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METRIZABILITY OF TOPOLOGICAL SEMIGROUPS ON LINEARLY ORDERED TOPOLOGICAL SPACES

ZIQIN FENG AND ROBERT HEATH

Abstract. The authors use techniques and results from the theory of generalized metric spaces to give a new, short proof that every connected, linearly ordered topological space that is a cancellative topological semigroup is metrizable, and hence embeddable in $\mathbb{R}$. They also prove that every separable, linearly ordered topological space that is a cancellative topological semigroup is metrizable, so embeddable in $\mathbb{R}$.

1. Background and introduction

A linearly ordered topological space (LOTS) L is a linearly ordered set L with the open interval topology. A cancellative topological semigroup on L is a semigroup with a continuous semigroup operation such that $ab = ac$, $ba = ca$, and $b = c$ are equivalent for any $a, b, c \in L$. A question that can be traced to both Nils Henrik Abel and Sophus Lie, and was listed as the second half of Hilbert's fifth problem, essentially asks whether a cancellative topological semigroup on a connected LOTS can be embedded in the real line. The history of the problem, and the various solutions and partial solutions and related questions, are most thoroughly documented by K. H. Hofmann and J. D. Lawson. In [7], they note Otto Hölder's (1901) contribution [p. 19], and continue,

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> Clifford pointed out ... that the arguments and results of Aczél and Tamari remain valid for general connected linearly ordered sets. Aczél later pointed out that cancellativity implies strict monotonicity ..., and thus his result could also be formulated for cancellative threads. Recently, Craigen and Palés ... have simplified the overall proof. [p. 24]

Additionally, a proof using generalized metric techniques is given by Ronald E. Barnhart in [3]. That proof, however, is restricted to the abelian case. Here, we give a fairly short proof using only generalized metric techniques. We also show that the theorem still holds if "connected" is replaced by "separable." One might assume that would be a corollary of the theorem in [7] that "'a totally ordered set can be embedded in $\mathbb{R}$ if and only if it contains a countable subset C such that for any $x < y$ there is a $c \in C$ with $x \leq c \leq y$'" [with no mention of a semigroup]. The assumption that that hypothesis follows from separability is seen to been false. The space obtained from $[0,1]$ by replacing each point by a pair of adjacent points (the double-arrow space) is a compact, separable LOTS that can't be embedded in $\mathbb{R}$, and of course does not have the aforementioned property.

2. Metrizability

Theorem 1. *Every connected LOTS L which is a cancellative topological semigroup is metrizable and hence, embeddable in* $\mathbb{R}$.

The theorem follows from the following propositions. Below, L satisfies the conditions of the theorem. Note that every closed and bounded subset of L is compact.

Proposition 2. *For any* $a \in L$, *the maps* f_a *given by* $f_a(x) = ax$ *and* g_a *given by* $g_a(x) = xa$ *are autohomeomorphisms of* L.

Proof: Fix $a \in L$. Since L is a topological semigroup, f_a and g_a are continuous by the properties of topological semigroups. Also by the cancellativity of L, f_a and g_a are both one-to-one. Take $b, c \in L$ with $b < c$. Then $[b, c]$ is compact and connected, so f_a maps $[b, c]$ into the closed interval with endpoints $f_a(b)$ and $f_a(c)$. Therefore, f_a maps open intervals to open intervals, and so f_a^{-1} is continuous. Similarly, g_a^{-1} is continuous. □

From Proposition 2, we know directly that f_a and g_a are both either order-preserving or order-reversing.

Proposition 3. *The space L is first countable.*

Proof: Pick $a \in L$ and an increasing sequence $\{a_\alpha : \alpha < \gamma\}$ which converges to a from the left. Pick a countable subsequence $\{a_n : n \in \omega\}$ of $\{a_\alpha : \alpha < \gamma\}$. Then since L is connected and $\{a_n\}$ has an upper bound, $b = \sup\{a_n : n \in \omega\}$ exists. If $b = a$, then we have nothing to do from the left. If not, consider f_a. Since f_a is a homeomorphism, $f_a(a_n)$ converges to ab. Also g_b is a homeomorphism which maps a to ab. Therefore, the preimage of $\{a_n b : n \in \omega\}$ is a sequence which converges to a from the left.

