TOPOLOGY PROCEEDINGS
Volume 32, 2008
Pages 89–100

http://topology.auburn.edu/tp/

On the Position of Ceder Space and McAuley Space

by

Masami Amono and Takemi Mizokami

Electronically published on April 7, 2008

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 32 (2008)
Pages 89-100

E-Published on April 7, 2008

ON THE POSITION OF CEDER SPACE AND McAULEY SPACE

MASAMI AMONO AND TAKEMI MIZOKAMI

Abstract. We give a simple proof that McAuley space and Ceder space have a σ-closure-preserving base, answering a problem of Stephen Watson, and show that they are non-Lašnev, D-spaces in the sense of Keiô Nagami.

1. Introduction

All spaces are assumed to be regular T_1-spaces. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote all natural numbers, rational numbers, real numbers, respectively. For a space X, let us denote the topology by $\tau(X)$ or τ. For a family $\mathcal{U}$ of subsets of a space X and for a subset A of X, we write

$$S(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\},$$

and for a point $p \in X$, we write

$$C(p,\mathcal{U}) = \bigcap\{U \in \mathcal{U} \mid p \in U\}.$$

For a subset A of X, we write the restriction of $\mathcal{U}$ to A by $\mathcal{U}|A$. In 1953, Jun-iti Nagata stated in [3] that Ceder space, defined below, has a σ-closure-preserving base. But he did not give a direct proof. In 1961, Jack G. Ceder defined M_1-spaces as a generalization of metric spaces, and he gave a direct proof that Ceder space is an M_1-space, i.e., that it has a σ-closure-preserving base [1]. But his proof is very complicated. So, in 1992, Stephen Watson showed that

2000 *Mathematics Subject Classification.* 54E20.
Key words and phrases. D-space, M_1-space, uniformly approaching.

both Ceder space and McAuley space, defined below, are obtained as special resolutions of the half plane at each point in $\mathbb{R}\times\{0\}$ into a 2-point space by suitable continuous mappings [6, Lemma 3.3.17] and proposed the problem to find a simple proof of Ceder's result [6, Problem 3.3.18].

In this paper, we give an answer to this problem. The essential point of our proof is based on the fact that if $\mathcal{U}$ is a locally finite family of subsets of a space X, then the family $\{\bigcup\mathcal{U}_0 \mid \mathcal{U}_0 \subset \mathcal{U}\}$ is closure-preserving in X, and we use the fact that $\mathbb{R}\times\{0\}$ has a *uniformly approaching anti-cover* in both McAuley space and Ceder space. This concept is due to Keiô Nagami [4].

Finally, we determine the position of both spaces in the sequence from metric spaces through M_1-spaces; that is, we show that they are D-space in the sense of Nagami [4], which is weaker than Lašnev spaces.

As for M_1-spaces and other related terms, refer to Gary Gruenhage [2].

2. McAuley space is an M_1-space

We state the topology of two spaces known already and called here McAuley space and Ceder space.

Definition 2.1. Let $X = X_0 \cup X_1$, where

$$X_0 = \{(x,0) \mid x \in \mathbb{R}\}, \quad X_1 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}.$$

We topologize X by defining neighborhood bases at each point $p \in X$ in the following two ways: First, (1) if $p = (x,y) \in X_1$, then p has a usual neighborhood base, and (2) if $p = (x,0) \in X_0$, then p has a neighborhood base $\{M(p,1/n) \mid n \in \mathbb{N}\}$, where

$$M\left(p,\frac{1}{n}\right) = \{p\} \cup \left\{(x',y') \in X \mid y' < \frac{1}{n}|x'-x| < \frac{1}{n^2}\right\}, \quad n \in \mathbb{N}.$$

Then the space X, thus defined, is called *McAuley space.*

Second, if (2) $p = (x,0)$ has a neighborhood base $\{C(p,1/n) \mid n \in \mathbb{N}\}$, where

$$C\left(p,\frac{1}{n}\right) =$$

$$\{p\} \cup \left\{(x',y') \in X \mid y' < n - \sqrt{n^2-(x'-x)^2},\ |x'-p| < \frac{1}{n}\right\},$$

then we call the space X, thus defined, *Ceder space*.

