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ON THE HOCHSTER DUAL OF A TOPOLOGICAL SPACE

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Abstract. This paper deals with a dual topology introduced by M. Hochster; this duality is applied to two particular classes of spaces related to spectral spaces: down-spectral spaces and up-spectral spaces. Dual topological properties are also introduced and studied.

0. Introduction

The ideas of duality in topology are due in a special case to J. de Groot [5]. The general case has been discussed by Ralph Kopperman in a remarkable paper [10].

We deal here with the "dual topology" on a space X introduced by M. Hochster in [8] and [9].

Definition 0.1. Let $\mathfrak{B}$ be a subbasis of a topological space X. By the *dual topology* on X determined by $\mathcal{B}$, we mean the topology on X which has $\mathcal{B}$ as a subbasis for its closed sets. The resulting topological space will be denoted by $X^\star$.

Note that throughout this paper, all topological spaces X will be assumed to have a basis $\mathcal{B}$ of compact open sets, and thus, $X^\star$ will denote the dual topology of Hochster determined by $\mathcal{B}$, i.e., the

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topology which has the complement of the compact open sets as an open basis.

The following definition is natural.

Definition 0.2. Let $\mathcal{P}$ and $\mathcal{Q}$ be two topological properties. We say that $\mathcal{Q}$ is a *dual property* of $\mathcal{P}$ if for each topological space X, the following equivalence holds: X is $\mathcal{P}$ if and only if $X^\star$ is $\mathcal{Q}$.

A property $\mathcal{P}$ is said to be *self-dual* if $\mathcal{P}$ is a dual of $\mathcal{P}$.

The main goal of this paper is the investigation of dual properties in spectral topology.

Before giving our main results, let us recall some elementary facts from the theory of spectral spaces.

Recall that a closed set C of a topological space X has a *generic point* if there is some $x \in C$ such that $C = \overline{\{x\}}$.

A subspace Y of a space X is said to be *irreducible* if the intersection of two nonempty open sets of Y is nonempty.

A topological space in which every nonempty irreducible closed set has a unique generic point is called a *sober space*.

According to Hochster [8], a topology $\mathcal{T}$ on a set X is said to be *spectral* if and only if the following axioms hold.

(i) $(X, \mathcal{T})$ is a sober space.

(ii) X is compact and has a basis of compact open sets.

(iii) The intersection of two compact open sets is compact.

Hochster has proved that a space X is spectral if and only if $X^\star$ is spectral [8]. Thus, according to our definition, "spectral" is a self-dual property.

We will investigate duality properties for some concepts introduced recently by Karim Belaid and Othman Echi in [1].

In order to obtain information concerning a longstanding (since 1976) open question about spectral sets stated by William J. Lewis and Jack Ohm [11], Belaid and Echi have introduced the following definitions.

Definition 0.3. Let X be a topological space.

(1) X is said to be an *up-spectral space* if it satisfies the axioms of a spectral space with the exception that X is not necessarily compact.

(2) X is said to be a *down-spectral space* if it satisfies the axioms of a spectral space with the exception that X does not have necessarily a generic point when it is irreducible.

Let $\mathcal{C}$ be a collection of topological spaces and $\mathcal{D} : \mathcal{C} \longrightarrow \mathcal{C}$ a correspondence between $\mathcal{C}$ and $\mathcal{C}$. Following [10], $\mathcal{D}$ is said to be a duality if $\mathcal{D}^3 = \mathcal{D}$.

The main result of this paper deals with Kopperman's duality on the class $\mathcal{UD}$ of all up-spectral spaces and down-spectral spaces: Let $\star : \mathcal{UD} \longrightarrow \mathcal{UD}$ be the correspondence which takes X to its Hochster dual $X^\star$. Then $\star$ is a Kopperman duality [Theorem 2.1].

1. Preliminary results

Proposition 1.1. *Let X be a topological space with a basis $\mathcal{B}$ of compact open sets. Then the following statements are equivalent.*

(i) *$X^\star$ is irreducible.*

(ii) *For each pair of open compact sets U and V of X such that $X = U \cup V$, we have $X = U$ or $X = V$.*

Proof: $(i) \Longrightarrow (ii)$ Is straightforward, since U and V are closed in $X^\star$.

