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CHARACTERIZATIONS OF SOME CLASSES OF DENDRITES WITH A CLOSED SET OF END POINTS

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Abstract. We investigate dendrites with a closed, countable set of end points. Such dendrites can be categorized according to the rank of their set of end points. We show that dendrites with a specific rank $\alpha + 1$ contain some particular dendrite M_α. As a consequence, we obtain a theorem that the rank of the set of end points of a dendrite with a closed set of end points cannot be increased under weakly confluent, and thus, confluent, open, or monotone mappings.

1. Introduction

In [6], Sophia Zafiridou examined universal elements in certain subsets of the class of dendrites with a closed set of end points of rank no larger than some ordinal α. In particular, she examined the subfamily of dendrites having no more than one point in the $(\alpha - 1)$-derivative of the set of end points, and the subset of *this* family having all points of order no larger than some κ. In addition, she showed that the class of dendrites with a set of end points of rank no larger than some α, and the class of dendrites with a closed, countable set of end points have no universal elements. In this paper, we construct a *smallest* element for the complement of the former class. More precisely, we show that for every ordinal α,

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there is a dendrite M_α that is contained in every dendrite with a closed set of end points of rank $\alpha+1$ or more. As a consequence, we show that the rank of the set of end points of a dendrite with a closed set of end points cannot be increased under weakly confluent, and thus confluent, open, or monotone mappings.

2. Preliminaries

In this paper, all spaces are assumed to be metric, and all ordinals countable.

We use the term *continuum* to mean a compact, connected space. A *dendrite* is a locally connected continuum that contains no simple closed curve, and we will assume that all dendrites under consideration are also nondegenerate. It is known that every subcontinuum of a dendrite is also a dendrite [4, §51, VI, Theorem 4, p. 301].

A *mapping* means a continuous function. A mapping $f : X \to Y$ between continua is said to be

- *monotone* if the preimage of each point is connected,
- *open* if the images of open sets are open,
- *confluent* if for each subcontinuum Q of Y, each component of $f^{-1}(Q)$ maps onto Q,
- *weakly confluent* if for each subcontinuum Q of Y, some subcontinuum of X maps onto Q.

The order of a point p in a dendrite X is the number of components of $X \setminus \{p\}$. Points of order one are called *end points*, and points of order three or more are called *ramification points.* The set of end points of a dendrite X is denoted by $E(X)$, and the set of ramification points is denoted $R(X)$. It is known that every point in a dendrite with a closed set of end points is of finite order [1, Theorem 3.3, p. 4], and that each subcontinuum of such a dendrite also has a closed set of end points [1, Theorem 3.2, p. 3].

For an ordinal α, the Cantor-Bendixson derivative of order α of a space E, denoted $E^{(\alpha)}$, is defined inductively as

- $E^{(0)} = E$,
- $E^{(\beta+1)} = \{e \in E | e \text{ is a limit point in } E^{(\beta)}\}$,
- $E^{(\gamma)} = \bigcap_{\beta<\gamma} E^{(\beta)}$ for limit ordinals γ.

The Cantor-Bendixson rank of E, denoted $\operatorname{rank}(E)$, is defined to be the least ordinal α such that $E^{(\alpha)}$ is empty. We will also use the notation $E^{(\alpha)}(X)$ to denote the α derivative of the set of end points of the dendrite X.

For compact spaces X, it is known that rank(X) exists if and only if X is countable. It is also known that if rank$(X) = \alpha$, then α is a successor ordinal and $X^{(\alpha-1)}$ is finite.

3. Main results

Fix the two points $p = \langle 0, 0\rangle$ and $e = \langle 1, 0\rangle$ of the plane. Define M_0 to be the straight line $\overline{ep}$ between them. Also fix a sequence $p_n \in \overline{ep}$ such that $p_j \in (p_i, e)$ for all $i < j$, and $\lim_{n\to\infty} p_n = e$.

Let $\alpha_0 > 0$ be an ordinal, and suppose that we have defined M_α for all $0 \leq \alpha < \alpha_0$. We will now construct M_{α_0}.

If α_0 is a successor ordinal, fix the sequence $\{\alpha_0^k\}_{k=1}^\infty$ to be constantly $\alpha_0 - 1$. If α_0 is a limit ordinal, fix $\{\alpha_0^k\}_{k=1}^\infty$ to be a strictly increasing sequence of ordinals such that $\lim_{k\to\infty} \alpha_0^k = \alpha_0$. For each k, let $M_{\alpha_0^k}(k)$ be a copy of $M_{\alpha_0^k}$ attaching to $\overline{ep}$ such that

(1) there is a homeomorphism $h : M_{\alpha_0^k} \to M_{\alpha_0^k}(k)$ such that $h(p) = p_k$;
(2) for any i, j such that $i \neq j$, the intersection $M_{\alpha_0^i}(i) \cap M_{\alpha_0^j}(j)$ is empty;
(3) $\lim_{k\to\infty} \text{diam}(M_{\alpha_0^k}(k)) = 0$.

