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BOOLEAN COMPACTIFICATIONS OF A SEMIGROUP

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Abstract. We call a semigroup compactification T of a (discrete) semigroup S Boolean if the underlying space of T is zero-dimensional. The class of Boolean compactifications of S is a complete lattice, in a natural way. Motivated by Numakura's theoerem, we prove that this lattice is isomorphic to the ideal lattice of the lattice of finite congruence relations of S. We describe in some detail the lattice of compactifications of S, for some particular examples of semigroups S.

1. Introduction

It is well-known that for every discrete semigroup S, the Stone-Čech compactification βS of S can be made in a natural way into a semigroup, too. Unlike S, βS has always idempotent elements, and they can be used to produce very elegant proofs of combinatorial results. A seemingly unpleasant fact about βS, however, is that the semigroup multiplication on βS is only right continuous, i.e. right multiplication with a fixed element x is continuous for all $x \in \beta S$, but left multiplication with x can fail to be continuous. The algebraic aspects of βS and their applications to combinatorics are presented in Hindman and Strauss's monograph [5].

Attention has also been paid to more general notions of compactifications of a semigroup. One possible line of generalization consists in starting with semigroups S with a topology in which multiplication is assumed to be right continuous, both left and right

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continuous or even continuous as a binary function from $S \times S$ to S; another one in requiring that multiplication in the compactifications of S is right continuous, both left and right continuous or continuous as a binary function. Cf. Chapter 21 in [5] and its references to earlier literature. A much more general study of semigroup compactifications can be found in Chapter 3 of the the monograph [1] by Bergelson, Junghenn, and Milnes.

In this paper, we study in some detail what we call Boolean compactifications of an arbitrary discrete semigroup S: we require the compactifying semigroups of S to be topological semigroups with a Boolean (i.e. compact and zero-dimensional) topology. Our results paraphrase a classical result by Numakura [8] stating that Boolean topological semigroups have a simple and beautiful structure: they are inverse limits of finite discrete semigroups. More exactly, we will use Numakura's theorem to describe *all* Boolean compactifications of a fixed semigroup S − they correspond to the ideals of the lattice of finite congruence relations on S, larger ideals giving rise to larger compactifications. In general, it seems easier to completely describe the lattice of finite congruence relations on S and its ideals than the lattice of Boolean compactifications of S. E.g. in the very special example of the semigroup $\mathbb{Z}$ of integers under addition, we obtain in this way a perfectly clear image of the lattice $\mathrm{BC}(\mathbb{Z})$ of Boolean compactifications of $\mathbb{Z}$. It turns out that they are intimately connected to the additive groups of the p-adic integers $\mathbb{Z}_p$, for prime numbers p.

After developing the relevant notions in Sections 2 and 3, we prove the main result in Section 4 by establishing an isomorphism L from the complete lattice of ideals on the lattice $\mathrm{fcr}(S)$ of finite congruence relations on S onto the lattice $\mathrm{BC}(S)$ of Boolean compactifications of S and describing the inverse mapping R of L. In Section 5, we draw conclusions from this theory, for arbitrary semigroups S. In Section 6, we describe the lattices $\mathrm{BC}(\mathbb{Z},+)$ and $\mathrm{BC}(\mathbb{N},+)$ in some detail and make remarks on $\mathrm{BC}(S)$ for some other semigroups S.

Let us recall some notation and notions from set theory, topology and Boolean algebra. We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of positive integers and $\omega = \{0, 1, 2, \dots\} = \{0\} \cup \mathbb{N}$. $|X|$ is the cardinality of a set X.

Compact spaces are always assumed to be Hausdorff. A subset of a topological space X is *clopen* if it is both closed and open; $\mathrm{Clop}(X) = \{A \subseteq X : A \text{ is clopen}\}$ is the Boolean algebra of clopen subsets of X. A *Boolean space* is a compact space X which has a base consisting of clopen sets, i.e. which has $\mathrm{Clop}(X)$ as a base. Note that closed subspaces of Boolean spaces and products of arbitrarily many Boolean spaces are Boolean.

For X and Y topological spaces, an *embedding* from X into Y is a homeomorphism $e : X \to Y$ from X onto the subspace $e[X]$ of Y. For X discrete, this means that e is one-one and $e(x)$ is an isolated point of Y, for every $x \in X$.

A *topological semigroup* is a structure $(S, \cdot, \mathcal{O})$ such that $(S, \cdot)$ is a semigroup, $\mathcal{O}$ is a topology on S, and the semigroup multiplication $\cdot : S \times S \to S$ is continuous. We call it a *Boolean topological semigroup* if its underlying space is Boolean.

For unexplained definitions and further results, cf. [3] in topology, [5] in (topological) semigroups, and [7] in Boolean algebras.

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2. The partial order of Boolean compactifications

In the following, we assume that a fixed discrete semigroup $(S, \cdot)$ is given.

Definition 2.1. A *Boolean compactification* of S is a pair (e, T) where T is a Boolean topological semigroup, $e : S \to T$ is a semigroup homomorphism, and the image $e[S]$ of S under e is dense in T. The class of all Boolean compactifications of S is denoted by $\mathrm{BC}(S)$.

Let us point out that we do not assume the map e to be one-one, let alone a topological embedding. We will address later the question whether S has a Boolean compactification (e, T) in which e is one-one or even an embedding. The answer depends largely on the properties of the semigroup S.

Remark 2.2. The class $\mathrm{BC}(S)$ is non-empty: let $T_{min} = \{t\}$ be a one-element semigroup (with $t \cdot t = t$) and $e_{min}(s) = t$ for all $s \in S$. We call $c_{min} = (e_{min}, T_{min})$ the trivial or the least Boolean compactification of S.

As usual in topology, we define the natural partial ordering (strictly speaking, a quasi-ordering) on the class of Boolean compactifications of S.

Definition and Remark 2.3. Given two Boolean compactifications $c = (e, T)$ and $c' = (e', T')$ of S, we define $c' \leq c$ if there is a continuous semigroup homomorphism $h : T \to T'$ such that $h \circ e = e'$.

In this case, the map $h : T \to T'$ is uniquely determined on $e[S]$ and hence on the whole of T, since $e[S]$ is dense in T; for the same reason it maps T onto T'. We call h the connecting map (between c and c') and sometimes denote it by $h_{cc'}$.

Definition 2.4. The Boolean compactifications $c = (e, T)$ and $c' = (e', T')$ of S are *equivalent* (and we write $c \approx c'$) if both $c' \leq c$ and $c \leq c'$ hold. This means that there is a uniquely determined continuous isomorphism $h : T \to T'$ satisfying $h \circ e = e'$, i.e. the connecting maps $h_{cc'}$ and $h_{c'c}$ are bijective and inverses of each other.