$(ii) \Longrightarrow (i)$ Let V_1 and V_2 be two nonempty open sets of $X^\star$. We have to prove that $V_1 \cap V_2 \neq \emptyset$. Since $\{X \setminus U \mid U$ is a compact open set of $X\}$ is a basis of $X^\star$, it is enough to take the particular case $V_1 = X \setminus U_1$ and $V_2 = X \setminus U_2$, with U_i a compact open set of X. It is clear that $V_1 \cap V_2 = X \setminus (U_1 \cup U_2) \neq \emptyset$, since $X \neq U_1$ and $X \neq U_2$. □

The following result shows that the property "compact" is a dual of the property "reducible or has a generic point."

Proposition 1.2. *Let X be a topological space with a basis $\mathcal{B}$ of compact open sets. Then the following statements are equivalent.*

(i) *X is a compact space.*

(ii) *$X^\star$ is reducible or has a generic point.*

Proof: $(i) \Longrightarrow (ii)$ Suppose that $X^\star$ is irreducible. Let us show that there exists $a \in X$ such that $X^\star = \overline{\{a\}}^\star$. Assume that for each $x \in X$, there is a compact open set U_x of X distinct from X such that $x \in U_x$. Then we have $X = \bigcup[U_x : x \in X]$, and since X is compact, there exists a finite subset Y of X such that $X = \bigcup[U_x : x \in Y]$. On the other hand, $X^\star$ is irreducible; this forces $X = U_x$ for some $x \in Y$. It follows that there exists $a \in X$ such that the only compact open set of X containing a is X. Therefore, $\overline{\{a\}}^{\star} = X^{\star}$.

$(ii) \Longrightarrow (i)$ If $X^{\star} = \overline{\{a\}}^{\star}$, then X is the unique closed set of $X^{\star}$ containing a. Since X has a basis of compact open sets, X is the unique compact open set of X containing a.

If $X^{\star}$ is reducible, then by Proposition 1.1, there exist two compact open sets U and V of X such that $X = U \cup V$ with $U \neq X$ and $V \neq X$. Therefore, X is a compact space. □

Proposition 1.3. *Let X be a topological space with a basis $\mathcal{B}$ of compact open sets. Then the following properties hold.*

(i) *If X has a generic point, then $X^{\star}$ is compact.*

(ii) *If $X^{\star}$ is compact, then X is reducible or has a generic point.*

(iii) *In particular, if X is irreducible, then $X^{\star}$ is compact if and only if X has a generic point.*

Proof: (i) Suppose that $X = \overline{\{a\}}$. Let $(C_i, i \in I)$ be a collection of closed sets of $X^{\star}$ such that $\bigcap[C_i : i \in I] = \emptyset$. Since the compact open sets of X constitute a basis of closed sets of $X^{\star}$, one may suppose that the C_i's are open compact in X. Since $a \notin \bigcap[C_i : i \in I]$, we have $a \notin C_i$ for some $i \in I$, but then $C_i = \emptyset$ because $X = \overline{\{a\}}$, proving that $X^{\star}$ is compact.

(ii) Suppose that X is irreducible. Let $\mathcal{F}$ be the collection of all nonempty compact open sets of X. Since $X^{\star}$ is compact and X is irreducible, then the intersection $\bigcap[U : U \in \mathcal{F}]$ is non empty. Pick a in $\bigcap[U : U \in \mathcal{F}]$; we easily get $X = \overline{\{a\}}$.

(iii) Follows immediately from (i) and (ii). □

The following example proves that "reducible or has a generic point" is not a dual property of "compact."

Example 1.4. A reducible topological space X such that $X^{\star}$ is not compact.

Let X_1 and X_2 be two disjoint infinite sets equipped with the cofinite topology. Set $X = X_1 \cup X_2$ endowed with the disjoint union topology.

Since X_1 and X_2 are closed, X is a reducible topological space.

Now $\{X_1 \setminus \{x\} : x \in X_1\}$ is a collection of compact open sets of X which do not meet, but all finite intersections of this collection are nonempty since X_1 is infinite. Thus, $X^{\star}$ is not compact.

Notation 1.5. Let X be a T_0-space. If X is reducible or has a generic point, set $X^\vee = X$. If not, set $X^\vee = X \cup \{\omega\}$ (where $\omega \notin X$) equipped with the topology whose closed sets are $X^\vee$ and all closed sets of X distinct from X.