Set

$$M_{\alpha_0} = \overline{ep} \cup \left(\bigcup_{k=1}^{\infty} M_{\alpha_0^k}(k) \right).$$

Clearly, M_α is a dendrite with a closed, countable set of end points for each α. Also note that for $\alpha > 0$, $E^{(\alpha)}(M_\alpha) = \{e\}$, and therefore, rank$(E(M_\alpha)) = \alpha + 1$.

Theorem 3.1. *Let X be a dendrite with a closed, countable set of end points. If M_α can be embedded into X, then* rank$(E(X)) \geq \alpha + 1$.

Proof: Since rank$(E(X)) > 0$ for any nondegenerate dendrite, the case $\alpha = 0$ is trivially true.

Let $h : M_\alpha \to X$ be an embedding. We will show that $h(E^{(\beta)}(M_\alpha)) \subseteq E^{(\beta)}(X)$ for all $\beta > 0$.

Note that the case $\beta = 1$ follows directly from the proof of Theorem 3.2 in [1], which we repeat here for convenience. Consider an arbitrary limit end point $\hat{e}$ of M_α, and let $\hat{e}_n$ be a sequence of endpoints of M_α such that $\lim_{n\to\infty} \hat{e}_n = \hat{e}$. We may assume that $h(\hat{e}_n) \notin E(X)$. For each n, if $h(\hat{e}_n)$ is an end point of X, then define $x_n = h(\hat{e}_n)$. If not, then choose some component C_n of $X \backslash h(M_\alpha)$ such that $h(\hat{e}_n) \in \operatorname{cl} C_n$, and choose $x_n \in C_n \cap E(X)$. Since $\{\operatorname{cl} C_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint continua in a hereditarily locally connected continuum, it forms a null sequence [5, Chapter 5, (2.6), p. 92]. Thus, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} h(\hat{e}_n) = h(\hat{e})$, and by closedness of $E(X)$, we have $h(\hat{e}) \in E^{(1)}(X)$.

Let β_0 be an ordinal, and suppose that $h(E^{(\beta)}(M_\alpha) \subseteq E^{(\beta)}(X)$ for all $1 \leq \beta < \beta_0$. We will show that this inclusion holds for $\beta = \beta_0$.

Case 1. β_0 is a successor ordinal.

By induction, we have $h(E^{(\beta_0-1)}(M_\alpha)) \subseteq E^{(\beta_0-1)}(X)$. Let $m \in E^{(\beta_0)}(M_\alpha)$, and m_n be a sequence of points from $E^{(\beta_0-1)}(M_\alpha)$ such that $m_n \to m$. The points $h(m_n)$ form a sequence in $E^{(\beta_0-1)}(X)$, and by continuity of h, we have $h(m_n) \to h(m)$, so $h(m) \in E^{(\beta_0)}(X)$. Since m was arbitrary, we conclude that $h(E^{(\beta_0)}(M_\alpha) \subseteq E^{(\beta_0)}(X)$.

Case 2. β_0 is a limit ordinal.

From the definition of the α-derivative for limit ordinals, and by induction, we have

$$h(E^{(\beta_0)}(M_\alpha)) = h\left(\bigcap_{\beta<\beta_0} E^{(\beta)}(M_\alpha)\right) \subseteq \bigcap_{\beta<\beta_0} h(E^{(\beta)}(M_\alpha)) \subseteq \bigcap_{\beta<\beta_0} E^{(\beta)}(X) = E^{(\beta_0)}(X)$$

Since $E^{(\alpha)}(M_\alpha)$ is nonempty, so is $h(E^{(\alpha)}(M_\alpha))$. Thus, by the inclusion above, the set $E^{(\alpha)}(X)$ is also nonempty, and therefore, $\operatorname{rank}(E(X)) \geq \alpha + 1$. □

Theorem 3.2. *For any dendrite X with a closed set of end points such that* $\operatorname{rank}(E(X)) \geq \alpha + 1$ *and for each isolated end point or* *ramification point $\hat{p}$ of X, there is an embedding of M_α into X such that p is mapped to $\hat{p}$.*

Proof: For $\alpha = 0$, the dendrite M_α is just an arc, so the theorem holds.

Let α_0 be an ordinal, and suppose that the theorem holds for all $0 \leq \alpha < \alpha_0$. We will show that it holds for $\alpha = \alpha_0$.

