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MICHAEL SELECTION THEOREM FOR MAX-PLUS COMPACT CONVEX SETS

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Abstract. We prove a counterpart of the Michael selection theorem for max-plus compact convex sets. The proof is based on the properties of Milyutin map of the spaces of idempotent probability measures.

1. Introduction

The Michael selection theorem for convex sets has many applications not only in topology but also in convex analysis as well as another parts of mathematics. In particular, implicitly, selection theorems often imply softness theorems, which, in turn, are ingredients of characterization results for maps in topology of infinite-dimensional manifolds (see [9]). Of interest are also selection theorems for generalized convex structures (see, also selection theorems for generalized convex structures (see, e.g., [10]). The aim of this note is to prove a selection theorem for the so-called max-plus (tropical) convex sets. For reader's convenience, we do not formulate the result in the most general setting.

Let $\mathbb{R}_{\max}$ denote the set $\mathbb{R}\cup\{-\infty\}$ endowed with the operation max. If $a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, then we let $a \oplus b = (\max\{a_1, b_1\}, \ldots, \max\{a_n, b_n\})$. If $\lambda \in \mathbb{R}$, then we let $\lambda \odot a = (\lambda + a_1, \ldots, \lambda + a_n)$. We extend the operation $\oplus$ over $\mathbb{R}_{\max}$ by

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letting $(-\infty) \oplus a = (-\infty, \dots, -\infty)$. A subset A in $\mathbb{R}^n$ is said to be *max-plus convex* if $\alpha \odot a \oplus \beta \odot b \in A$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$. The max-plus convexity (or tropical convexity, in another terminology) introduced in [12] finds many applications in research on optimization problems [13], abstract convex analysis [8], and phylogenetic analysis [2]. Many results of convex geometry have their counterparts in the max-plus case (see [3], [1] and the references therein).

2. Preliminaries

By I we denote the idempotent probability measure functor (see [11]). Below we provide some necessary information on the properties of I for the convenience of the reader.

Let $C(X)$ denote the set of all continuous functions defined on a compact Hausdorff space X. A functional $\mu\colon C(X) \to \mathbb{R}$ is called an *idempotent probability measure* (a Maslov measure) if

(1) $\mu(c_X) = c$;
(2) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$;
(3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$.

(Here c_X denotes the constant function on X taking the value c, $\oplus$ is the pointwise maximum, and $\odot$ the addition of a constant and a function.) We endow the $I(X)$ with the weak* topology. A base of this topology is formed by the sets

$$O(\mu; \varphi_1, \dots, \varphi_n; \varepsilon) = \{\nu \in I(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon,\ i = 1, \dots, n\},$$

where $\varphi_1, \dots, \varphi_n$ run through the set $C(X)$, $n \in \mathbb{N}$, and $\varepsilon > 0$. If $f\colon X \to Y$ is a continuous map of compact Hausdorff spaces, then we define $I(f)\colon I(X) \to I(Y)$ as follows: $I(f)(\mu)(\varphi) = \mu(\varphi f)$, $\mu \in I(X)$, $\varphi \in C(Y)$.

For any $x \in X$, we denote by $\delta_x \in I(X)$ the functional acting by the formula $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$.

Let $\mu \in I(X)$. The *support* of μ (written $\mathrm{supp}(\mu)$) is the minimal closed subset A of X satisfying the condition: if $\varphi, \psi \in C(X)$ and $\varphi|A = \psi|A$, then $\mu(\varphi) = \mu(\psi)$.

The following result is proved in [11].

Theorem 2.1. *Let X be a compact metrizable space. Then there exists a zero-dimensional compact metrizable space X and a continuous map $f\colon X \to Y$ for which there exists a continuous map $s\colon Y \to I(X)$ such that* $\mathrm{supp}(y) \subset f^{-1}(y)$*, for every $y \in Y$.*

This theorem is a counterpart in the theory of idempotent probability measures of the classical Milyutin theorem on maps admitting averaging operators for continuous functions [5].

The following is a counterpart, for the idempotent probability measures, of the barycenter map of probability measures in linear topological spaces. Let $A \subset \mathbb{R}^n$ be a compact max-plus convex subset. We denote by $x_1, \dots, x_n$ the coordinate functions $\mathbb{R}^n \to \mathbb{R}$. Given $\mu \in I(A)$, we let $\beta(\mu) = (\mu(x_1), \dots, \mu(x_n))$. It is proved in [11] that the map β is well-defined and continuous. The point $\beta(\mu) \in \mathbb{R}^n$ is called the *idempotent barycenter* of μ.

3. Selection theorem

Theorem 3.1. *Let $F\colon X \to Y$ be a lower semicontinuous max-plus convex valued map of compact metrizable spaces X, Y and $Y \subset \mathbb{R}^n$. Then this map admits a continuous selection.*

Proof. Let $g\colon Z \to X$ be a continuous map of a zero-dimensional compact metric space Z for which there exists a continuous map $s\colon X \to I(Z)$ with the property $I(g)(s(x)) = \delta_x$, $x \in X$.

By the zero-dimensional Michael selection theorem [4], there exists a continuous selection $h\colon Z \to Y$ of the multivalued map $Fg\colon Z \to Y$. The map $x \mapsto I(h)(s(x))$ maps X into $I(Y)$ and we have $\mathrm{supp}(I(h)(s(x))) \subset F(x)$, for every $x \in X$. Therefore, the idempotent barycenter $f(x) = \beta((I(h)(s(x)))$ is defined and is an element of the set $F(x)$. Thus, the map f is a continuous selection of F. □

The following definition is due to E. Shchepin [7].

Definition 3.2. A map $f\colon X \to Y$ is said to be *soft* provided that for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \big\downarrow & & \big\downarrow f \\ Z & \xrightarrow[\psi]{} & Y \end{array} \tag{3.1}$$

such that Z is a paracompact space and A is a closed subset of Z there exists a map $\Phi\colon Z \to X$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$.

It is known [7] that the open maps with convex preimages are soft. The idea of the proof works also in the max-plus convex case.

Theorem 3.3. *Let $X \subset \mathbb{R}^n$ be a compact space and $f\colon X \to Y$ be an open map onto a compact metrizable space with max-plus convex preimages. Then the map f is soft.*

Proof. Let A be a closed subset of a paracompact space Z and $\varphi\colon A \to X$, $\psi\colon Z \to Y$ making diagram (3.1) commutative. Define a multivalued map $F\colon Z \to X$ as follows

$$F(z) = \begin{cases} \{\varphi(z)\}, & \text{if } z \in A, \\ f^{-1}(\psi(z)), & \text{if } z \notin A. \end{cases}$$

It is easy to see that the map F is lower semicontinuous and max-plus convex valued. By Theorem 3.3, there is a continuous selection $\Phi\colon Z \to X$ of F. Clearly, Φ is a required map. □

4. Remarks and open questions

The idea to apply Milyutin maps in order to derive the convex selection theorem from the zero-dimensional selection theorem belongs to E. Shchepin [6].

The results of this note can be generalized for max-plus convex subsets in spaces more general than euclidean ones, e.g., in the spaces $\mathbb{R}^\tau$, for arbitrary cardinal number τ. We leave to the reader the problem of generalization of the selection theorem over noncompact max-plus convex sets.

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