Remark 1.6. If $X^\vee = X \cup \{\omega\}$, then the following properties hold.

(1) $\overline{\{\omega\}}^\vee = X^\vee$; in particular $X^\vee$ is an irreducible space.

(2) For each $x \in X$, $\overline{\{x\}}^\vee = \overline{\{x\}}$.

(3) Let O be an open set of $X^\vee$ containing ω. Then there exists a nonempty open set U of X such that $O = U \cup \{\omega\}$.

(4) The nonempty irreducible closed sets of $X^\vee$ are $X^\vee$ and the nonempty irreducible closed sets of X.

Recall that, following [7], a continuous map $q : X \longrightarrow Y$ is said to be a *quasihomeomorphism* if for each open set U of X there exists a unique open set V of Y such that $U = q^{-1}(V)$; this means that the mapping $\theta(q) : \mathcal{O}(X) \longrightarrow \mathcal{O}(y)$ (where $\mathcal{O}(X)$ is the family of all open sets of X) which takes an open set V of Y to $q^{-1}(V)$ is bijective.

Proposition 1.7. *Let X be a T_0-space. Then the canonical embedding $i : X \hookrightarrow X^\vee$ is a quasihomeomorphism.*

Proof: It is enough to prove the result where $X^\vee = X \cup \{\omega\}$.

Since the open sets of $X^\vee$ are $\emptyset$ and $U \cup \{\omega\}$, where U is a nonempty open set of X, it is clear that the mapping $\theta(i) : \mathcal{O}(X^\vee) \longrightarrow \mathcal{O}(X)$, which takes an open set V of $X^\vee$ to $i^{-1}(V) = V \cap X$, is bijective. Thus, i is a quasihomeomorphism. □

Let X be a topological space. We say that X is *semisober* if it satisfies the following properties.

(a) X is a T_0-space.

(b) Each nonempty irreducible closed set of X distinct from X has a generic point.

The following result provides a link between sober and semisober spaces.

Proposition 1.8. *Let X be a T_0-space. Then the following statements are equivalent.*

(i) *X is a semisober space.*

(ii) *$X^\vee$ is a sober space.*

Proof: The result is obvious when $X^{\vee} = X$. Let us suppose that $X^{\vee} = X \cup \{\omega\}$.

$(i) \Longrightarrow (ii)$ Let C be a nonempty irreducible closed set of $X^{\vee}$. Following Remark 1.6(4), $C = X^{\vee}$ or C is a nonempty irreducible closed set of X.

If $C = X^{\vee}$, then $C = \overline{\{\omega\}}$.

If C is an irreducible closed set of X, then necessarily $C \neq X$. Hence, there exists $x \in C$ such that $C = \overline{\{x\}} = \overline{\{x\}}^{\vee}$.

$(ii) \Longrightarrow (i)$ Let C be a nonempty irreducible closed set of X distinct from X. Then C is a nonempty irreducible closed set of $X^{\vee}$. Hence, there exists $x \in C$ such that $C = \overline{\{x\}}^{\vee} = \overline{\{x\}}$. $\square$

Down-spectral spaces and spectral spaces are linked via Proposition 1.10, but first we need a technical straightforward lemma which may be derived immediately from [7, Chap. 0, Corollaire 2.7.6].

Lemma 1.9. *Let $q : X \longrightarrow Y$ be a quasihomeomorphism and U an open set of Y. Then the following statements are equivalent.*

(i) U is compact.

(ii) $q^{-1}(U)$ is compact.

Now, combining Proposition 1.7, Proposition 1.8, and Lemma 1.9, we easily get the following result.

Proposition 1.10. *Let X be a T_0-space. Then, X is a down-spectral space if and only if $X^{\vee}$ is a spectral space.*

We need also some further notations.

Notation 1.11. Let X be a T_0-space. If X is compact, set $X^{\wedge} = X$. If not, set $X^{\wedge} = X \cup \{\psi\}$ equipped with the topology whose open sets are $X^{\wedge}$ and all open sets of X.

Remark 1.12. Let X be a T_0-space. Then the following properties hold.

(1) The canonical embedding $i : X \hookrightarrow X^{\wedge}$ is open.