We give a geometric description of these neighborhood bases. $M(p, 1/n)$ is the "bow-tie" centered at p which is the union of two triangles, one with vertices $p = (x, 0), (x + 1/n, 0)$ and $(x + 1/n, 1/n^2)$, the other with vertices $p, (x-1/n, 0)$, and $(x-1/n, 1/n^2)$. Except for the point p itself, the top edges of the triangles are not included in $M(p, 1/n)$, and neither are the right and left edges. $C(p, 1/n)$ is similar, except the two "triangles" have a top edge which is an arc of the circle of radius n in the upper half plane which is tangent to the x-axis at p.

Definition 2.2 ([4])**.** Let X be a space and F a closed subset of X. An open cover $\mathcal{U}$ of $X \setminus F$ is called a *uniformly approaching anti-cover* of F in X if for each open subset O of X,

$$\overline{S(X \setminus O, \mathcal{U})} \cap F \cap O = \emptyset.$$

It is easy to see that an open cover $\mathcal{U}$ is a uniformly approaching anti-cover of F in X if and only if for each open set O of X there exists an open set O' of X such that

$$O' \cap F = O \cap F, \quad S(O', \mathcal{U}) \subset O,$$

or there exists a subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that

$$V = (O \cap F) \cup \left(\bigcup \mathcal{U}_0\right)$$

is an open neighborhood of $O \cap F$ in X such that $V \subset O$.

And it is also easy to see that $\mathcal{U}$ is a uniformly approaching anti-cover of F in X if and only if for each point $p \in F$ and each basic neighborhood O of p in X, the following holds true:

$$\overline{S(X \setminus O, \mathcal{U})} \cap F \cap O = \emptyset.$$

We recall Nagami's discussion [4] that every closed subset has a uniformly approaching anti-cover, which we refer to here as the Fundamental Method.

Fundamental Method ([4])**.** Let (X, d) be a metric space and F be a closed subset of X. For each $p \notin F$, let

$$U(p) = B\left(p, \frac{r(p)}{3}\right),$$

where $r(p) = d(p, F)$ and $B(p, \varepsilon)$ is an open ball with center and radius ε. Then $\mathcal{U} = \{U(p) \mid p \in X \setminus F\}$ is an open cover of $X \setminus F$. To see that $\mathcal{U}$ is uniformly approaching, let O be an open set of X. Let

$$V = \{p \in O \mid d(p, F \cap O) < d(p, X \setminus O)\}.$$

Then V is an open set of X such that $V \cap F = O \cap F \subset V$ and

$$S(X \setminus O, \mathcal{U}) \cap V = \emptyset.$$

Proposition 2.3. *In McAuley space X, X_0 has a uniformly approaching anti-cover.*

Proof: Let τ_{met} be the topology generated by the usual metric d on the Euclidean plane $\mathbb{R}^2$. Then τ_{met} is the subtopology of the topology τ of X and the following are easy to see.

(1) X_0 is a closed subset of (X, τ_{met});
(2) $\tau_{met}|X_0 = \tau|X_0$ and $\tau_{met}|(X \setminus X_0) = \tau|(X \setminus X_0)$.

Since (X, τ_{met}) is a metric space, X_0 has a uniformly approaching anti-cover $\mathcal{U}$ in (X, τ_{met}), constructed by the Fundamental Method stated above. We show that $\mathcal{U}$ is also a uniformly approaching anti-cover of X_0 in (X, τ). To this end, it suffices to show that for each $M(p, 1/k)$, there exists $m \in \mathbb{N}$ such that $k < m$ and

$$M\left(p, \frac{1}{m}\right) \cap S\left(X \setminus M\left(p, \frac{1}{k}\right), \mathcal{U}\right) = \emptyset.$$

Without loss of generality, we can assume $p = (0, 0)$. Obviously,

$$G = \left\{(x, y) \in X \mid y < \frac{1}{k}|x|\right\} = M\left(p, \frac{1}{k}\right) \setminus \{p\}$$

is open in (X, τ_{met}). Let

$$W = \left\{(x, y) \in X \mid y < \frac{1}{k + \sqrt{k^2 + 1}}|x|,\ y + |x| < \frac{1}{k}\right\}.$$