Let X be a dendrite with a closed set of end points such that $\text{rank}(E(X)) \geq \alpha_0 + 1$, and let $\hat{p}$ be any isolated end point or ramification point of X. Choose $\hat{e} \in E^{(\alpha_0)}(X)$. Note that if $\hat{p}$ is a ramification point, then letting C be the closure of the component of $X \setminus \hat{p}$ that contains $\hat{e}$, C is a neighborhood of $\hat{e}$, and thus, the rank of $\hat{e}$ in $E(C)$ is the same as the rank in $E(X)$. Also note that $\hat{p}$ is an isolated end point in C. Thus, we may assume, without loss of generality, that $\hat{p}$ is an isolated end point in X.

Let $\{\hat{p}_n\}_{n=1}^{\infty}$ be the set of ramification points in $\hat{e}\hat{p}$, ordered so that $\hat{p}_j \subseteq (\hat{p}_i, \hat{e})$ for every $i < j$. For each n, denote by X_n the union of all closures of components of $X \backslash \hat{e}\hat{p}$ that contain the point $\hat{p}_n$.

Let $\{\alpha_0^k\}_{k=1}^{\infty}$ be the sequence of ordinals fixed in the definition of M_{α_0}. We claim that for each k, there are infinitely many X_n such that $\text{rank}(E(X_n)) \geq \alpha_0^k$. If not, then $E^{(\alpha_0^k)}(X_n)$ is nonempty for at most finitely many X_n. Thus, for any sequence of end points $\{\hat{e}_n\}_{n=1}^{\infty} \subseteq E^{(\alpha_0^k)}(X) \setminus \{\hat{e}, \hat{p}\}$ such that $\hat{e}_n \to e$ (of which at least one exists, since $\hat{e} \in E^{(\alpha_0)}(X)$), there must be a subsequence that lies completely in one X_n. Since $\hat{e} \notin E(X_n)$ for any n, this contradicts the fact that X_n has a closed set of end points, and the claim is shown.

Therefore, we may fix a subsequence X_{n_k} of X_n so that $\text{rank}(E(X_{n_k})) \geq \alpha_0^k$ for all k.

Since each $\hat{p}_n$ is a ramification point of X and since X has a closed set of end points, $\hat{p}_n$ is not a limit end point of X_n for any n. Thus, by induction, there is an embedding $h_k : M_{\alpha_0^k} \to X_{n_k}$ for each k such that $h_k(p) = \hat{p}_n$. Let $h : M_{\alpha_0} \to X$ be such that $h|_{ep}$ is a homeomorphism with $\hat{e}\hat{p}$ and $h(p) = \hat{p}$. Also define $h|_{M_{\alpha_0^k}(k)} = h_k$ for all k. Clearly, h is the required embedding. □

Combining theorems 3.1 and 3.2, we have the following characterization.

Corollary 3.3. *Let X be a dendrite with a closed set of end points. Then* $\operatorname{rank}(E(X)) \geq \alpha+1$ *iff X contains a copy of the dendrite M_α.*

Theorem 3.4. *If X, Y are dendrites with a closed set of end points and $f : X \to Y$ is a weakly confluent surjection, then* $\operatorname{rank}(E(Y)) \leq \operatorname{rank}(E(X))$.

Proof: Let $\hat{e}$ be an arbitrary point of $E^{(1)}(Y)$, and let $\hat{p}_n$ be a sequence of points of $R(Y)$ such that $\hat{p}_n \to \hat{e}$. By [3, Theorem II.1], we may choose $x_n \in \operatorname{cl}(R(X))$ such that $f(x_n) = \hat{p}_n$ for each n. Possibly taking a subsequence, we may assume that x_n is convergent and set $x = \lim_{n\to\infty} x_n$. By [1, Corollary 3.5], we have $\operatorname{cl}(R(X)) \subseteq E(X) \cup R(X)$, so x is either a limit point of $R(X)$ or of $E(X)$. In either case, the point x is in $E^{(1)}(X)$. By continuity of f, the sequence $f(x_n)$ converges to $f(x)$, but by construction, the limit of $f(x_n)$ is $\hat{e}$. Thus, $f(x) = \hat{e}$, and since $\hat{e}$ was arbitrary, we conclude that $E^{(1)}(Y) \subseteq f(E^{(1)}(X))$.

Suppose that $\operatorname{rank}(E(X)) = \alpha + 1$ for some ordinal α.

Case 1. $\alpha < \omega$.

By (4.11) and (4.12) in [2] and from the inclusion above, we have

$$E^{(\alpha+1)}(Y) = [E^{(1)}(Y)]^{(\alpha)} \subseteq [f(E^{(1)}(X))]^{(\alpha)} \subseteq f(E^{(\alpha+1)}(X)).$$

Case 2. $\alpha \geq \omega$.