(2) The open compact sets of $X^{\wedge}$ are $X^{\wedge}$ and the open compact sets of X.

(3) $\overline{\{\psi\}}^{\wedge} = \{\psi\}$, and for each $x \in X$, $\overline{\{x\}}^{\wedge} = \overline{\{x\}} \cup \{\psi\}$.

(4) Let F be a nonempty subset of $X^{\wedge}$. Then $\overline{F}^{\wedge} = \overline{F} \cup \{\psi\}$. Hence, the closed sets of $X^{\wedge}$ are $\emptyset$ and $G \cup \{\psi\}$, where G is closed in X.

The following result shows that the property "sober" of a space X depends uniquely on the same property of a nonempty set U of X and its complement $X \setminus U$.

First, we need to recall a technical lemma.

Lemma 1.13 ([4, Proposition 7, p. 122]). *Let X be a topological space and U a nonempty open subset of X. Then the mapping $V \longmapsto \overline{V}$ defines a bijection from the set of irreducible nonempty closed subsets of U onto the set of irreducible nonempty closed subsets of X meeting U. The inverse bijection is $Z \longmapsto Z \cap U$.*

Theorem 1.14. *Let X be a T_0-space and U a nonempty open set of X. Then the following statements are equivalent.*
(i) X is sober.
(ii) U and $X \setminus U$ are sober.

Proof: $(i) \Longrightarrow (ii)$ Clearly, any closed set of a sober space is sober.

Let us prove that U is sober. In fact, this has been done in [1, Lemma 5.3], but, for the sake of completeness, we provide a proof. Let F be a nonempty irreducible closed subset of U. According to Lemma 1.13, $\overline{F}$ is an irreducible closed subset of X. Hence, $\overline{F}$ has a generic point x. Let us write $\overline{F} = \overline{\{x\}}^X$ and note that $x \in U$. We have $x \in F = \overline{F} \cap U$, and thus $F = \overline{\{x\}}^U$.

$(ii) \Longrightarrow (i)$ Let C be a nonempty irreducible closed set of X.

We consider two cases.

Case 1. Suppose that $C \cap U \neq \emptyset$. Then $C \cap U$ is a nonempty irreducible closed set of U. Hence, there exists $x \in C \cap U$ such that $C \cap U = \overline{\{x\}}^U$.

Let us prove that $C = \overline{\{x\}}$. Of course, $\overline{\{x\}} \subseteq C$.

Conversely, let $y \in C$ and V be an open set of X containing y.

Since C is irreducible (in X), $C \cap U \cap V \neq \emptyset$. Thus, $\overline{\{x\}} \cap U \cap V \neq \emptyset$. Therefore, $x \in V$.

Case 2. Suppose that $C \cap U = \emptyset$.

In this case, C is a nonempty irreducible closed set of the sober closed set $X \setminus U$. Thus, C has a generic point in $X \setminus U$; i.e, there exists $x \in C$ such that $C = \overline{\{x\}}$. □

If X is a T_0-space, then X is open in $X^{\wedge}$ and $X^{\wedge} \setminus X$ is sober. Then, according to Theorem 1.14, we get the following corollary.

Corollary 1.15. *Let X be a T_0-space. Then the following statements are equivalent.*

(i) *X is a sober space.*

(ii) *$X^{\wedge}$ is a sober space.*

Proposition 1.16. *Let X be a T_0-space. Then the following statements are equivalent.*

(i) *X is an up-spectral space.*

(ii) *$X^{\wedge}$ is a spectral space.*

Proof: According to Corollary 1.15 and the fact that $X^{\wedge}$ is a compact space, it suffices to show that X has a basis of compact open sets if and only if $X^{\wedge}$ has a basis of compact open sets.

Indeed, let $\mathcal{B}$ ($\mathcal{B}'$, respectively) be a basis of compact open sets of X ($X^{\wedge}$, respectively) closed under finite intersections. Then, using Remark 1.12(2), $\mathcal{B}_\infty = \mathcal{B} \cup \{X^{\wedge}\}$ ($\mathcal{B}'_1 = \mathcal{B}' \setminus \{X\}$, respectively) is a basis of compact open sets of $X^{\wedge}$ (X, respectively). □

2. Duality

We are in a position to state the main result of this paper.