By similar reasoning, we get a countable sequence $\{b_n : n \in \omega\}$ converging to a from the right. Then we get that $\{(a_n, b_n) : n \in \omega\}$ is a countable local base at a. □

Proposition 4. *The sequence $\{a^n : n \in \omega\}$ is either constant or strictly monotone and unbounded for any $a \in L$.*

Proof: Three cases arise.

Case 1. Assume $a = a^2$. Then the sequence is obviously constant.

Case 2. Assume $a < a^2$. Since f_a is order-preserving, we know we need only to show $a^2 < a^3$. Suppose, for contradiction, $a^3 < a^2$.

If $p, q \in [a, a^2]$ and $p < q$, then we have the following two conditions.

i) If $ap < p$, then $aq < ap < p < q$. Therefore, $aq < q$.

ii) If $aq > q$, then $ap > aq > q > p$. Therefore, $ap > p$.

Thus, we can take $I = \inf\{p : p \in [a, a^2], ap < p\}$ and $S = \sup\{p : p \in [a, a^2], ap > p\}$. It is obvious that $I, S \in [a^3, a^2]$ and $I \leq S$.

Consider the relationship between I and S. If $I = S$, then $aI = I$, and this contradicts $a < a^2$. If $I < S$, then for any m with $I \leq m \leq S$, we have $am = m$, and again, we get a contradiction.

Case 3. Assume $a^2 < a$. Using an argument similar to that in Case 2, we can show $a^3 < a^2$, as required.

Thus, $\{a^n : n \in \omega\}$ is strictly monotone in Case 2 and Case 3. Unboundedness is easy to prove by contradiction. □

Note that from the above proof, if $a < a^2$, then $\{x \in L : a \le x\}$ is a union of almost disjoint homeomorphic closed intervals. Also note that we now know f_a and g_a are both order-preserving.

Proposition 5. *Take $a \in L$ and assume, without loss of generality, $a < a^2$ and $a \neq \min L$. Then $L_a = [a, \infty)$ is metrizable.*

Proof: By Proposition 2, there exists sequences $x_1 < x_2 < \cdots$ and $y_1 > y_2 > \cdots$ that both converge to a. Define $g_n(a) = (x_n, y_n)$. For each $p \in [a^3, a^4]$, take $q \in [a^2, a^3]$ with $p = aq$ and let $g_n(p) = (x_n q, y_n q)$. Now we show that the neighborhood system $\{g_n(p), n \in \omega, p \in (a^3, a^4)\}$ satisfies the requirements of semi-metrizability.

Suppose $y \in [a^3, a^4]$ and for each n, $y \in g_n(p_n) = g_n(aq_n) = (x_n q_n, y_n q_n)$. Without loss of generality, assume $q_n \longrightarrow z$, then $x_n q_n \longrightarrow az$ and $y_n q_n \longrightarrow az$. Thus, since $y \in (x_n q_n, y_n q_n)$ for each n, we have that $y = az$. Therefore, $p_n \longrightarrow y$. It follows that $[a^3, a^4]$ is semi-metrizable and hence, it is metrizable by the equivalence of the semi-metrizability and metrizability in LOTS [4]. Hence, $L_a = \{x \in L : a \le x\}$ is metrizable. And since L_a is connected and locally compact, it is separable. Hence, L_a is embeddable in $\mathbb{R}$. □

Proof of Theorem 1: Here, without loss of generality, we can assume there is an $a \in L$ with $a < a^2$. Next, we will prove the theorem in three cases.

Case 1. $\min L = m$. Then it is easy to see that $m \le m^2$. If $m < m^2$, then L is metrizable by Proposition 5. Otherwise, by Proposition 3, we can find $\{x_n : n \in \omega\}$ which converges to m from the right. Then for each $n \in \omega$, $x_n < (x_n)^2$. Hence, L_{x_n} is metrizable for each n by Proposition 5. Therefore, L is metrizable.

