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ON THE SEMI-OPEN INDUCED MAPPINGS

JAVIER CAMARGO

ABSTRACT. An example is given of a non-open mapping f such that $C(f)$ is semi-open, giving a negative answer to the question posed in *Semi-openness and almost-openness of induced mappings* by Xianjiu (Xian Jiu) Huang, Fanping Zeng, and Gengrong Zhang [Appl. Math. J. Chinese Univ. Ser. B **20** (2005), no. 1, 21–26]. Furthermore, some relationships between $f, C_n(f)$ and $HS_n(f)$ are proved.

1. INTRODUCTION

It is known that the mapping $f : X \to Y$ between topological spaces is open if $f(U)$ is an open subset of Y for each open subset U in X. In [7], Xianjiu (Xian Jiu) Huang, Fanping Zeng, and Gengrong Zhang defined semi-open mappings. It is evident that every open mapping is semi-open. Furthermore, they demonstrated that if $C(f)$ is a semi-open mapping, then f is also semi-open. Also given is an example of an open mapping f such that the induced mapping $C(f)$ is not semi-open, and they asked the following question.

Question 1.1. [7, p. 26]. Let $f : X \to Y$ be a surjective mapping between continua X and Y. Suppose that $C(f)$ is semi-open; does it then follow that f is open?

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In this paper, we give an answer to this question. In section 3, we prove that if f is defined between arcs or between simple closed curves and $C(f)$ is semi-open, then f is a homeomorphism. Moreover, in section 4, we present an example where we give a negative answer to Question 1.1. Finally, in section 5, we show that $C_n(f)$ is semi-open if and only if $HS_n(f)$ is semi-open for each positive integer n.

2. Definitions

If (X,d) is a metric space, then given $A \subset X$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $B_d(A,\varepsilon)$, the interior of A is denoted by $Int_X(A)$, and its closure is denoted by $Cl_X(A)$. The symbol $\mathbb{N}$ denotes the set of positive integers. A *continuum* is a nonempty, compact, connected metric space, and a *mapping* is assumed to be a continuous function. Given a continuum X with metric d, we consider the following hyperspaces of X.

(1) $2^X = \{A \subset X : A \text{ is closed and nonempty}\}$,
(2) $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}, n \in \mathbb{N}$,
(3) $F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}, n \in \mathbb{N}$,

each with the Hausdorff metric H, defined by

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

for each $A, B \in 2^X$. (See [2, p. 9] or [4, p. 58] for general information about hyperspaces.) It is known that the collection of sets $\langle U_1, U_2, ..., U_l\rangle$ form a base on 2^X with the Vietoris topology (see [2, p. 3]), where $U_1, U_2, ..., U_l$ are open sets in X and

$$\langle U_1, U_2, ..., U_l\rangle = \{A \in 2^X : A \subset \cup_{i=1}^{l} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i\}.$$

It is known that the Vietoris topology coincides with the topology induced by the Hausdorff metric [2, Theorem 3.1].

If $p \in X$, we denote by $C_p(X)$ the set

$$C_p(X) = \{A \in C(X) : p \in A\}.$$

It is not difficult to prove that $C_p(X)$ is closed in $C(X)$.

Notation 2.1. Given a continuum X, let $\langle U_1, U_2, ..., U_l\rangle_n$ denote the set $\langle U_1, U_2, ..., U_l\rangle \cap C_n(X)$.

Definition 2.2. Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be semi-open provided that $Int_Y(f(U)) \neq \emptyset$ for each nonempty open subset U in X.

Obviously, each open mapping is semi-open. However, in [7], an example is given where the converse is not true.

Definition 2.3. Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be quasi-interior at a point $p \in X$ provided that there exists a point $q \in U$ such that $f(q) \in Int_Y(f(U))$ for each open subset U in X containing p.

The following result may be found in [7, Lemma 2.1].