In a Boolean compactification (e, T) of S, the space T can have cardinality at most $2^{2^{|S|}}$, since $e[S]$ is dense in T. It follows that, up to equivalence of compactifications, $\mathrm{BC}(S)$ is a *set*. We will often speak about $\mathrm{BC}(S)$, having in mind the quotient set $\mathrm{BC}(S)/\approx$. E.g. we speak about the partial order $(\mathrm{BC}(S), \leq)$, having in mind $\mathrm{BC}(S)/\approx$ with the induced partial ordering. This partial order has a least element, the trivial compactification of S as defined in 2.2. It has, in fact, a much better structure: it is a complete lattice. The analogue, for the partial order of all compactifications of an arbitrary completely regular space, seems to have been noticed first by Bing (cf. [2], [3]).

Proposition 2.5. *The partial order* $(\mathrm{BC}(S), \leq)$ *is a complete lattice. I.e. every subset of* $\mathrm{BC}(S)$ *has a least upper bound and a greatest lower bound.*

Proof. We prove that every subset M of $\mathrm{BC}(S)$ has a least upper bound; this suffices because we know that $(\mathrm{BC}(S), \leq)$ has a least element. Write $m = (e_m, T_m)$ for every $m \in M$.

Consider the cartesian product $\prod_{m\in M} T_m$ of the (Boolean topological) semigroups T_m, a Boolean topological semigroup with coordinatewise multiplication and the product topology, and the homomorphism $e : S \to \prod_{m\in M} T_m$ mapping each $x \in S$ to $(e_m(x))_{m\in M}$. Letting T be the closure of $e[S]$ in $\prod_{m\in M} T_m$, we see that (e, T) is a Boolean compactification of S; in particular, T is a subsemigroup of $\prod_{m\in M} T_m$ because the latter is a topological semigroup. And for $m \in M$, the projection $pr_m : T \to T_m$ mapping every element of T to its m'th coordinate shows that $(e_m, T_m) \leq (e, T)$.

Assume that (ε, Y) is another upper bound of M in $\mathrm{BC}(S)$, with the aim of showing that $(e, T) \leq (\varepsilon, Y)$. For every $m \in M$, we have the connecting map $h_m : Y \to T_m$ which satisfies $h_m \circ \varepsilon = e_m$. This gives the continuous homomorphism $h : Y \to \prod_{m\in M} T_m$ defined by $h(y) = (h_m(y))_{m\in M}$ and satisfying $h \circ \varepsilon = e$. Moreover, h maps $\varepsilon[S]$ into $e[S] \subseteq T$ and thus Y into T. So $(e, T) \leq (\varepsilon, Y)$. □

We will write $\sup M$ for the least upper bound (e, T) of M in $\mathrm{BC}(S)$ or even for its underlying Boolean topological semigroup T. In particular, $\mathrm{BC}(S)$ has a greatest element $c_{max} = \sup(\mathrm{BC}(S))$.

The statement that S has a greatest Boolean compactification can be obtained as a special case of a more general fact, the adjoint functor theorem (cf. [4]). More precisely, consider the inclusion functor F from the category $\mathcal{B}$ of Boolean topological semigroups to the category $\mathcal{S}$ of Hausdorff topological semigroups. Now F has a left adjoint $G : \mathcal{S} \to \mathcal{B}$, and for arbitrary $S \in \mathcal{S}$, $G(S)$ is the greatest Boolean compactification of S.

In the situation of the preceding proof, it seems natural to ask whether the closure T of $e[S]$ in $\prod_{m\in M} T_m$ can coincide with the whole of $\prod_{m\in M} T_m$, i.e. whether $e[S]$ can be dense in $\prod_{m\in M} T_m$. The answer depends largely on the given subset M of $\mathrm{BC}(S)$, respectively on its structure under the partial order $\leq$ of $\mathrm{BC}(S)$. It will be convenient to have an extra notion for this situation.

Remark and Definition 2.6. Assume M is a subset of $\mathrm{BC}(S)$ and write $m = (e_m, T_m)$ for $m \in M$. Then $e[S]$ is dense in $\prod_{m\in M} T_m$ iff, for every finite subset J of M and every family $(U_m)_{m\in J}$ of non-empty open subsets $U_m \subseteq T_m$, there is some $x \in S$ such that $e_m(x) \in U_m$, for all $m \in J$. We call M resp. the family $(e_m, T_m)_{m\in M}$ of compactifications *independent* in this case.

E.g. if M contains two elements m and n with $n < m$, then for every $x \in S$, $e_n(x) = h_{mn}(e_m(x))$ is determined by $e_m(x)$; so if T_n has at least two elements, M cannot be independent. The theory developed in Section 4 will explain the notion of independence in much finer detail. In the present situation, we can make just one more remark.

Remark 2.7. To make things precise, we assume here that M is an arbitrary index set and, for $m \in M$, c_m is a Boolean compactification of S. The family $(c_m)_{m \in M}$ is independent iff for every finite $J \subseteq M$, the subfamily $(c_m)_{m \in J}$ is independent.

If, for $m \in M$, c'_m is another Boolean compactification of S satisfying $c'_m \leq c_m$ and $(c_m)_{m \in M}$ is independent, then so is $(c'_m)_{m \in M}$.

3. Finite congruence relations

There is an obvious connection between finite Boolean compactifications and finite congruence relations of a semigroup S.

Definition and Remark 3.1. We call a Boolean compactification (e, T) of S *finite* if its compactifying space T is finite. It follows then that T is discrete (being Hausdorff) and that the homomorphism $e : S \to T$ is onto. Conversely, every homomorphism e from S onto a finite semigroup T gives the finite Boolean compactification (e, T) where we equip T with the discrete topology. I.e. the finite Boolean compactifications of S are simply the finite (discrete) homomorphic images of S.

We denote the subclass (respectively sub*set*, up to $\approx$) of $\mathrm{BC}(S)$ consisting of all finite Boolean compactifications by $\mathrm{FC}(S)$. It is an ideal (in the sense of lattice theory) of the lattice $\mathrm{BC}(S)$: if $c' \leq c$ in $\mathrm{BC}(S)$ and the compactification c is finite, then so is c'. And for any $c = (e, T)$ and $c' = (e', T')$ in $\mathrm{FC}(S)$, the least upper bound of c and c' in $\mathrm{BC}(S)$ is again finite, being a subspace of $T \times T'$. In particular, $\mathrm{FC}(S)$ is a sublattice of $\mathrm{BC}(S)$ – however, it is not a complete lattice, except in very special cases. Cf. 3.4 for a counterexample.

Definition 3.2. Call a congruence relation r on S *finite* if the set S/r of congruence classes of S with respect to r is finite. As usual in universal algebra, we have a correspondence between finite congruence relations and finite homomorphic images, i.e. finite Boolean compactifications, of S.

More precisely, for a finite congruence relation r on S, we obtain the finite Boolean compactification

$$\gamma(r) = (\pi_r, S/r)$$

of S, where S/r is the quotient semigroup of S under r and $\pi_r : S \to S/r$ is the canonical epimorphism mapping $x \in S$ to its congruence class $x/r = \{y \in S : (x, y) \in r\}$. And conversely, every finite Boolean compactification (e, T) of S induces the finite congruence relation $r = \{(x, y) \in S \times S : e(x) = e(y)\}$ on S.