Theorem 2.1. *Let $\mathcal{UD}$ be the class of all up-spectral spaces and down-spectral spaces. Let $\star : \mathcal{UD} \longrightarrow \mathcal{UD}$ be the correspondence which takes X in $\mathcal{UD}$ to its Hochster dual $X^\star$. Then $\star$ is a duality (in the sense of Kopperman).*

In order to prove this result, we need a sequence of lemmata.

Lemma 2.2. *Let X be a topological space with a basis of compact open sets closed under finite intersections. Then $(X^\star)^\vee$ and $(X^\wedge)^\star$ are homeomorphic.*

Proof: There are two cases to be discussed.

Case 1. If X is compact, then by Proposition 1.2, $X^\star$ is reducible or has a generic point. Thus, $(X^\wedge)^\star = (X^\star)^\vee = X^\star$.

Case 2. If X is not compact, then $X^\star$ is irreducible with no generic point, by Proposition 1.2. Set $(X^\star)^\vee = X \cup \{\omega\}$ and $X^\wedge = X \cup \{\psi\}$, and consider the map $f : (X^\wedge)^\star \longrightarrow (X^\star)^\vee$ which takes x to x for $x \in X$ and ψ to ω.

Let us show that f is a homeomorphism.

(1) *f is a continuous map.*

Let O be a nonempty open set of $(X^\star)^\vee$. Then $\omega \in O$, and there exists a nonempty open set U of $X^\star$ such that $O = U \cup \{\omega\}$. Hence, there exists a family $(V_i : i \in I)$ of compact open sets of X such that $U = \bigcup[X \setminus V_i : i \in I]$. Thus,

$$f^{-1}(O) = \bigcup[X \setminus V_i : i \in I] \cup \{\psi\} = \bigcup[X^\wedge \setminus V_i : i \in I].$$

Now, by Remark 1.12(2), each V_i is a compact open set of $X^\wedge$. It follows that $f^{-1}(O)$ is open in $(X^\wedge)^\star$.

(2) *f is an open map.*

Let O be a nonempty open set of $(X^\wedge)^\star$. Then there exists a family $(U_i : i \in I)$ of compact open sets of $X^\wedge$ such that $O = \bigcup[X^\wedge \setminus U_i : i \in I]$. By Remark 1.12(2), U_i is either $X^\wedge$ or a compact open set of X. Thus, $f(O) = \bigcup[(X \setminus U_i) \cup \{\psi\} : i \in I]$ is an open set of $(X^\star)^\vee$.

Therefore, f is a homeomorphism.

And the proof is complete. □

Lemma 2.3. *Let X be an irreducible space with no generic point, then $(X^\star)^\wedge$ and $(X^\vee)^\star$ are homeomorphic.*

Proof: According to Proposition 1.3, $X^\star$ is not compact. Set $(X^\star)^\wedge = X \cup \{\psi\}$ and $X^\vee = X \cup \{\omega\}$, and consider the map $f : (X^\vee)^\star \longrightarrow (X^\star)^\wedge$ which takes x to x for $x \in X$ and ω to ψ.

(1) *f is a continuous map.*

Let O be a nonempty open set of $(X^\star)^\wedge$ distinct from $(X^\star)^\wedge$. Then there exists a family $(U_i, i \in I)$ of compact open sets of X such that $O = \bigcup[X \setminus U_i : i \in I]$. Thus,

$$f^{-1}(O) = \bigcup[X^\vee \setminus (U_i \cup \{\omega\}) : i \in I].$$

By Proposition 1.7 and Lemma 1.9, $(U_i \cup \{\omega\} : i \in I)$ is a family of compact open sets of $X^\vee$. Consequently, $f^{-1}(O)$ is an open set of $(X^\vee)^\star$.

(2) *f is an open map.*

Let O be a nonempty open set of $(X^\vee)^\star$ distinct from $(X^\vee)^\star$. Then there exists a family $(U_i, i \in I)$ of compact open sets of X such that

$$O = \bigcup[X^\vee \setminus (U_i \cup \{\omega\}) : i \in I] = \bigcup[X \setminus U_i : i \in I].$$

Hence, $f\,(O) = \bigcup[X \setminus U_i : i \in I]$ is an open set of $(X^\star)^\wedge$.

