gamma_j(x', y')) \leq \mu_j(\Gamma_j),$$

and $|x' - y'| \geq \min_{u>1}(\gamma_t^{(j)}, S_{e^*}(\gamma_u^{(p)}))$. As in the proof of Case 2, we also get $|f_i(x) - f_i(y)| = o(|x - y|)$.

By the above two steps, we have

$$|f_h(x) - f_h(y)| = o(|x - y|),$$

where f_h is constant when restricted on $\gamma_1^{(h)}$ or $\gamma_{2(\#\Pi_h)+1}^{(h)}$. For endpoints a, b, we have $|f_h(a) - f_h(b)| = \mu_h(\Gamma_h) = x_h > 0$. Therefore, $\Gamma_h \in \Omega$. □

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