We denote the set of finite congruence relations on S by $\mathrm{fcr}(S)$.

Remark and Definition 3.3. Two finite Boolean compactifications c and c' are equivalent iff they induce the same congruence relation. I.e. mapping each finite congruence relation r to the compactification $\gamma(r)$ defined as above gives a bijection $\gamma : \mathrm{fcr}(S) \to \mathrm{FC}(S)/\approx$. For the sake of brevity, we write again $\mathrm{FC}(S)$ for $\mathrm{FC}(S)/\approx$, and

$$\gamma : \mathrm{fcr}(S) \to \mathrm{FC}(S).$$

Under the bijection γ, the lattice structure of $\mathrm{FC}(S)$ corresponds to a lattice structure on $\mathrm{fcr}(S)$ which turns out to be the most natural one: for r and s in $\mathrm{fcr}(S)$, $\gamma(r) \leq \gamma(s)$ holds in the partial order $\leq$ on $\mathrm{FC}(S)$ iff $s \subseteq r$ (as relations on S, i.e. subsets of $S \times S$), i.e iff elements of S identified by s are also identified by r. And in this case, the connecting homomorphism from $\gamma(s)$ to $\gamma(r)$ is obviously the canonical epimorphism $\pi_{sr} : S/s \to S/r$ mapping x/s to x/r, for $x \in S$.

We therefore define the partial ordering $\leq$ on $\mathrm{fcr}(S)$ by putting $r \leq s$ iff $s \subseteq r$ (iff $\gamma(r) \leq \gamma(s)$), making $\gamma : \mathrm{fcr}(S) \to \mathrm{FC}(S)$ a lattice isomorphism. E.g., the least element of $\mathrm{fcr}(S)$ under this partial ordering is $S \times S$.

Moreover for $r, s \in \mathrm{fcr}(S)$, the greatest lower bound of r and s in $\mathrm{fcr}(S)$ is the congruence relation of S generated by $r \cup s$, and their least upper bound is $r \cap s$ - for the latter assertion, note that $r \cap s$ is a congruence relation on S and it is finite because the congruence class $x/(r\cap s)$ of any $x \in S$ with respect to $r\cap s$ is simply $x/r \cap x/s$.

Example 3.4. Consider the (semi-)group $\mathbb{Z}$ of integers under addition; we will study this example in more detail in Section 6. The finite congruence relations on $\mathbb{Z}$ are well-known; they are simply the congruence relations $\equiv_d$ (congruence modulo d), for $d \in \mathbb{N}$. For $d, e \in \mathbb{N}$, $\equiv_d \leq \equiv_e$ holds in the lattice $\mathrm{fcr}(\mathbb{Z})$ iff d divides e. I.e. $\mathrm{fcr}(\mathbb{Z})$ is isomorphic to $\mathbb{N}$ with the partial ordering of divisibility.

In particular, $\mathrm{fcr}(\mathbb{Z})$ has no greatest element, and thus the lattices $\mathrm{fcr}(\mathbb{Z})$ and $\mathrm{FC}(\mathbb{Z})$ are not complete.

4. Boolean compactifications and ideals of $\mathrm{fcr}(S)$

We set up in this section a one-one correspondence between Boolean compactifications of a semigroup S (up to equivalence as in 2.4) and ideals in the lattice $\mathrm{fcr}(S)$ of finite congruence relations on S. The Boolean compactification corresponding to an ideal I of $\mathrm{fcr}(S)$ will be the inverse limit of the (finite and discrete, hence Boolean topological) semigroups S/r, $r \in I$. We briefly recall the definition of an inverse limit over a directed set, i.e a partially ordered set in which every finite subset has an upper bound. Cf. [3] for details on inverse limits of topological spaces.

Definition 4.1. Assume $(M, \leq)$ is a directed set, T_m is a set for $m \in M$, and for $l \leq m$ in M, we are given a map $\pi_{ml} : T_m \to T_l$ such that π_{mm} is the identity function on T_m, for $m \in M$, and for $k \leq l \leq m$, the equation $\pi_{mk} = \pi_{lk} \circ \pi_{ml}$ holds. We call $\mathcal{S} = ((T_m)_{m\in M}, (\pi_{ml}))_{l\leq m})$ an *inverse system* over M in this case. The *inverse limit* of $\mathcal{S}$ is the set

$$X = \mathrm{liminv}\ \mathcal{S} = \{x \in \prod_{m\in M} T_m : \pi_{ml}(x_m) = x_l, \text{ for } l \leq m \text{ in } M\}.$$

For $m \in M$, there is a canonical map $\pi_m : X \to T_m$ – the projection from $\prod_{m\in M} T_m$ to T_m, restricted to X; for $l \leq m$ in M, we have $\pi_{ml} \circ \pi_m = \pi_l$.

If the sets T_m are Hausdorff spaces resp. semigroups and the bonding maps π_{ml} are continuous resp. homomorphisms, then X is a closed subspace resp. a subsemigroup of $\prod_{m\in M} T_m$, and the canonical maps $\pi_m : X \to T_m$ are continuous resp. homomorphisms. In particular, the inverse limit of an inverse system of Boolean topological semigroups is a Boolean topological semigroup.

Remark 4.2. When dealing with inverse limits, we often use the following canonical base $\mathcal{B}$ for the inverse limit space, assuming the preceding notation. For $m \in M$ and U an open subset of T_m, $B_{mU} = \{x \in X : x_m \in U\}$ is an open subset of X; we put $\mathcal{B} = \{B_{mU} : m \in M \text{ and } U \subseteq T_m \text{ open}\}$. We say that B_{mU} is represented over m. Note that B_{mU} is then also represented over every $n \geq m$ in M, because $B_{mU} = B_{nV}$, for $V = \pi_{nm}^{-1}[U]$.

Consequently, if the spaces T_m are Boolean and $X = a_1 \cup \dots \cup a_k$ is a partition of X into clopen subsets, then there are $m \in M$ and a clopen partition $T_m = U_1 \cup \dots \cup U_k$ of T_m such that $a_i = B_{m,U_i}$, for $1 \leq i \leq k$.

Every directed subset M of the lattice $(\mathrm{BC}(S), \leq)$ of Boolean compactifications of S gives rise to an inverse system over M, in the following way.

Example 4.3. Assume M is a directed subset of $(\mathrm{BC}(S), \leq)$; we write $m = (e_m, T_m)$ for $m \in M$. For $l \leq m$ in M, we have the connecting map $h_{ml} : T_m \to T_l$ as defined in 2.3. And by the uniqueness statement of 2.3, $\mathcal{S}(M) = ((T_m)_{m \in M}, (h_{ml}))_{l \leq m})$ is an inverse system over M.

Consider, in addition to the inverse limit X of $\mathcal{S}(M)$, the least upper bound $(e, T) = \sup M$ of M in $\mathrm{BC}(S)$, as constructed in the proof of 2.5. Both X and the underlying Boolean semigroup T of $\sup M$ are closed subsemigroups of the product semigroup $P = \prod_{m \in M} T_m$. Recall from 2.5 that $e(x) = (e_m(x))_{m \in M}$ holds for $x \in S$ and that T is the closure of $\{e(x) : x \in S\}$ in P.