Therefore, f is a homeomorphism.

And the proof is complete. □

Lemma 2.4. *Let X and Y be two T_0-spaces. Then the following properties hold.*

(1) *If X and Y are homeomorphic, then $X^{\wedge}$ and $Y^{\wedge}$ are homeomorphic.*

(2) *If X and Y are both not compact, then the following statements are equivalent.*

(i) *X and Y are homeomorphic.*

(ii) *$X^{\wedge}$ and $Y^{\wedge}$ are homeomorphic.*

Proof: (1) Straightforward.

(2) It is sufficient to show $(ii) \Longrightarrow (i)$.

Indeed, since X and Y are not compact, set $X^{\wedge} = X \cup \{\omega\}$, $Y^{\wedge} = Y \cup \{\psi\}$, and $g : X^{\wedge} \longrightarrow Y^{\wedge}$ a homeomorphism between $X^{\wedge}$ and $Y^{\wedge}$. Thus, $g(\{\omega\}) = \{g(\omega)\}$ is a closed set of $Y^{\wedge}$; and since $\{\psi\}$ is the unique closed point of $Y^{\wedge}$, we get $g(\omega) = \psi$. Therefore, the map $f : X \longrightarrow Y$, such that $f(x) = g(x)$ for each $x \in X$, is a homeomorphism. □

Lemma 2.5. *Let X and Y be two T_0-spaces. Then the following properties hold.*

(1) *If X and Y are homeomorphic, then $X^{\vee}$ and $X^{\vee}$ are homeomorphic.*

(2) *If X and Y are both irreducible with no generic point, then the following statements are equivalent.*

(i) *X and Y are homeomorphic.*

(ii) *$X^{\vee}$ and $Y^{\vee}$ are homeomorphic.*

Proof: (1) Straightforward.

(2) It is sufficient to prove $(ii) \Longrightarrow (i)$. Let X and Y be two irreducible spaces with no generic point. Set $X^{\vee} = X \cup \{\omega\}$, $Y^{\vee} = Y \cup \{\psi\}$, and $g : X^{\vee} \longrightarrow Y^{\vee}$ a homeomorphism. We get $g\left(X^{\vee}\right) = g\left(\overline{\{\omega\}}^{\vee}\right) = \overline{\{g(\omega)\}}^{\vee} = Y^{\vee} = \overline{\{\psi\}}^{\vee}$ and thus, $g(\omega) = \psi$. Therefore, it is easily seen that the map $f : X \longrightarrow Y$, such that $f(x) = g(x)$ for each $x \in X$, is a homeomorphism. □

Proof of Theorem 2.1: We break the proof into four steps.

<u>Step 1.</u> X is an up-spectral space if and only if $X^{\star}$ is a down-spectral space.

Proof of Step 1 : Let X be an up-spectral space. By Proposition 1.16, $X^{\wedge}$ is a spectral space and thus, by [8], $(X^{\wedge})^{\star}$ is spectral. Hence, $(X^{\star})^{\vee}$ is also spectral, by Lemma 2.2. Therefore, Proposition 1.10 shows that $X^{\star}$ is down-spectral.

Conversely, suppose that $X^{\star}$ is down-spectral. Then by Proposition 1.10, $(X^{\star})^{\vee}$ is a spectral space and thus, Lemma 2.2 shows that $(X^{\wedge})^{\star}$ is spectral. Therefore, $X^{\wedge}$ is spectral [8]. Proposition 1.16 completes the proof of this step.

Step 2. If X is a down-spectral space, then $X^{\star}$ is an up-spectral space.

Proof of Step 2 : First, let us remark that if X is reducible or has a generic point, then X is spectral and thus, $X^{\star}$ is spectral. In particular, $X^{\star}$ is up-spectral.

Now, let X be an irreducible down-spectral space with no generic point. It follows from Proposition 1.10 that $X^{\vee}$ is a spectral space; hence, $(X^{\vee})^{\star}$ is also spectral [8]. By Lemma 2.3, $(X^{\star})^{\wedge}$ is a spectral space. Consequently, $X^{\star}$ is an up-spectral space, by Proposition 1.16.

Step 3. If X is an up-spectral space, then $X^{\star\star} = X$.

Proof of Step 3 : First, let us remark that the result is straightforward when X is compact.