For a transfinite ordinal γ, it is clear from the definition that $(E^{(1)})^{(\gamma)} = E^{(\gamma)}$. Thus, similar to case 1, we have

$$E^{(\alpha+1)}(Y) = [E^{(1)}(Y)]^{(\alpha+1)} \subseteq [f(E^{(1)}(X))]^{(\alpha+1)} \subseteq f(E^{(\alpha+1)}(X)).$$

Since $E^{(\alpha+1)}(X)$ is empty, so is $f(E^{(\alpha+1)}(X))$, and therefore by the two cases above, $E^{(\alpha+1)}(Y)$ is empty. Thus, we conclude that $\operatorname{rank}(E(Y)) \leq \alpha + 1 = \operatorname{rank}(E(X))$. □

Corollary 3.5. *The rank of the set of end points of a dendrite with a closed set of end points cannot be increased by*

(1) *taking subdendrites,*
(2) *open mappings,*
(3) *monotone mappings,*
(4) *confluent mappings.*

Proof: Item (1) follows from the fact that each subcontinuum of a dendrite is a retract of that dendrite, and every retraction is weakly confluent. All open mappings on compact spaces [5, Theorem 7.5, p. 148], all confluent mappings, and all monotone mappings are weakly confluent, confirming items (2), (3), and (4). □

4. The hierarchy of weakly confluent mappings

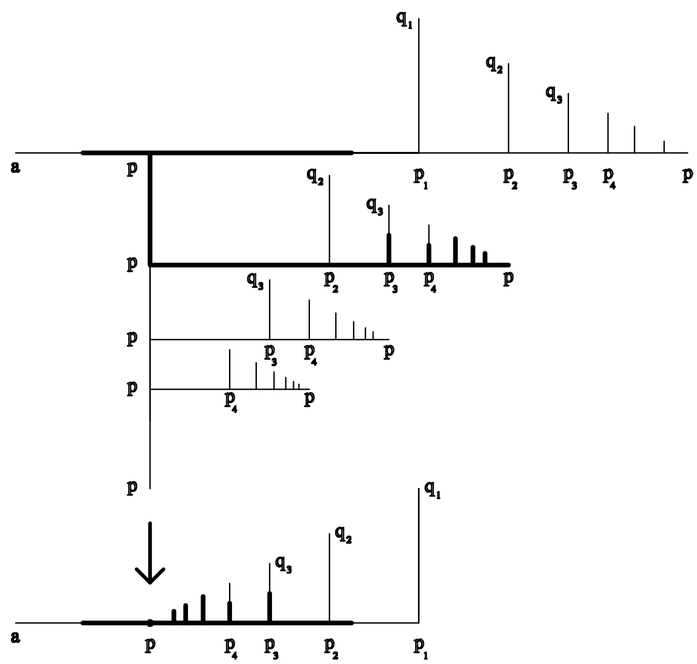
In [2], J. J. Charatonik, W. J. Charatonik, and J. R. Prajs studied mapping hierarchies for dendrites. Let us recall basic definitions and some facts established in that paper.

Given a class $\mathbb{F}$ of mappings and two dendrites X and Y, we say that $Y \leq_{\mathbb{F}} X$ if there is a surjection $f \in \mathbb{F}$ mapping X onto Y. If the class $\mathbb{F}$ contains homeomorphisms and is closed under compositions, then the relation $\leq_{\mathbb{F}}$ is a quasi-ordering on the class of dendrites, i. e., it is reflexive and transitive. Denote by $\mathbb{M}$ the class of monotone maps, by $\mathbb{C}$ the class of confluent maps, and by $\mathbb{W}$ the class of weakly monotone maps. The authors show, among many other things, that the quasi-orders $\leq_{\mathbb{M}}$ and $\leq_{\mathbb{C}}$ are identical [2, Corollary 5.7], and they ask if the quasi-order $\leq_{\mathbb{W}}$ is identical with the previous two (see [2, Question 5.12]). Here, we answer the question in the negative by showing an example of two dendrites X and Y such that there is no monotone (equivalently, confluent) map from X onto Y, but there is a weakly confluent one.

Example 4.1. *There are dendrites X and Y such that there is no confluent mapping from X onto Y, but there is a weakly confluent one.*

Proof: The continua X and Y are shown in the figure below. Points in X are labeled according to their image in Y, and the mapping is linear between labeled points. To see that the mapping is weakly confluent, consider a subcontinuum Q of Y. If Q is right of the point p, there is a subcontinuum in the upper right corner of X that maps onto Q. A typical continuum containing the point p and a continuum in X that is mapped onto it are highlighted in the figure.

To see that there is no monotone map from X onto Y, observe that Y is precisely the dendrite W defined in [1, p. 3], while X does not contain a copy of W. The existence of such a map would contradict Theorem 6.1 in [1, p. 12]. □



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