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SETS OF ENDPOINTS OF CHAINABLE CONTINUA

JULIEN DOUCET

This paper is dedicated to James T. Rogers, Jr., who has encouraged and helped many. I am very fortunate to have been one of his students. His colleagues and students owe him more than can be expressed.

Abstract. We prove that the set of endpoints $\mathrm{Ep}(X)$ of any chainable continuum X is a G_δ subset of X and, hence, $\mathrm{Ep}(X)$ is topologically complete.

1. Introduction

In [3], the set of endpoints, $\mathrm{Ep}(X)$, of a chainable continua X was investigated. It was shown that the cardinality of $\mathrm{Ep}(X)$ could be any finite cardinal number, $\aleph_0$, or $\mathfrak{c}$. For each of these cardinalities, X can be decomposable or indecomposable. If $\mathrm{Ep}(X)$ is finite, it must be (metrically and topologically) complete. If $\mathrm{Card}\big(\mathrm{Ep}(X)\big) = \aleph_0$ or $\mathfrak{c}$, it was shown that X could be decomposable or indecomposable and complete or incomplete (in the metric sense) in each of these cases. Jan van Mill noted that the examples for incompleteness were metrically incomplete, but topologically complete. He posed the following question: Can $\mathrm{Ep}(X)$ be topologically incomplete? (See [3, Question 9.1, p. 58].) We answer this question by proving that $\mathrm{Ep}(X)$ is a G_δ subset of X and, hence, $\mathrm{Ep}(X)$ is topologically complete.

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2. Basic metric spaces, continua, and chains

A *continuum* is a compact, connected, nonvoid metric space. For a metric space (X, d), the *diameter* of $S \subseteq X$ is defined by $\operatorname{diam}(S) = \sup_{x,y \in S} d(x, y)$.

A *chain* C is a finite sequence $L_1, L_2, L_3, \ldots, L_n$ of open sets in a space X such that $L_i \cap L_j \neq \emptyset \iff |i - j| \leq 1$. Each L_i is called a *link*. If $\epsilon > 0$ and the diameter of each link is less than ϵ, then the chain is called an ϵ-chain. A continuum is *chainable* if for each $\epsilon > 0$, the continuum can be covered by an ϵ-chain. For example, $I = [0, 1]$ and the $\sin(1/x)$-curve,

$$\big\{(x, \sin \frac{1}{x})\big| 0 < x \leq \frac{2}{\pi}\big\} \cup \big\{(0, y)\big| - 1 \leq y \leq 1\big\},$$

are both chainable. $\top$ and $\bigcirc$ are not chainable.

Theorem 2.1 (Bing)**.** *Each chainable continuum can be embedded in the plane so that it can be chained with chains whose links are round circles.*

The *mesh* of a chain $C\colon L_1, L_2, L_3, \ldots, L_n$ is defined by $\operatorname{mesh} C = \sup_{1 \leq i \leq n} \operatorname{diam}(L_i)$. A point x of a chainable continuum X is an *endpoint* of X if there exists a sequence $\{C_i\}$ of chains covering X such that

(i) mesh $C_i < \frac{1}{2^i}$, and
(ii) x belongs to the first link of each chain C_i.

A continuum M is said to be *irreducible between two points p and q* if p and q are contained in M and no proper subcontinuum of M contains p and q.

R. H. Bing [2, Section 5, p. 660] proved that if p is a point of a chainable continuum M, then the following three conditions are equivalent.

(A) Each nondegenerate subcontinuum of M containing p is irreducible from p to some other point.
(B) If each of two subcontinua of M contains p, one of the subcontinua contains the other.
(C) For each positive number ϵ, there is an ϵ-chain covering M such that only the first link of the chain contains p.

A metric space is *topologically complete* if it is homeomorphic to a complete metric space. $(0, 1)$ is not complete with its usual metric topology, but it is topologically complete since $(0,1)$ is homeomorphic to $\mathbb{R} = (-\infty, \infty)$ with its usual metric, which is complete.

3. What is known about completeness of Ep(X)

In [3], it was shown that the set of endpoints could have any cardinality (0, 1, 2, ..., n, ..., $\aleph_0$, $\mathfrak{c}$). For each cardinality of the set of endpoints, it was also shown that the chainable continua can be decomposable or indecomposable. For finite cardinality of the set of endpoints, the continua obviously has to be complete (metrically and topologically). It was shown that in the case of infinite cardinality, the continuum can be metrically complete or incomplete. After a conference talk given by the author, van Mill noticed that all of the examples of Ep(X) which were incomplete were also topologically complete. He posed the following question: Is there an example of a chainable continuum whose set of endpoints is topologically incomplete? (See [3, Question 9.1, p. 58].) In Theorem 4.2, we show that the answer to this question is negative. The following table summarizes the situation.

Decomposable or Indecomposable Chainable Continua Set of Endpoints				
	Metrically		Topologically	
Cardinality	Complete	Incomplete	Complete	Incomplete
Any finite	Yes	No	Yes	No
$\aleph_0$	Yes	Yes	Yes	No*
$\mathfrak{c}$	Yes	Yes	Yes	No*

* See Theorem 4.2.

4. Topological Completeness of Ep(X)

Definition. A set is a G_δ *set* if it is a countable intersection of open sets.

In [6, p. 408], Kazimierz Kuratowski referred to the Alexandrov Theorem. He referenced [1] and [4]. From this, we have that G_δ sets (in complete metric spaces) are topologically complete.

Theorem 4.1 (Alexandrov)**.** *Every G_δ-subset of a complete space is homeomorphic to a complete space (i.e., it is complete in the topological sense).*

We use the Alexandrov Theorem to prove our main result.

Theorem 4.2. *For any chainable continuum X,* $\mathrm{Ep}(X)$ *is a* G_δ *set and, hence, topologically complete.*

Proof: Let X be a chainable continuum. Let $x \in \mathrm{Ep}(X)$. For each $n \in \mathbb{N}$, there exists a chain $C_n^x : L_{n1}^x, L_{n2}^x, L_{n3}^x, \ldots, L_{nm_n^x}^x$ covering X such that $x \in L_{n1}^x$ and mesh $C_n^x < \frac{1}{n}$. Let $U_n = \bigcup_{x \in \mathrm{Ep}(X)} L_{n1}^x$.

Then U_n is open for each $n \in \mathbb{N}$. Let $V = \bigcap_{n=1}^{\infty} U_n$. Then V is a G_δ set.

Claim. $\mathrm{Ep}(X) = V$.

Let $x \in \mathrm{Ep}(X)$. For all $n \in \mathbb{N}$ there exists a chain C_n^x such that x is in the first link of C_n^x and mesh $C_n^x < \frac{1}{n}$. Then $x \in U_n$ for all $n \in \mathbb{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} U_n = V$. Therefore, $\mathrm{Ep}(X) \subseteq V$.

Let $v \in V$. Let $m \in \mathbb{N}$. Then $v \in U_m$ and, consequently, there $x \in \mathrm{Ep}(X)$ such that $v \in L_{m1}^x$, the first link of chain C_m^x and mesh $C_m^x < \frac{1}{m}$. Hence, $v \in \mathrm{Ep}(X)$. Therefore, $V \subseteq \mathrm{Ep}(X)$. The rest now follows easily from the Alexandrov Theorem. □

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