Proposition 4.4. *For M a directed subset of $(\mathrm{BC}(S), \leq)$, the inverse limit X of $\mathcal{S}(M)$ coincides with the underlying Boolean semigroup T of the least upper bound (e, T) of M in $\mathrm{BC}(S)$. In particular, (e, X) is a Boolean compactification of S and the least upper bound of M in $(\mathrm{BC}(S), \leq)$.*

Hence we will also write $\sup M$ *for* $\operatorname{liminv} \mathcal{S}(M)$.

Proof. To prove that $T \subseteq X$, note that $e(x) \in X$ for every $x \in S$, by the commutativity properties of the connecting maps h_{ml}, and that X is a closed subset of P.

To prove that $X \subseteq T$, let U be an open subset of P which intersects X, with the aim of finding some $x \in S$ such that $e(x) \in U$. We can assume that

$$U = \{p \in P : p(k) \in U_k, \text{ for all } k \in K\},$$

where K is a finite subset of M and U_k is open in T_k, for $k \in K$. Fix some upper bound m of K, in M, and consider the open subset

$$D = \bigcap_{k \in K} h_{mk}^{-1}[U_k]$$

of T_m. It suffices to show that D is non-empty. For then, pick $x \in S$ with $e_m(x) \in D$; for all $k \in K$, x satifies $e_k(x) = h_{mk}(e_m(x)) \in U_k$ and thus $e(x) \in U$. To see that $D \neq \emptyset$, pick some $p \in U \cap X$. Then $p_m \in D$ because $p \in X$ and thus for all $k \in K$, $h_{mk}(p_m) = p_k \in U_k$ holds. $\square$

Under the lattice isomorphism $\gamma : \mathrm{fcr}(S) \to \mathrm{FC}(S)$, every ideal I in the lattice of finite congruence relations of S gives rise to the ideal $M = \gamma[I]$ in the lattice of finite compactifications of S, and hence to the Boolean compactification liminv $\mathcal{S}(M)$ of S.

Definition 4.5. We write $\mathrm{Id}(S)$ for the set of all ideals of the lattice $\mathrm{fcr}(S)$. It is a complete lattice, under set theoretic inclusion: the least upper bound of a subset $\mathcal{I}$ of $\mathrm{Id}(S)$ is the ideal of $\mathrm{fcr}(S)$ generated by $\bigcup \mathcal{I}$.

We define a map

$$L : \mathrm{Id}(S) \to \mathrm{BC}(S)$$

by putting, for $I \in \mathrm{Id}(S)$,

$$L(I) = \mathrm{liminv}\, \mathcal{S}(\gamma[I]) = \sup \gamma[I],$$

the least upper bound of $\gamma[I]$ in $\mathrm{BC}(S)$. We will also write

$$L(I) = \mathrm{liminv}\, (S/r)_{r \in I},$$

notationally ignoring the bonding maps $\pi_{sr} : S/s \to S/r$, for $r \leq s$ in I, and the canonical map $e : S \to L(I)$ given by $e(x) = (x/r)_{r \in I}$.

The canonical base described in 4.2 for an arbitrary inverse limit gives the clopen base $\mathcal{B} = \{U_{rx} : r \in I, x \in S\}$ for $L(I)$, where $U_{rx} = \{z \in X : z_r = x/r\}$.

The map L is obviously order preserving in the following sense. If $I \subseteq J$ are ideals of $\mathrm{fcr}(S)$, then $\gamma[I] \subseteq \gamma[J]$ and hence $L(I) = \sup \gamma[I] \leq \sup \gamma[J] = L(J)$ holds in $\mathrm{BC}(S)$.

We begin to describe the map R inverse to $L : \mathrm{Id}(S) \to \mathrm{BC}(S)$.

Definition and Remark 4.6. Assume $c = (e, T)$ is a Boolean compactification of S. The set of those finite compactifications d of S which satisfy $d \leq c$, in $\mathrm{BC}(S)$, is clearly an ideal of the lattice $\mathrm{FC}(S)$. Under the lattice isomorphism $\gamma : \mathrm{fcr}(S) \to \mathrm{FC}(S)$ of 3.3, it corresponds to an ideal of the lattice $\mathrm{fcr}(S)$:

$$R(c) = \{r \in \mathrm{fcr}(S) : \gamma(r) \leq c\}.$$

This gives an order preserving map

$$R : \mathrm{BC}(S) \to \mathrm{Id}(S).$$

Because of part (b) of the following proposition, the congruence relations in $R(c)$ are said to be *realized* by c.

The congruence relations realized by c are described in the following proposition. Note that, for $c = (e, T) \in \mathrm{BC}(S)$, $r \in \mathrm{fcr}(S)$, and $x \in S$, the congruence class x/r of x is a subset of S and its image $e[x/r]$ under e is a subset of T. For $(x, y) \notin r$, x/r and y/r are disjoint subsets of S, but their images $e[x/r]$ and $e[y/r]$ in T need not be disjoint since e is not assumed to be one-one.

Proposition 4.7. *Assume that* $c = (e, T) \in \mathrm{BC}(S)$ *and* $r \in \mathrm{fcr}(S)$. *The following are equivalent.*

(a) r *is realized by* c, *i.e.* $\gamma(r) \leq c$

(b) for $x, y \in S$ *such that* $(x, y) \notin r$, *the closures of* $e[x/r]$ *and* $e[y/r]$ *in* T *are disjoint*

(c) there is an equivalence relation $\bar{r}$ *on* T *with closed equivalence classes which extends* r *via* e, *i.e. for* $x, y \in S$, $(x, y) \in r$ *iff* $(e(x), e(y)) \in \bar{r}$.

Proof. (a) implies (b). Let $h : T \to S/r$ be the connecting map witnessing $\gamma(r) \leq c$. For $x \in S$, h maps $e(x)$ to $\pi_r(x) = x/r$. So h maps the whole of $e[x/r]$ to the point x/r of S/r; by continuity, it maps the closure a_x of $e[x/r]$ to x/r. Thus a_x is the preimage of the point $x/r \in S/r$ under h, and for $(x, y) \notin r$, a_x and a_y are disjoint.

(b) implies (c). In the situation of (b), we have the clopen partition $T = \bigcup_{x\in S} a_x$ where a_x is the closure of $e[x/r]$ in T and $a_x = a_y$ iff $(x, y) \in r$ and $a_x \cap a_y = \emptyset$ otherwise; in particular, $e(x) \in a_x$ holds for every x. Let $\bar{r}$ be the equivalence relation on T with equivalence classes a_x, $x \in S$. Then for $x, y \in S$, $(e(x), e(y)) \in \bar{r}$ iff $a_x = a_y$ iff $(x, y) \in r$.