Then W is an open subset of X such that

$$W = \{p \in G \mid d(p, X_0) < d(p, X \setminus G\}.$$

By the Fundamental Method, we have

$$S(X \setminus G, \mathcal{U}) \cap W = \emptyset.$$

Take $m \in \mathbb{N}$ such that $m > k + \sqrt{k^2+1}$. Then $M(p, 1/m)$ is an open neighborhood of p in X such that

$$M\left(X \setminus M\left(p, \frac{1}{k}\right), \mathcal{U}\right) \cap M\left(p, \frac{1}{m}\right) = \emptyset.$$

Hence, $\mathcal{U}$ is a uniformly approaching anti-cover of X_0 in X. □

Corollary 2.4. *In Ceder space X, X_0 has a uniformly approaching anti-cover.*

Proof: The discussion is almost the same as in the case of McAuley space. Let $\mathcal{U}$ be the same family as in the above proof. Then we show that $\mathcal{U}$ is a uniformly approaching anti-cover of X_0 in this space X. To see it, it suffices to show that for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$C\left(p, \frac{1}{m}\right) \subset W \cup \{p\},$$

where

$$W = \left\{p \in X \mid d(p, X_0) < d(p, X \setminus C\left(p, \frac{1}{n}\right)\right\}.$$

Without loss of generality, we can show the case when $n = 1$ and $p = (0, 0)$. By simple calculation, W coincides with

$$\{(x, y) \in X \mid 4y < x^2, \quad y + |x| < 1\}.$$

Then for a sufficiently large $m \in \mathbb{N}$, the following holds true.

$$C\left(p, \frac{1}{m}\right) \subset W \cup \{p\}.$$

By the Fundamental Method, we have

$$S\left(X \setminus C\left(p, \frac{1}{n}\right), \mathcal{V}\right) \cap C\left(p, \frac{1}{m}\right) = \emptyset.$$

This completes the proof. □

The proof of Theorem 2.5 below gives an answer to the problem proposed by Watson.

Problem 2.5 ([6, Problem 3.3.18])**.** Find a simple proof of the next theorem.

An essential point in the next proof is that if $\mathcal{U}$ is a locally finite cover of a space X, then $\{\bigcup\mathcal{U}_0 \mid \mathcal{U}_0 \subset \mathcal{U}\}$ is closure-preserving in X.

Theorem 2.6 (Nagata [3], Ceder [1]). *McAuley space and Ceder space are M_1-spaces.*

Proof: Since X_1 is a metrizable open subspace, there exists a σ-closure-preserving family $\mathcal{U}_0$ (in X) of open subsets of X forming a neighborhood base at each point of X_1 in X. By Proposition 2.3, there exists a uniformly approaching anti-cover $\mathcal{V}$ of X_0 in X. Without loss of generality, we can assume that $\mathcal{V}$ is locally finite in X_1. Let $(a,b) \in \mathbb{Q}^2,\ a < b$ be arbitrary. Let $\Delta(a,b)$ be the totality of subfamilies $\mathcal{V}(\delta)$ of $\mathcal{V}$ such that

$$U(\delta) = ((a,b) \times \{0\}) \cup \left(\bigcup \mathcal{V}(\delta)\right)$$

is an open neighborhood of $(a,b) \times \{0\}$ in X such that

$$\overline{U(\delta)} \subset \pi^{-1}[a,b],$$

where $\pi : X \to \mathbb{R}$ is the projection. Then

$$\mathcal{V}(a,b) = \{U(\delta) \mid \mathcal{V}(\delta) \in \Delta(a,b)\}$$

is a closure-preserving family of open subsets of X. It is easy to see that

$$\mathcal{W} = \bigcup\{\mathcal{V}(a,b) \mid (a,b) \in \mathbb{Q}^2,\ a < b\}$$

is a σ-closure-preserving family of open subsets of X forming a neighborhood base at each point of X_0 in X. Thus, we have a σ-closure-preserving base $\mathcal{U} \cup \mathcal{W}$ for X, proving that X is an M_1-space. □

3. The position of McAuley and Ceder spaces

Definition 3.1 ([4]). A space X is called a *D-space* if X is a paracompact σ-space such that each closed subset has a uniformly approaching anti-cover in X.

The following implications are well known.