Lemma 2.4. *Let $f : X \to Y$ be a mapping between continua X and Y. Then f is semi-open if and only if it is quasi-interior at each point of X.*

Let $f : X \to Y$ be a mapping between continua. Then $2^f : 2^X \to 2^Y$, given by $2^f(A) = f(A)$ for each $A \in 2^X$, is called the *induced mapping between 2^X and 2^Y*. In [2, p. 106], it is proved that 2^f is a mapping. Let $n \in \mathbb{N}$; the mapping $C_n(f) : C_n(X) \to C_n(Y)$ given by $C_n(f) = 2^f|_{C_n(X)}$ is called the *induced mapping between the n-fold hyperspaces $C_n(X)$ and $C_n(Y)$*.

Let X be a continuum. We denote by $HS_n(X)$ the quotient space $C_n(X)/F_n(X)$, for each $n \in \mathbb{N}$.

Notation 2.5. Given a continuum X and $n \in \mathbb{N}$, let q_X^n denote the quotient mapping from $C_n(X)$ onto $HS_n(X)$, and let F_X^n denote the point $q_X^n(F_n(X))$.

Remark 2.6. Note that $q_X^n|_{C_n(X)\setminus F_n(X)}$ is a homeomorphism between $C_n(X) \setminus F_n(X)$ and $HS_n(X) \setminus \{F_X^n\}$.

Let $f : X \to Y$ be a mapping between continua and let $n \in \mathbb{N}$. We define $HS_n(f) : HS_n(X) \to HS_n(Y)$ by

$$HS_n(f)(\chi) = \begin{cases} q_Y^n(C_n(f)((q_X^n)^{-1}(\chi))), & \text{if } \chi \neq F_X^n \\ F_Y^n, & \text{if } \chi = F_X^n \end{cases},$$

called the *induced mapping between the n-fold hyperspace suspensions $HS_n(X)$ and $HS_n(Y)$*. Note that by definition, $q_Y^n \circ C_n(f) = HS_n(f) \circ q_X^n$. (For details on $HS_n(X)$ and $HS_n(f)$, see [3, p. 145].)

For each continuum X, we write $C(X), HS(X)$, and q_X, instead of $C_1(X), HS_1(X)$, and q_X^1, respectively.

3. Semi-open induced mappings

In this section, we study relationships between $C(f)$ and f, when f is defined between arcs or between simple closed curves.

Proposition 3.1. *Let $f : [0,1] \to [0,1]$ be a surjective mapping. If $C(f)$ is semi-open, then f is monotone.*

Proof: Suppose that f is not monotone. Let p and q be points of $[0,1]$ such that $f(p) = f(q)$ and which belong to different components of $f^{-1}(f(p))$. Assume that $p < q$. Thus, $f([p,q])$ is nondegenerate. Therefore, there exists a non-cut point $y \in f([p,q])$ such that $y \neq f(p)$. Let $x \in [p,q]$ such that $f(x) = y$. Let V_1 and V_2 be disjoint open subsets of $[0,1]$ such that $f(p) \in V_1$ and $y \in V_2$.

Since $[0,1]$ is locally connected, there exist three open and connected subsets U_1, U_2, and U_3 of $[0,1]$ such that

(1) $p \in U_1, q \in U_3$, $x \in U_2 \setminus (U_1 \cup U_3)$, and $U_1 \cap U_3 = \emptyset$;
(2) $[p,q] \subset U_1 \cup U_2 \cup U_3$;
(3) $f(U_1)$ and $f(U_3)$ are subsets of V_1.

Note that $[p,q] \in \langle U_1, U_2, U_3 \rangle$, and if $A \in \langle U_1, U_2, U_3 \rangle_1$, then $x \in A$. Furthermore, by (3), $f(A \setminus [p,q]) \subset V_1$. Thus, y is a non-cut point of $f(A)$.

We prove that $Int_{C([0,1])}(C(f)(\langle U_1, U_2, U_3 \rangle_1)) = \emptyset$. Let $B \in \langle U_1, U_2, U_3 \rangle_1$ and let $\epsilon > 0$. By [7, Theorem A], f is semi-open. Since $U_1 \cap U_3 = \emptyset$ and $B \cap U_i \neq \emptyset$ for each $i = 1, 3$, $Int_{[0,1]}(B) \neq \emptyset$. Thus, $f(B)$ is nondegenerate.