(c) implies (a). Assume that $\bar{r}$ extends r, as assumed in (c). For $x \in S$, write a_x for the equivalence class of $e(x)$ with respect to $\bar{r}$. Then $T = \bigcup_{x\in S} a_x$ because $S = \bigcup_{x\in S} x/r$, $e[S] = \bigcup_{x\in S} e[x/r]$ (a finite union because there are only finitely many congruence classes x/r), and T is the closure of $e[S]$. And $a_x = a_y$ holds iff $(e(x), e(y)) \in \bar{r}$ iff $(x, y) \in r$ iff $x/r = y/r$.

Thus we obtain a function $h : T \to S/r$ satisfying $h \circ e = \pi_r$ by mapping the points of a_x to the point x/r of S/r, for $x \in S$. h is continuous because the sets a_x are closed. And π_r is a semigroup homomorphism, so by continuity of the semigroup multiplication on T, h is a homomorphism, too. The map h witnesses that $\gamma(r) \leq c$. □

We are ready to prove the main result on Boolean compactifications of a semigroup S.

Theorem 4.8. *The maps L and R are inverses of each other. Thus L is an isomorphism from* $(\mathrm{Id}(S), \leq)$*, the lattice of ideals of* $\mathrm{fcr}(S)$*, onto* $(\mathrm{BC}(S), \leq)$*, the lattice of Boolean semigroup compactifications of S.*

Proof. Assume first that $c = (e, T)$ is a Boolean compactification of S and consider the ideal $I = R(c)$ with the aim of showing that $L(I) = c$, i.e. that c is, up to equivalence of compactifications, the least upper bound of $\gamma[I]$ in $\mathrm{BC}(S)$. By the very definition of I, the ideal of $\mathrm{fcr}(S)$ realized by c, c is an upper bound of $\gamma[I]$. Now Numakura's theorem (cf. [8]) says that T is the inverse limit of some directed system of finite discrete semigroups, i.e. there is a directed subset I_0 of $\mathrm{fcr}(S)$ such that $c = (e, T) = \sup \gamma[I_0]$. Thus $I_0 \subseteq R(c) = I$ and $c = \sup \gamma[I_0] = \sup \gamma[I]$.

For the converse, assume that I is an ideal of $\mathrm{fcr}(S)$ and consider $c = L(I) = \sup \gamma[I]$; we have to show that $R(c) = I$. Clearly, every $r \in I$ is realized by c; so assume ρ is a finite equivalence relation on S realized by c with the aim of proving that $\rho \in I$. We will show that $r \subseteq \rho$ holds for some $r \in I$. Write $c = (e, X)$ and $\gamma(\rho) = (\pi_\rho, S/\rho)$.

Now $\gamma(\rho) \leq c$; so let $h : X \to S/\rho$ be the continuous homomomorphism satisfying $h \circ e = \pi_\rho$ and fix a set of representatives $\{x_1, \ldots, x_k\}$ in S modulo ρ. By continuity of h, the preimages of the points x_i/ρ under h constitute a clopen partition $X = a_1 \cup \cdots \cup a_k$ of X; by Remark 4.2, fix some $r \in I$ and a clopen partition $T_r = U_1 \cup \cdots \cup U_k$ of $T_r = S/r$ such that $a_i = B_{r,U_i}$, for $1 \leq i \leq k$. Thus for $p \in X$, $p \in a_i$ holds iff $p_r \in U_i$. Also note that for $x \in S$ and $p = e(x)$, the r'th coordinate of p is $\pi_r(x) = x/r$ and its ρ'th coordinate is $\pi_\rho(x) = x/\rho$. So for $i \in \{1, \ldots, k\}$, we have $(x, x_i) \in \rho$ iff $x/r \in U_i$. So for x and y in S, $(x, y) \in r$ implies that x/r and y/r are both in U_i, for some i. For this i, both $(x, x_i) \in \rho$ and $(y, x_i) \in \rho$ hold, hence $(x, y) \in \rho$. Thus we have shown that $r \subseteq \rho$, i.e. that $\rho \leq r$ holds in the lattice $\mathrm{fcr}(S)$. It follows that $\rho \in I$, since I is an ideal of $\mathrm{fcr}(S)$. □

5. Consequences of the Main Theorem

In this section, we draw several conclusions from Theorem 4.8: we characterize properties of a Boolean compactification $c \in \mathrm{BC}(S)$ by properties of its corresponding ideal $I = R(c) \in \mathrm{fcr}(S)$. Respectively, we describe properties of the Boolean compactification $L(I)$ by properties of I.

Remark 5.1. In Section 2, completeness of the lattice $\mathrm{BC}(S)$ gave the existence of a greatest Boolean compactification c_{max} of S. By 4.8, it corresponds to the greatest ideal I of $\mathrm{fcr}(S)$, i.e. to $I = \mathrm{fcr}(S)$. Thus $c_{max} = (e_{max}, T_{max})$ where T_{max} is the inverse limit of $(S/r)_{r \in \mathrm{fcr}(S)}$ and $e_{max}(x) = (x/r)_{r \in \mathrm{fcr}(S)}$, for $x \in S$.

Definition and Remark 5.2. Let us call a Boolean compactification (e, T) of S *injective* resp. *topological* iff $e : S \to T$ is injective, resp. an embedding. We assumed that S is given the discrete topology, so (e, T) is topological iff e is one-one and $e(x)$ is an isolated point of T, for all $x \in S$. Obviously, if $c = (e, T)$ is injective resp. topological and $c \leq c'$ in the partial order of Boolean compactifications, then so is c'. Thus S has an injective resp. topological compactification iff its greatest Boolean compactification c_{max} is injective resp. topological.

Remark 5.3. Assume $c = (e, T)$ is a Boolean compactification of S, where I is an ideal of $\mathrm{fcr}(S)$. Clearly, c is injective iff the congruence relations in I separate points, i.e. for $x \neq y$ in S, there is some $r \in I$ such that $(x, y) \notin r$ (r *separates* x and y). And c is topological iff for every $x \in S$, there is some $r \in I$ which *isolates* x, i.e. $x/r = \{y \in S : (x, y) \in r\} = \{x\}$.

In particular, S has an injective Boolean compactification iff for $x \neq y$ in S, some $r \in \mathrm{fcr}(S)$ separates x and y. Similarly, S has a topological Boolean compactification iff for every x in S, some $r \in \mathrm{fcr}(S)$ isolates x.

For the next remark, we recall from topology the definition of the weight $w(X)$ of a topological space X: it is the minimum of the cardinalities of open bases of X. For X an infinite Boolean space, $w(X)$ is the cardinality of the base $\mathrm{Clop}(X)$ of X.

Let I be an ideal of $\mathrm{fcr}(S)$; we want to compute the weight of the Boolean compactification $L(I)$ corresponding to I. If I is finitely generated, then it has a greatest element r, $I = \{s \in \mathrm{fcr}(S) : s \leq r\}$ is finite (cf. 5.5 below), and $L(I)$ is canonically isomorphic to the finite semigroup S/r. This leaves the case of an infinite I.

Remark 5.4. Assume I is an infinite ideal of $\mathrm{fcr}(S)$. Then the weight of $L(I)$ is $|I|$.