> Metric space $\Longrightarrow$ Lašnev space $\Longrightarrow$ D-space in the sense of Nagami [4] $\Longrightarrow$ free L-space in the sense of Nagami [5] $\Longrightarrow$ M_3-μ-space $\Longrightarrow$ perfect image of an M_0-space $\Longrightarrow$ M_1-space.

We give the position of both McAuley space and Ceder space in the above sequence.

Theorem 3.2. *McAuley space and Ceder space are non-Lašnev D-spaces.*

Proof: Let X be McAuley space. Then obviously, X is a paracompact σ-space. Let F be a closed subset of X and let

$$F_0 = X_0 \cap F, \quad F_1 = X_1 \cap F.$$

By Proposition 2.3, there exists a uniformly approaching anti-cover $\mathcal{V}$ of X_0 in X. Without loss of generality, we can assume that $\mathcal{V}$ is locally finite in X_1. Since F is a closed subset of X, for each point $p = (x, 0) \in X_0 \setminus F_0$, there exists $k(x) \in \mathbb{N}$ such that

$$M\left(p, \frac{1}{k(x)}\right) \cap F = \emptyset.$$

Take $m(x) \in \mathbb{N}$ such that $m(x) > 3k(x)$ and

$$\begin{aligned} M\left(p, \frac{1}{m(x)}\right) &\cap [\mathbb{C}((x + 1/k(x), k(x)), k(x)) \\ &\cup \mathbb{C}((x - 1/k(x)), k(x)), k(x))] = \emptyset, \end{aligned} \tag{3.1}$$

where $\mathbb{C}(q, r)$'s are closed balls with center q and radius r. Let

$$U(p) = M\left(p, \frac{1}{m(x)}\right), \quad p = (x, 0) \in X_0 \setminus F_0.$$

Then the family

$$\{U(p) \mid p = (x, 0) \in X_0 \setminus F_0\}$$

has the following properties.

Claim 1. $\{U(p) \cap X_0 \mid p = (x, 0) \in X_0 \setminus F_0\}$ is a uniformly approaching anti-cover of F_0 in X_0.

Since $m(x) > 3k(x)$, we have

$$\frac{1}{m(x)} < \frac{1}{3k(x)}, \quad (x, 0) \in X_0 \setminus F_0\,;$$

then we can use the Fundamental Method to show the next claim.

Claim 2.

$$U = \bigcup \{U(p) \mid p = (x, 0) \in X_0 \setminus F_0\} \tag{3.2}$$

is an open set of X such that

$$U \cap X_0 = X_0 \setminus F_0 \subset U \subset X \setminus F\,,$$

and if O is an open set of X with $O \cap F_0 \neq \emptyset$, then

$$\overline{\pi(U \setminus O)} \cap (F_0 \cap O) = \emptyset,$$

where $\pi : X \to X_0$ is the projection.

Assume the contrary; then there exists

$$(x_0, 0) \in \overline{\pi(U \setminus O)} \cap (F_0 \cap O).$$

Since X_0 is a Fréchet space, there exists a sequence $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ of points of $U \setminus O$ such that $\lim_{n\to\infty} x_n = x_0$. Recalling the definition of U in (3.2), for each $n \in \mathbb{N}$, there exists $(x'_n, 0) \in X_0 \setminus F_0$ such that

$$(x_n, y_n) \in U((x'_n, 0)).$$

Since O is an open neighborhood of $(x_0, 0)$ in X, there exists $l \in \mathbb{N}$ such that

$$M\left((x_0, 0), \frac{1}{l}\right) \subset O.$$

By (3.1), we can take $m_0 \in \mathbb{N}$ such that $m \geq m_0$ implies

$$M\left((x'_m, 0), \frac{1}{m(x'_m)}\right) \subset M\left((x_0, 0), \frac{1}{l}\right),$$

which implies $(x_n, y_n) \in O$. This contradicts $(x_n, y_n) \in U \setminus O$ and proves the claim.