We take $D = f(B) \setminus (y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2})$. Without loss of generality, we may assume that $\frac{\epsilon}{2}$ is less than the diameter of $f(B)$. Thus, since y is a non-cut point of $f(B)$, D is a nonempty subcontinuum of $[0,1]$. Therefore, D is not a point of $C(f)(\langle U_1, U_2, U_3 \rangle_1)$ and clearly, $H(f(B), D) < \epsilon$. Thus, $Int_{C([0,1])}(C(f)(\langle U_1, U_2, U_3 \rangle_1)) = \emptyset$. But this contradicts our assumption that $C(f)$ is semi-open. □

Proposition 3.2. *Let $f : S^1 \to S^1$ be a surjective mapping. If $C(f)$ is semi-open, then f is monotone.*

Proof: Suppose that f is not monotone. Let p and q be points of S^1 such that $f(p) = f(q)$ and which belong to different components of $f^{-1}(f(p))$. Let $\widehat{pq}$ be an arc for which p and q are non-cut points. We consider two cases.

Case 1. $f(\widehat{pq})$ is a proper subcontinuum of S^1. An argument similar to that in the proof of Proposition 3.1 shows that there exists a nonempty open set $\langle U_1, U_2, U_3\rangle_1$ such that

$$Int_{C(S^1)}(C(f)(\langle U_1, U_2, U_3\rangle_1)) = \emptyset.$$

Case 2. $f(\widehat{pq}) = S^1$. Let U be an open proper subset of S^1 such that $\widehat{pq} \subset U$. Let V and W be open subsets of $U \setminus \widehat{pq}$ such that $\langle U, V, W\rangle_1 \neq \emptyset$, and if $A \in \langle U, V, W\rangle_1$, then $\widehat{pq} \subset A$. Thus, $C(f)(\langle U, V, W\rangle_1) = \{S^1\}$. Therefore,

$$Int_{C(S^1)}(C(f)(\langle U, V, W\rangle_1)) = \emptyset.$$ □

The following theorem gives a positive answer to Question 1.1 when f is defined between arcs or between simple closed curves.

Theorem 3.3. *Let $f : X \to X$ be a surjective mapping where $X \in \{[0,1], S^1\}$. If $C(f)$ is semi-open, then f is a homeomorphism.*

Proof: Let $f : X \to X$ be a surjective mapping such that $C(f)$ is semi-open. By [7, Theorem A], f is also semi-open, and by propositions 3.1 and 3.2, f is monotone. Let $y \in X$. Since f is monotone, $f^{-1}(y)$ is connected. Note that $f^{-1}(y)$ is degenerate, because if there exists an open subset $U \subset f^{-1}(y)$, then $Int_Y(f(U)) = \emptyset$, but this implies the contradiction that f is semi-open. Therefore, f is one-to-one and, in fact, a homeomorphism. □

4. Example

In this section, we present an example where we give a negative answer to Question 1.1.

Example 4.1. We give a map from a simple triod onto an arc such that $C(f)$ is a semi-open map but f is not open.

We define $X_1 = \{(x,y) : y = -x + \frac{1}{2}, \frac{1}{4} \leq x \leq \frac{1}{2}\}$, $X_2 = \{(x,y) : y = x - \frac{1}{2}, 0 \leq x \leq \frac{1}{2}\}$, and $X_3 = \{(x,0) : \frac{1}{2} \leq x \leq 1\}$ and denote $X = X_1 \cup X_2 \cup X_3$. Let $f : X \to [0,1]$ be a map defined by $f(x,y) = x$. Clearly, f is neither open nor monotone.

We prove that $C(f)$ is semi-open. By Lemma 2.4, it suffices to show that $C(f)$ is quasi-interior at each point of $C(X)$.