For a proof, write $L(I) = (e, X)$ where $X = \mathrm{liminv}(S/r)_{r \in I}$. Let $\mathcal{B}$ be the canonical base of X as defined in 4.2. Clearly, $|\mathcal{B}| = |I|$; on the other hand, $|\mathcal{B}| = |\mathrm{Clop}(X)|$ since by compactness, every clopen subset of X is a finite union of sets in $\mathcal{B}$. Thus $w(X) = |\mathrm{Clop}(X)| = |I|$.

In particular, the greatest Boolean compactification c_{max} of S has weight $|\mathrm{fcr}(S)|$, if $\mathrm{fcr}(S)$ is infinite.

Remark 5.5. Note that for every finite congruence relation r on S, there are only finitely many $s \in \mathrm{fcr}(S)$ satisfying $s \leq r$ – this is because $s \leq r$ means that every congruence class of s is the union of congruence classes of r. Thus if M is an infinite subset of $\mathrm{fcr}(S)$ and I is the ideal of $\mathrm{fcr}(S)$ generated by M, then $|I| = |M|$.

It follows that for every cardinal κ satisfying $\omega \leq \kappa \leq |\mathrm{fcr}(S)|$, there is a Boolean compactification of S with weight κ: take some $M \subseteq \mathrm{fcr}(S)$ with $|M| = \kappa$ and put $c = L(I)$ where I is the ideal of $\mathrm{fcr}(S)$ generated by M.

In 2.6, we defined the notion of an independent family of Boolean compactifications of S. We can now explain in more detail which families are independent.

Remark 5.6. (a) Let M be any index set and for $m \in M$, let r_m be a finite congruence relation on S, giving rise to the finite Boolean compactification $c_m = \gamma(r_m) = (\pi_{r_m}, S/r_m)$ (cf. 3.2). Here S/r_m has the discrete topology and the singletons $\{z/r_m\}$, $z \in S$, constitute an open basis of S/r_m. Hence the family $(c_m)_{m \in M}$ is independent iff for every finite subset J of M and for every family $(x_m)_{m \in J}$ of elements of S indexed by J, there is some $x \in S$ such that $(x, x_m) \in r_m$ holds for all $m \in J$, i.e. iff the congruence classes x_m/r_m, $m \in J$, have non-empty intersection. We call the family $(r_m)_{m \in M}$ independent in this case.

(b) Assume now that, for $m \in M$, $c_m = (e_m, T_m)$ is an arbitrary Boolean compactification of S. Then the family $(c_m)_{m \in M}$ is independent iff, for every sequence $(r_m)_{m \in M}$ such that $r_m \in R(c_m)$, the family $(r_m)_{m \in M}$ is independent as defined in (a). (Recall that $R(c_m)$ is the ideal of $\mathrm{fcr}(S)$ corresponding to c_m by 4.5, i.e. $r \in R(c_m)$ iff r is realized by c_m.)

One direction of this follows from 2.7 because $r_m \in R(c_m)$ means that $\gamma(r_m) \leq c_m$, for all $m \in M$. For the other direction, we assume by 4.8, for the sake of concreteness, that $c_m = L(I_m) = \mathrm{liminv}\,(S/r)_{r \in I_m}$ and $e_m(x) = (x/r)_{r \in I_m}$ as in 4.2, where $I_m = R(c_m)$ is an ideal of $\mathrm{fcr}(S)$. Let $J \subseteq M$ be be finite and, for $m \in J$, V_m a non-empty open subset of $\mathrm{liminv}\,(S/r)_{r \in I_m}$, with the aim of finding some $x \in S$ such that $e_m(x) \in V_m$ holds for all $m \in J$. By 4.2, we may assume that each V_m has the form $V_m = U_{r_m, x_m}$ where $r_m \in I_m$ and $x_m \in S$. By assumption, the congruence relations r_m, $m \in J$, are independent, and thus there exists some $x \in S$ such that $(x, x_m) \in r_m$ holds for $m \in J$. But this implies that $e_m(x)(r_m) = x/r_m = x_m/r_m$ holds for $m \in J$, so $e_m(x) \in V_m$.

Stone's duality theory establishes a bijective correspondence (up to homeomorphism resp. isomorphism) between Boolean spaces and Boolean algebras: the algebra dual to a Boolean space T is $\mathrm{Clop}\, T$, the algebra of all clopen subsets of T; the space dual to a Boolean algebra A is the set $\mathrm{Ult}\, A$ of all ultrafilters of A with the Stone topology.

More exactly, the topological Boolean compactifications (e, T) of an infinite discrete space S can be described by subalgebras of the power set algebra $\mathcal{P}(S)$ of S (cf. Section 8.3 in [7]) as follows. Call A an intermediate algebra if it is a subalgebra of $\mathcal{P}(S)$ including the algebra A_{fc} which consists of all finite and all cofinite subsets of S (the finite-cofinite algebra over S). The topological Boolean compactifications of S correspond to the intermediate algebras in a bijective manner, larger compactifications corresponding to larger algebras. In particular, the greatest intermediate algebra $\mathcal{P}(S)$ corresponds to the Stone-Čech compactification of S, and the least intermediate algebra A_{fc} corresponds to the one-point compactification.

We now describe the (underlying spaces of) Boolean semigroup compactifications c of a discrete semigroup S by their dual Boolean algebras $A_c \subseteq \mathcal{P}(S)$.

Definition and Remark 5.7. (a) For a finite congruence relation r on S, put

$$A_r = \{a \subseteq S : a \text{ is a union of congruence classes of } r\},$$

a finite subalgebra of $\mathcal{P}(S)$; the atoms of A_r are the congruence classes of r. Note that $r \leq s$ holds in $\mathrm{fcr}(S)$ iff A_r is a subalgebra of A_s. Thus for I an ideal in $\mathrm{fcr}(S)$,

$$A_I = \bigcup_{r \in I} A_r$$

is a Boolean subalgebra of $\mathcal{P}(S)$.

(b) For $c = (e, T)$ a Boolean semigroup compactification of S, the Boolean algebra dual to T is, by Remark 4.2 (up to isomorphism),

$$A_c = A_I$$

where $I = R(c)$ is the ideal of $\mathrm{fcr}(S)$ corresponding to c. And if c' is another Boolean compactification of S, then $c \leq c'$ holds in $\mathrm{BC}(S)$ iff $R(c) \leq R(c')$ holds in $\mathrm{Id}\ S$, i.e. iff A_c is a subalgebra of $A_{c'}$.

(c) Conversely, assume that A is a Boolean subalgebra of $\mathcal{P}(S)$ such that $A = \bigcup_{r \in M} A_r$, for some directed subset M of $\mathrm{fcr}(S)$ (we call such an algebra a *congruence subalgebra* of $\mathcal{P}(S)$). Then A gives rise to the Boolean semigroup compactification $c = (e, T)$

where $T = \mathrm{Ult}\ A$ and $e(s) = \{a \in A : s \in a\}$, for $s \in S$. And the ideal $R(c)$ corresponding to c is the ideal of $\mathrm{fcr}(S)$ generated by M.