Since $\mathcal{V}$ is uniformly approaching anti-cover of X_0 in X, there exists an open neighborhood U' of $X_0 \setminus F_0$ in X such that

$$\overline{S(U', \mathcal{V})} \cap X_1 \subset U.$$

Also, by this property, for each $p = (x, 0) \in X_0 \setminus F_0$, there exists $\mathcal{V}(x) \subset \mathcal{V}$ such that

$$V(p) = \left(x - \frac{1}{m(x)}, x + \frac{1}{m(x)}\right) \times \{0\} \cup \left(\bigcup \mathcal{V}(x)\right)$$

is an open neighborhood of $(x - 1/m(x), x + 1/m(x)) \times \{0\}$ in X and

$$V(p) \subset U' \cap U(p). \tag{3.3}$$

Note that

$$V(p) \subset \pi^{-1}\left(\left(x - \frac{1}{m(x)}, x + \frac{1}{m(x)}\right) \times \{0\}\right), \tag{3.4}$$

and that $\{V(p) \mid p = (x, 0) \in X_0 \setminus F_0\}$ is an open cover of $X_0 \setminus F_0$.

Since X_1 is a metric space, there exists a uniformly approaching anti-cover $\mathcal{W}$ of F_1 in X_1, which is locally finite in $X_1 \setminus F_1$. For each $p \in X_1 \setminus F_1$, let

$$W(p) = C(p, \mathcal{V}) \cap C(p, \mathcal{W}).$$

Then $W(p)$ is an open neighborhood of p in X such that

$$W(p) \cap F = \emptyset, \quad W(p) \cap X_0 = \emptyset.$$

Thus, we can finally construct an open cover

$$\mathcal{G} = \{V(p) \mid p \in X_0 \setminus F_0\} \cup \{W(p) \mid p \in X_1 \setminus F_1\}$$

of $X \setminus F$ in X.

Claim 3. $\mathcal{G}$ is a uniformly approaching anti-cover of F in X.

To show it, let O be an open set of X. We consider the following two cases.

Case 1. $O \cap F_0 = \emptyset$.

Since $\mathcal{W}$ is a uniformly approaching anti-cover of F_1 in X, there exists an open set O' of X such that

$$O \cap F \subset O', \quad S(O', \mathcal{W}) \subset O \setminus \overline{S(U', \mathcal{V})}.$$

By (3.3), we easily see

$$S(X \setminus O, \mathcal{G}) \cap O' = \emptyset.$$

Case 2. $O \cap F_0 \neq \emptyset$.

Since $\mathcal{W}$ is a uniformly approaching anti-cover of F_1 in X, there exists an open set Q of X such that

$$Q \cap F = O \cap F_1 \subset Q \subset S(Q, \mathcal{W}) \subset O \setminus \overline{S(U', \mathcal{V})}. \tag{3.5}$$

By Claim 2 and then by Claim 1, we can find an open set O_0 of X_0 satisfying

$$\begin{aligned} &F_0 \subset O_0 \subset O \cap X_0, \ \overline{\pi(U' \setminus O)} \cap O_0 = \emptyset, \\ &S(O_0, \{V(p) \cap X_0 \mid p \in X_0 \setminus F_0\}) \subset (O \cap X_0) \setminus \overline{\pi(U' \setminus O)}. \end{aligned} \tag{3.6}$$

Since $\mathcal{V}$ is uniformly approaching, there exists an open set P of X such that

$$(3.7) \qquad P \cap X_0 = O_0 \subset P,\ S(P, \mathcal{V}) \subset O,\ P \subset \pi^{-1}(O_0).$$

Then by (3.4), we have checked that

$$(3.8) \qquad V(p) \cap P \neq \emptyset \iff (V(p) \cap X_0) \cap O_0 \neq \emptyset$$

for each $p = (x, 0) \in X_0 \setminus F_0$.

Finally, we show that $S(P \cup Q, \mathcal{G}) \subset O$. Let $V(p),\ p \in X_0 \setminus F_0$, be arbitrarily chosen from $\mathcal{G}$ and suppose $V(p) \cap (P \cup Q) \neq \emptyset$. By (3.3) and (3.5), we have $V(p) \cap Q = \emptyset$. Therefore, $V(p) \cap P \neq \emptyset$. By (3.8),

$$(V(p) \cap X_0) \cap O_0 \neq \emptyset,$$

which, combined with (3.6), implies $V(p) \cap (U' \setminus O) = \emptyset$. Therefore, by (3.3), we have $V(p) \subset O$.