Let $A \in C(X)$. We consider two cases.

Case 1. $(\frac{1}{2}, 0) \notin A$. Since $A \notin C_{(1/2,0)}(X)$ and $C_{(1/2,0)}(X)$ is closed, there exists $r > 0$ such that $B_H(A, r) \cap C_{(1/2,0)}(X) = \emptyset$.

Suppose that $A \subset X_1$. Let $D \in B_H(A,r)$ and let $s > 0$ such that $B_H(D,s) \subset B_H(A,r)$ and $B_H(D,s) \cap C_{(1/4,1/4)}(X) = \emptyset$.

We prove that

$$B_H(f(D), \frac{s}{2}) \subset C(f)(B_H(A,r)).$$

Let $E \in B_H(f(D), \frac{s}{2})$. Notice that

$$B_H(f(D), \frac{s}{2}) \cap (C_{1/4}([0,1]) \cup C_{1/2}([0,1])) = \emptyset.$$

Hence, $\{\frac{1}{4}, \frac{1}{2}\} \cap E = \emptyset$. Thus, $f^{-1}(E) \cap X_1$ is a connected set such that $f(f^{-1}(E) \cap X_1) = E$.

Now, by definition of f,

$$H(f^{-1}(E) \cap X_1, D) \leq 2H(E, f(D)) < s.$$

Therefore,

$$f^{-1}(E) \cap X_1 \in B_H(D,s) \subset B_H(A,r)$$

and $E \in C(f)(B_H(A,r))$. Similarly, if $A \subset X_2$ or $A \subset X_3$.

Case 2. $(\frac{1}{2}, 0) \in A$. Let $r > 0$. Define $D_0 = A \cup Cl_X(B_d((\frac{1}{2},0), \frac{r}{3}))$. Clearly, $D_0 \in C(X)$ and $H(A, D_0) \leq \frac{r}{3}$. Let $D \in B_H(D_0, \frac{r}{6})$ and let $s > 0$ such that $B_H(D,s) \subset B_H(D_0, \frac{r}{6})$ and $B_H(D,s) \cap C_{(1/4,1/4)}(X) = \emptyset$. Notice that $(\frac{1}{2}, 0) \in K$ for each $K \in B_H(D,s)$.

We prove that $B_H(f(D), \frac{s}{2}) \subset C(f)(B_H(A,r))$. Let x_1, x_2, and x_3 be points in $[0,1]$ such that

$$\begin{aligned} D = &\{(x,y) : y = -x + \tfrac{1}{2}, x_1 \leq x \leq \tfrac{1}{2}\} \\ &\cup \{(x,y) : y = x - \tfrac{1}{2}, x_2 \leq x \leq \tfrac{1}{2}\} \cup \{(x,0) : \tfrac{1}{2} \leq x \leq x_3\}. \end{aligned}$$

By construction, $\frac{1}{4} + \frac{s}{2} \leq x_1 < \frac{1}{2}$, $x_2 < \frac{1}{2}$, and $x_3 > \frac{1}{2}$. Thus, $f(D) = [\min\{x_1, x_2\}, x_3]$.

Suppose first that $x_1 = \min\{x_1, x_2\}$. Observe that $\frac{1}{4} \notin K$ for each $K \in B_H(f(D), \frac{s}{2})$. Let $E \in B_H(f(D), \frac{s}{2})$ and let y_1 and y_2 be points in $[0,1]$ such that $E = [y_1, y_2]$. We define

$$\begin{aligned} E_0 = &\{(x,y) : y = -x + \tfrac{1}{2}, y_1 \leq x \leq \tfrac{1}{2}\} \\ &\cup \{(x,y) : y = x - \tfrac{1}{2}, \max\{x_2, y_1\} \leq x \leq \tfrac{1}{2}\} \cup \{(x,0) : \tfrac{1}{2} \leq x \leq y_2\}. \end{aligned}$$

Clearly, $f(E_0) = E$. Since $H(f(D), E) < \frac{s}{2}$, $H(E_0, D) < s$. Thus, $E_0 \in B_H(D, s) \subset B_H(D_0, \frac{r}{6})$ and

$$H(E_0, A) \leq H(E_0, D_0) + H(D_0, A) \leq \frac{r}{6} + \frac{r}{3} < r.$$

Therefore, $E_0 \in B_H(A, r)$ and $E \in C(f)(B_H(A, r))$.