Case 2. $\min L$ does not exist and there is some $b \in L$ with $b^2 < b$. Then we can take $m = \inf\{a : a < a^2\}$. This follows because we can get $c < c^2$ from $a < c$ and $a < a^2$ from Proposition 4. Then it is easy to check $m = m^2$. Let $x_1 < x_2 < \cdots$ and $y_1 > y_2 > \cdots$ both converge to m, and let $R_{x_i} = \{a \in L : a \le x_i\}$. A proof similar to that of Proposition 5 shows that R_{x_i} is metrizable for each i. Since L_{y_i} is also metrizable, we get that L is metrizable.

Case 3. $\min L$ does not exist and $a < a^2$ for any $a \in L$. If there is a countable co-initial decreasing sequence $\{x_n : n \in \omega\}$ which is unbounded, then $L = \bigcup_{n \in \omega} L_{x_n}$ is metrizable because L_{x_n} is metrizable for each $n \in \omega$. If not, we take $\{x_\alpha : \alpha \in \omega_1\}$ which is strictly decreasing and unbounded below without countable coinitial subsequence. Then we take $a \in L$. Consider the set $\{(x_\alpha)^m : m \in \omega, \alpha \in \omega_1\}$. Then we can find n_0, m_0, and α_0 such that $(x_\alpha)^{m_0} \in [a^{n_0}, a^{n_0+1}]$ for $\alpha > \alpha_0$. This is because $x < y \Rightarrow x^n < y^n$. Then we can suppose $\{(x_\alpha)^{m_0} : \alpha > \alpha_0\}$ converges to $b \in [a^{n_0}, a^{n_0+1}]$. This contradicts the first countability of L. So we get a contradiction. □

Next, we have another nice theorem about the metrizability of a separable LOTS.

Theorem 6. *Every separable LOTS which is also a cancellative topological semigroup is metrizable.*

Proof: Recall that a separable LOTS is metrizable if and only if its set of endpoints (points with either an immediate predecessor or an immediate successor) is at most countable. Also, note that in a separable LOTS every uncountable set contains a limit point, since every LOTS is monotonically normal

Assume L is a separable LOTS with uncountably many endpoints, and let L be a cancellative topological semigroup. Since L is separable, L has at most countably many isolated points. So we may assume, without loss of generality, that L has no isolated points (because the sum of nonisolated points can not be isolated, and L is a semigroup). Then the endpoints occur in pairs of adjacent points. Let E be the set of all such adjacent point pairs, $x = (x_1, x_2)$ with $x_1 < x_2$.

Let L^* be the space obtained by identifying each pair (x_1, x_2) to a point x. Then L^* is metrizable with metric δ which induces a pseudometric d on L. Let $E = \{(x_1, x_2) : d(x_1, x_2) = 0\}$.

For each $x = (x_1, x_2) \in E$, we know $x_1 + x_2 \notin \{2x_1, 2x_2\}$ by the cancellativity of addition. Notice that $2x_1 = 2x_2$ is possible. So if $diam_d\{x_1 + x_2, 2x_1, 2x_2\} = 0$, then we can get $2x_1 = 2x_2$, and (p, q) or (q, p) is in E if $x_1 + x_2 = p$ and $2x_1 = q$.

Consider the subset, $F = \{x \in E : diam_d\{x_1 + x_2, 2x_1, 2x_2\} > 0\}$, of L^*. Two cases arise.

Case 1. F is uncountable. Then since L is separable and monotonically normal, F has cluster points in L. That leads to a contradiction to the continuity of the operation and the distance function.

Case 2. F is countable. Then, without loss of generality, we can assume $B' = \{x \in E : (2x_1, x_1+x_2) \in E\}$ is uncountable [otherwise, $\{x \in E : (x_1+x_2, 2x_1) \in E\}$ is uncountable]. Then, by separability, all but countably many points of B' are limit points of B'. By the continuity of the semigroup operation, for each $x \in B'$, there is $n_x \in \omega$ such that if $\delta(x,t) < 1/n_x$, then $t_1 + t_2 \leq 2x_1 = 2x_2 < x_1 + x_2$. Then there is $\varepsilon > 0$ such that $G = \{x \in B' : 1/n_x > \varepsilon\}$ is uncountable. Now we can pick $x, z \in G$ such that $\delta(x,z) < \varepsilon$. It follows that

$$z_1 + z_2 < x_1 + x_2 < z_1 + z_2,$$

which is a contradiction.

Thus, L is metrizable, and the proof is complete. □

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