On the other hand, let $W(p),\ p \in X_1 \setminus F_1$, be arbitrarily chosen from $\mathcal{G}$ and suppose $W(p) \cap (P \cup Q) \neq \emptyset$. If $W(p) \cap P \neq \emptyset$, then by (3.7), we have $W(p) \subset O$, and if $W(p) \cap Q \neq \emptyset$, then by (3.5), we have $W(p) \subset O$. In either case, we have the required inclusion. Hence, X is a D-space.

The case for Ceder space is the same as above.

That both spaces are not metrizable is easily obtained from the Second Category Property of $\mathbb{R}$ and is well known. Since both spaces are first countable, they are not Lašnev spaces. This completes the proof. □

We cannot extend this result to general spaces which are the union $X = X_0 \cup X_1$ of a closed metrizable subset X_0 and metrizable subspace X_1 such that X_0 has a uniformly approaching anti-cover.

Example 3.3. *There is a non-D-space $X = X_0 \cup X_1$ which is an M_1-space, such that X_0 is a metrizable subset and has a uniformly approaching anti-cover, and X_1 is a metrizable subset of X.*

Proof: For each $n \in \mathbb{N}$, let $X(n)$ be a copy of the half plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

with the usual topology, and write it as

$$X(n) = \{((x, y), n) \mid (x, y) \in H\}.$$

Let X be the quotient space obtained from the topological sum $\oplus\{X(n) \mid n \in \mathbb{N}\}$ by identifying all $((x,0),n)$, $n \in \mathbb{N}$, with a single point $< x, 0 >$. Let $f : \oplus\{X(n) \mid n \in \mathbb{N}\} \to X$ be the natural mapping. Let

$$X_0 = \{< x, 0 > \mid x \in \mathbb{R}\}, X_1 = X \setminus X_0.$$

Then obviously, X_0 is a closed metrizable subset and X_1 is metrizable. Since X_0 has a uniformly approaching anti-cover in a metric space $f(X(n))$, X_0 has a uniformly approaching anti-cover in X.

We show that X is not a D-space. To this end, we show that $\{< 0, 0 >\}$ does not have a uniformly approaching anti-cover. Let $\mathcal{U}$ be any open cover of $X \setminus \{< 0, 0 >\}$. Let $n \in \mathbb{N}$. There exists $U(n) \in \mathcal{U}$ such that $< 1/n, 0 >\in U(n)$. Since $U(n)$ is open in X, there exists $\varepsilon_n > 0$ such that

$$W(n) = f(\{((x,y),n) \in X(n) \mid y < \varepsilon_n\})$$

is an open neighborhood of X_0 in $f(X(n))$ such that $U(n) \setminus W(n) \neq \emptyset$. Then

$$W = f(\bigcup\{W(n) \mid n \in \mathbb{N}\})$$

is an open neighborhood of X_0 in X such that $U(n) \setminus W \neq \emptyset$ for each $n \in \mathbb{N}$. This implies that

$$< 0, 0 >\in \overline{S(X \setminus W, \mathcal{U})}.$$

Hence, $\mathcal{U}$ is not a uniformly approaching anti-cover of $\{< 0, 0 >\}$ in X. □

References

[1] Jack G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

[2] Gary Gruenhage, *Generalized metric spaces*, in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 423–501.

[3] Jun-iti Nagata, *A contribution to the theory of metrization*, J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 185–192.

[4] Keiô Nagami, *The equality of dimensions*, Fund. Math. **106** (1980), no. 3, 239–246.

[5] ———, *Dimension of free L-spaces*, Fund. Math. **108** (1980), no. 3, 211–224.

[6] [W.] Stephen Watson, *The construction of topological spaces: Planks and resolutions*, in Recent Progress in General Topology. Ed. M. Hušek and J. van Mill. Amsterdam: Elsevier Science Pub., B.V., 1992. 637–757.

(Amono) Joint Graduate School (Ph.D. Program); Hyogo University of Teacher Education; Yashiro, Hyogo 673-1494, Japan

(Mizokami) Department of Mathematics; Joetsu University of Education; Joetsu, Niigata 943-8512, Japan

E-mail address: mizokami@juen.ac.jp