Suppose that X is not compact. Then $X^{\star}$ is irreducible with no generic point (see Proposition 1.2). By Proposition 1.16, $X^{\wedge}$ is spectral and thus, $(X^{\wedge})^{\star\star} = X^{\wedge}$ [8]. Therefore, according to Lemma 2.2 and Lemma 2.3, $(X^{\star\star})^{\wedge}$ and $X^{\wedge}$ are homeomorphic.

Remark that $X^{\star\star}$ is not compact. Indeed, if not, $X^{\star\star}$ would be spectral, as would X, by Hochster, contradicting the fact that X is not compact. Therefore, using Lemma 2.4 and its proof, $X^{\star\star} = X$.

Step 4. If X is a down-spectral space, then $X^{\star\star} = X$.

Proof of Step 4 : If X is reducible or has a generic point, then X is spectral and thus, $X^{\star\star} = X$.

If X is an irreducible space with no generic point, then X is not spectral and consequently, $X^{\star\star}$ is not spectral. Since $X^{\star\star}$ is a down-spectral space, it is an irreducible space with no generic point.

On the other hand, by Proposition 1.10, $X^{\vee}$ is a spectral space and thus, $(X^{\vee})^{\star\star} = X^{\vee}$ [8].

Combining lemmas 2.2 and 2.3, it is easily seen that $(X^{\star\star})^{\vee}$ and $X^{\vee}$ are homeomorphic.

Finally, Lemma 2.5 and its proof show that $X^{\star\star} = X$.

And the proof is complete. □

Example 2.6. *If $X^{\star}$ is up-spectral, then X need not be down-spectral.*

Let Y be an infinite set and $\omega \notin Y$. Set $X = Y \cup \{\omega\}$. Equip X with the topology whose closed sets are X, C, Y, and $C \cup \{\omega\}$, where the sets C are all finite subsets of Y. Clearly, X is a Noetherian space (i.e., each open set of X is compact).

We claim that $X^{\star}$ is up-spectral; however, X is not down-spectral.

Proof: Clearly, X is not semisober, since Y is an irreducible closed set of X with no generic point. Thus, X is not down-spectral.

Let us prove that $X^{\star}$ is endowed with the discrete topology.

Indeed, we have $\{\omega\} = X \setminus Y$, and Y is an open compact set of X. Hence, $\{\omega\}$ is an open set of $X^{\star}$.

Now let $y \in Y$; then $\{y\} = X \setminus ((Y \setminus \{y\}) \cup \{\omega\})$ and $(Y \setminus \{y\}) \cup \{\omega\}$ is an open compact set of X. Thus, $\{y\}$ is open in $X^{\star}$. It follows that $X^{\star}$ is a discrete space.

Of course, $X^{\star}$ satisfies the following properties.

(a) $X^{\star}$ is sober (since it is Hausdorff).

(b) $\mathcal{B} = \{\{t\} \mid t \in X\}$ is a basis of compact open sets of $X^{\star}$.

(c) The intersection of two compact open sets of $X^{\star}$ is compact.

Therefore, $X^{\star}$ is up-spectral. □

The following result gives a class of topological spaces in which "up-spectral" is a dual property of "down-spectral."

Theorem 2.7. *Let X be an irreducible T_0-space. Then the following statements are equivalent.*

(i) X is down-spectral.

(ii) $X^{\star}$ is up-spectral.

Proof: $(i) \Longrightarrow (ii)$ See Step 2 of the Proof of Theorem 2.1.

$(ii) \Longrightarrow (i)$ Let X be an irreducible T_0-space such that $X^{\star}$ is up-spectral.

Case 1. X has a generic point. In this case, by Proposition 1.3, $X^{\star}$ is compact. Since $X^{\star}$ is an up-spectral space, $X^{\star}$ is spectral. Thus, X is spectral [8]. In particular, X is down-spectral.

Case 2. X has no generic point. According to Lemma 2.3, $(X^\star)^\wedge$ and $(X^\vee)^\star$ are homeomorphic. Since $X^\star$ is an up-spectral space, by Proposition 1.16, $(X^\star)^\wedge$ is spectral and so is $(X^\vee)^\star$. Hence, $X^\vee$ is spectral [8], and finally, Proposition 1.10 shows that X is down-spectral. □

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