Now, if $x_2 = \min\{x_1, x_2\}$, then we define

$$E_0 = \{(x, y) : y = -x + \tfrac{1}{2}, \max\{x_1, y_1\} \leq x \leq \tfrac{1}{2}\}$$
$$\cup \{(x, y) : y = x - \tfrac{1}{2}, y_1 \leq x \leq \tfrac{1}{2}\} \cup \{(x, 0) : \tfrac{1}{2} \leq x \leq y_2\}.$$

Similarly, $f(E_0) = E$, $E_0 \in B_H(A, r)$, and $E \in C(f)(B_H(A, r))$.

5. Semi-openness and induced mapping

The goal of this section is to show that $C_n(f)$ is semi-open if and only if $HS_n(f)$ is semi-open, for each $n \in \mathbb{N}$ (Theorem 5.2).

Proposition 5.1. *Let X be a continuum and let $n \in \mathbb{N}$, then the quotient mapping $q_X^n : C_n(X) \to HS_n(X)$ is semi-open.*

Proof: Let $\mathcal{U}$ be an open subset of $C_n(X)$. Since

$$Cl_{C_n(X)}(C_n(X) \setminus F_n(X)) = C_n(X)$$

and $F_n(X)$ is closed in $C_n(X)$, there exists an open subset $\mathcal{V}$ of $C_n(X)$ such that $\mathcal{V} \subset \mathcal{U} \setminus F_n(X)$. Thus, by Remark 2.6, $q_X^n(\mathcal{V})$ is an open subset in $HS_n(X)$ and clearly, $q_X^n(\mathcal{V}) \subset q_X^n(\mathcal{U})$. Therefore, $Int_{HS_n(X)}(q_X^n(\mathcal{U})) \neq \emptyset$. □

It is very easy to prove that if f and g are semi-open mappings, then $f \circ g$ is also semi-open. Furthermore, if $f \circ g$ is a semi-open mapping, then f is also a semi-open mapping.

Theorem 5.2. *Let $f : X \to Y$ be a mapping between continua and let $n \in \mathbb{N}$. Then $C_n(f)$ is semi-open if and only if $HS_n(f)$ is semi-open.*

Proof: Let $f : X \to Y$ be a mapping between continua such that $C_n(f)$ is semi-open. By Proposition 5.1, $q_Y^n \circ C_n(f)$ is semi-open. Since $q_Y^n \circ C_n(f) = HS_n(f) \circ q_X^n$, $HS_n(f)$ is also semi-open.

Now, we assume that $HS_n(f)$ is semi-open. Let $\mathcal{U}$ be an open subset of $C_n(X)$. Since $Cl_{C_n(X)}(C_n(X) \setminus F_n(X)) = C_n(X)$ and $F_n(X)$ is closed in $C_n(X)$, there exists an open subset $\mathcal{V}$ of $C(X)$

such that $\mathcal{V} \subset \mathcal{U} \setminus F_n(X)$. Therefore, $Int_{HS_n(Y)}(HS_n(f)(q_X^n(\mathcal{V}))) \neq \emptyset$. Thus, there exists a nonempty open subset $\mathcal{W}$ of $HS_n(Y)$ such that $\mathcal{W} \subset Int_{HS_n(Y)}(HS_n(f)(q_X^n(\mathcal{V}))) \setminus \{F_Y^n\}$. Therefore, since $q_Y^n|_{C_n(Y)\setminus F_n(Y)}$ is a homeomorphism, $(q_Y^n)^{-1}(\mathcal{W}) \subset C_n(f)(\mathcal{U})$, and the proof is complete. □

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