(d) Note that the knowledge of the Boolean algebra $A = A_c$ as defined in (b), up to isomorphism, determines the space T up to homeomorphism, but not the embedding $e : S \to T$ resp. the multiplicative structure of the semigroup T. However if A is realized as a subalgebra of $\mathcal{P}(S)$ which is, in addition, a congruence subalgebra, then $e : S \to T = \mathrm{Ult}A$ and hence the multiplication on T (the continuous extension of the multiplication on S to T , via e), can be reconstructed from A as in (c).

(e) For c a Boolean compactification of S and A_c as defined in (b), we clearly have the following equivalences: c is an injective compactification of S iff A_c separates the points of S; similarly, c is a topological compactification of S iff $\{s\} \in A_c$ holds for all $s \in S$. And c is a one-point compactification of S iff A_c is the finite-cofinite subalgebra of $\mathcal{P}(S)$.

6. Examples. The structure of BC($\mathbb{Z}$) and BC($\mathbb{N}$)

We present a variety of examples of semigroups with largely different behaviour. Some of them have no injective resp. no topological compactification, others have a least topological compactification which can nicely be described.

Example 6.1. Assume that S is a semigroup in which every element has only finitely many factors. Here for a and x in S, we call x a factor of a if there is a finite non-empty subset $\{x_1, \ldots, x_n\}$ of S such that $a = x_1 \cdot \cdots \cdot x_n$ and $x \in \{x_1, \ldots, x_n\}$. We show that in this case there is a least ideal I of $\mathrm{fcr}(S)$ isolating the points of S, i.e. S has a least topological Boolean compactification. Let us first point out that the assumption on S (every element of S has only finitely many factors) holds, e.g., if S is a free semigroup, a free commutative semigroup, or the set of finite subsets of some infinite set J, under the operation of union (a free semilattice over $|J|$ generators, cf. [6]).

For a proof, we let, for $a \in S$, $f(a)$ be the set of all factors of a and note that $a \in f(a)$. The equivalence relation $r(a)$ on S with equivalence classes $\{x\}$, for $x \in f(a)$, plus $S \setminus f(a)$, is a finite congruence relation on S isolating a. Thus the ideal I of $\mathrm{fcr}(S)$ generated by the relations $r(a)$, $a \in S$, isolates the points of S.

It is the least ideal of $\mathrm{fcr}(S)$ with this property. In fact, assume J is an ideal of $\mathrm{fcr}(S)$ isolating points and let $a \in S$, with the aim of proving $r(a) \in J$. But for every $x \in f(a)$, there is an element $r_x \in J$ isolating x, and the least upper bound r of the r_x, $x \in f(a)$, in $\mathrm{fcr}(S)$ is an element of J satisfying $r \subseteq r(a)$; so $r(a) \in J$.

It is plain that, for each of the relations $r(a)$ and thus for every $r \in I$, the congruence classes of R are either singletons or cofinite in S. Therefore A_I, the congruence subalgebra of $\mathcal{P}(S)$ corresponding to the ideal I, is the finite-cofinite algebra over S and the least topological Boolean compactification of S is a one-point compactification.

Example 6.2. Let $(L, \leq)$ be a linear order, a semigroup under the operation $\max(x, y)$. The finite congruence relations of L correspond to the partitions of L into finitely many convex subsets. Thus for $a \in S$, there is a least element $r(a)$ of $\mathrm{fcr}(L)$ isolating a, the equivalence relation with the classes $\{a\}$, $\{x \in L : x < a\}$ and $\{x \in L : a < x\}$. The ideal I of $\mathrm{fcr}(L)$ generated by the $r(a)$, $a \in S$, gives the least topological Boolean compactification c_{top} of L. The elements of the congruence subalgebra A_I of $\mathcal{P}(S)$ are those subsets of S which are finite unions of intervals of L (with or without endpoints). Therefore, c_{top} is (in most cases) strictly larger than the one-point compactification of S.

In the special case that L is the set $\mathbb{N}$ of natural numbers with its usual well-ordering, the dual Boolean algebra of the least topological compactification of $(\mathbb{N}, \max)$ happens to coincide with the dual algebra of the least topological compactification of $(\mathbb{N}, +)$ - by the descriptions above, both of them are the finite-cofinite algebra over $\mathbb{N}$. In fact, this is also the least injective Boolean compactification of $(\mathbb{N}, \max)$.

However in general there are injective compactifications of $(L, \max)$ which are strictly less than c_{top}, but no least one. Let us consider this, for simplicity, for the case that L is a dense linear order without end points. In this case let, for $a \in L$, $s(a)$ be the finite congruence relation of L with congruence classes $\{x \in L : x < a\}$ and $\{x \in L : a \leq x\}$. For every dense subset D of L, the ideal I_D of $\mathrm{fcr}(S)$ generated by $\{s(a) : a \in D\}$ gives an injective compactification c_D of $(L, \max)$, and if D' is a proper dense subset of D, then $c_{D'}$ is strictly less then c_D.

Example 6.3. Assume that S is an arbitrary infinite set and the semigroup multiplication on S is defined by $x \cdot y = x$, i.e. S is a left zero semigroup. Then every equivalence relation on S is a congruence relation of $(S, \cdot)$. It follows that the one point compactiication of S is the least topological one (corresponding to the congruence subalgebra A_{fc} of $\mathcal{P}(S)$), there is no least injective compactification of S, and the greatest Boolean compactification of S is the Stone-Čech compactification (because its corresponding congruence algebra is the whole of $\mathcal{P}(S)$).

Semigroups with infinite subgroups show a completely different behaviour.

Example 6.4. Assume G is an infinite group with identity 1_G. The semigroup congruence relations on G are, in fact, group congruence relations, so the finite semigroup congruence relations on G correspond to the normal subgroups N of G with G/N finite. G has an injective Boolean compactification iff there is a set $\mathcal{N}$ of normal subgroups with G/N finite for all $N \in \mathcal{N}$ and $\bigcap \mathcal{N} = \{1_G\}$. But no Boolean compactification of G is topological since for each normal subgroup N of G with G/N finite, N is infinite and so are all congruence classes with respect to N, i.e. the sets $x \cdot N$, $x \in G$.

Similarly, if S is a semigroup with an infinite subgroup $(G, 1_G)$, then no Boolean compactification of S is topological: for every finite congruence relation r on S, the restriction of r to G is a finite congruence relation on G and hence the congruence classes of r on G are infinite.

The additive group $(\mathbf{Z}, +)$ of integers has no topological Boolean compactification, by 6.4. But its Boolean compactifications can be nicely described, and most of them are injective.

Example 6.5. (a) The finite congruence relations on $\mathbf{Z}$ have the form $r(d)$ (congruence modulo d), for $d \in \mathbf{N}$. For $d, d' \in \mathbf{N}$, $r(d) \leq r(d')$ means that d divides d'. And $\sup(r(d), r(d')) = r(m)$ where m is the least common multiple of d, d'.

(b) The ideals of $\mathrm{fcr}(\mathbf{Z})$ are in one-one correspondence with the "height" functions $h : P \to \omega + 1$. Here P is the set of primes and $\omega + 1$ is the set $\{0, 1, 2, \ldots, \omega\}$ of all natural numbers, plus ω (infinity). The height function h corresponding to $I \in \mathrm{fcr}(\mathbf{Z})$ is given by $h(p) = \omega$ if $r(p^k) \in I$ holds for all $k \in \omega$, and $h(p) = m$

if $r(p^k) \in I$ holds exactly for $k \leq m$. Conversely, the ideal I corresponding to a height function h is generated by the congruence relations $r(p^k)$ where $p \in P$, $k \in \omega$, and $k \leq h(p)$.

(c) For a prime p and a natural number m, we write $Z_{p,m}$ for the quotient group $\mathbf{Z}/p^m$. This is the Boolean compactification of $\mathbf{Z}$ corresponding to the height function h with $h(p) = m$ and $h(q) = 0$, for all $q \neq p$. Similarly, write $Z_{p,\omega}$ for the additive group of p-adic integers – the Boolean compactification corresponding to h with $h(p) = \omega$ and $h(q) = 0$, for all $q \neq p$.

(d) Assume $c = (e, T)$ is a Boolean compactification of $\mathbf{Z}$, $I = R(c)$ the corresponding ideal of $\mathrm{fcr}(\mathbf{Z})$, and h is the height function corresponding to I. Then

$$T \cong \prod_{p \in P} Z_{p,h(p)}.$$

In particular, for c_{max}, the greatest Boolean compactification of $\mathbf{Z}$, we obtain $h(p) = \omega$ for all $p \in P$, and

$$T_{max} \cong \prod_{p \in P} Z_{p,\omega}.$$

For a proof, consider the ideals $I(p)$ of $\mathrm{fcr}(S)$, for $p \in P$, where $I(p) = \{r(p^k) : k \leq h(p)\}$. The compactification of $(\mathbb{Z}, +)$ corresponding to $I(p)$ is then $\mathbb{Z}_{p,h(p)}$. By 5.6 and the Chinese Remainder Theorem, this family of compactifications is independent, and the assertion follows from 2.6.

The greatest Boolean semigroup compactification c_{max} of $\mathbb{Z}$ corresponds to the subalgebra of $\mathcal{P}(\mathbb{Z})$ consisting of all periodic sets. Here we call $a \subseteq \mathbb{Z}$ periodic if there is some $d \in \mathbb{N}$ such that, for all $x \in \mathbb{Z}$, $x \in a$ iff $x + d \in a$; we call d a modulus of a.

(e) For $c = (e, T)$, I and h as in (d), it is easily seen that c is injective iff either there is a prime p with $h(p) = \omega$ or there are infinitely many primes p with $h(p) > 0$. In this case, T is a Boolean space with countable weight and no isolated points, hence homeomorphic to the Cantor space.

We finally turn to the additive semigroup $\mathbb{N}$ of natural numbers. By 6.1, we know that $(\mathbb{N}, +)$ has a least topological compactification which is, in fact, a one point compactification.

Example 6.6. (a) The examples 6.5 and 6.1 motivate two special types of finite congruence relations on $\mathbb{N}$. For $d \in \mathbb{N}$, let $r(0, d)$

be the relation of being congruent modulo d. And for $k \in \omega$, let $r(k,1)$ be the relation defined by $(x,y) \in r(k,1)$ iff $(1 \leq x,y \leq k$ and $x = y)$ or $k < x,y$.

Define $r(k,d)$ to be the least upper bound of $r(0,d)$ and $r(k,1)$ in the lattice fcr($\mathbb{N}$). I.e. $(x,y) \in r(k,d)$ iff $(x,y) \in r(0,d) \cap r(k,1)$ iff $(1 \leq x,y \leq k$ and $x = y)$ or $(k < x,y$ and x,y are congruent modulo d). The quotient semigroup of $\mathbb{N}$ with respect to $r(k,d)$ is isomorphic to the finite semigroup $S = B \cup R$ where $B = \{1,\ldots,k\}$, $R = \{k+1,\ldots,k+d\}$ and, for $x,y \in S$, $x+y$ computed in S is the sum $x+y$ computed in $\mathbb{N}$ if it happens to be in S; otherwise, it is the unique $z \in R$ congruent to $x+y$ modulo d.

(b) Every finite congruence relation on $\mathbb{N}$ has the form $r(k,d)$ where $k \in \omega$ and $d \in \mathbb{N}$ are uniquely determined. Moreover, given (k_1,d_1) and (k_2,d_2) in $\omega \times \mathbb{N}$, the least upper bound of $r(k_1,d_1)$ and $r(k_2,d_2)$ in fcr($\mathbb{N}$) is $r(k,d)$ where $k = \max(k_1,k_2)$ and d is the least common multiple of d_1 and d_2. Thus, $r(k_1,d_1) \leq r(k_2,d_2)$ holds in fcr($\mathbb{N}$) iff $k_1 \leq k_2$ and d_1 divides d_2. It follows that the lattice fcr($\mathbb{N}$) is isomorphic to the product of the lattices $(\omega,\leq)$ and $(\mathbb{N},|)$ where $|$ is the divisibility relation.

(c) We define two ideals $J_B = \{r(b,1) : b \in \omega\}$ and $J_R = \{r(0,d) : d \in \mathbb{N}\}$ of fcr($\mathbb{N}$); every element of fcr($\mathbb{N}$) is the least upper bound of an element of J_B and an element of J_R. Therefore, every ideal I of fcr($\mathbb{N}$) is generated as an ideal by its subideals $\{r(k,1) : k \in B(I)\}$ and $\{r(0,d) : d \in R(I)\}$ where $B(I) = \{k \in \omega : r(k,1) \in I\}$ and $R(I) = \{d \in \mathbb{N} : r(1,d) \in I\}$ are ideals of the lattices $(\omega,\leq)$ and $(\mathbb{N},|)$. This gives a complete description of the of ideals of fcr($\mathbb{N}$) and hence of the Boolean semigroup compactifications of $\mathbb{N}$.

(d) Consider an ideal I of fcr($\mathbb{N}$), $c = L(I)$ its associated Boolean compactification and A the corresponding congruence subalgebra of $\mathcal{P}(\mathbb{N})$. Then A is generated, as a subalgebra of $\mathcal{P}(\mathbb{N})$, by the singletons $\{x\}$ where $x \in B(I)$ and the congruence classes of $\mathbb{N}$ modulo d where $d \in R(I)$. Hence c is a topological compactification iff $B(I) = \omega$. In this case, the Boolean algebra A is determined, up to isomorphism, by the fact that it is atomic (the atoms being the singletons $\{x\}$, $x \in \omega$) and its quotient under division by the ideal generated by the atoms is the finite Boolean algebra with d atoms if $D(I) = \{e \in \mathbb{N} : e \mid d\}$ and the countable atomless Boolean algebra if $D(I)$ is infinite.

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